Implementation of the three-neutron quantization condition

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“Three relativistic neutrons in a finite volume”
arXiv:2303.10219

Motivation

Neutron-rich nuclei are increasingly sensitive to three-nucleon forces. To better understand properties of these nuclei (shell structure, spectroscopy, dripline, etc) we need to understand the three nucleon interaction.

Neutron star properties are also constrained by three-nucleon forces.

A theoretical understanding of the three-nucleon interaction also provides insight into the many-body problem in **nuclear matter.**

The three-neutron case provides a starting point for understanding more general three-particle interactions.

The three-neutron quantization condition

A quantization condition (QC) relates the finite-volume spectrum to infinite-volume scattering amplitudes.

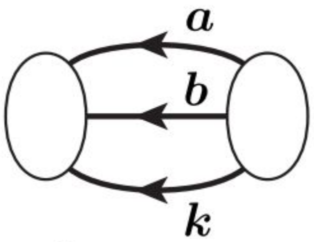
QC's are derived by isolating the part of the (finite-volume) correlator which scale as powers of the volume ~$(1/L)^n$.

A two-particle QC takes the form: $\det\left[1 + \boldsymbol{K}_2 \boldsymbol{F}_2\right] = 0$

For three particles, we need to also quantify the three-particle interactions and which two particles enter the two-particle interactions $\implies G, K_{df,3}$

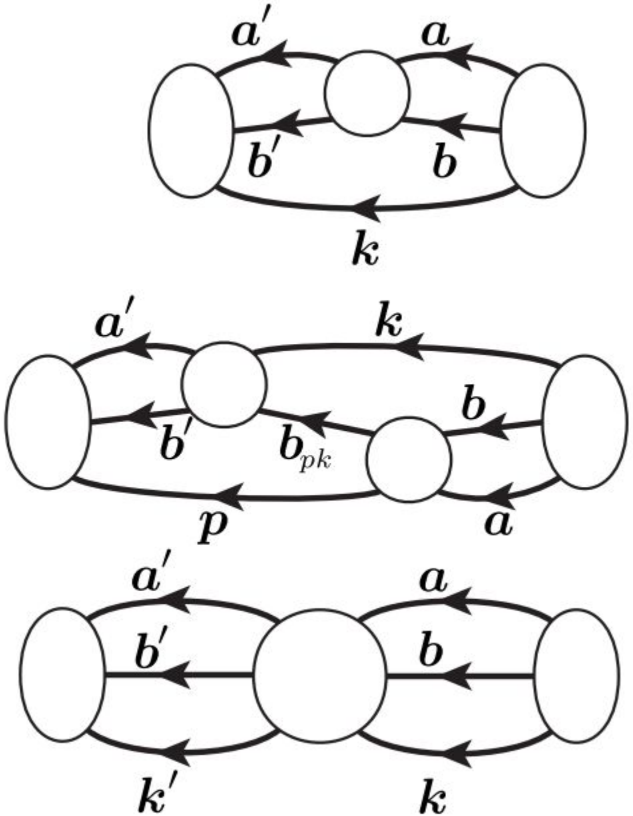
To complete the QC, integral equations are used to determine infinite volume amplitudes from K_2 and $K_{df,3}$.

The three-neutron quantization condition



$$\det_{\boldsymbol{k},\ell,m,\boldsymbol{m}_s^*} \left[1 - \mathbf{K}_{\text{df},3}\mathbf{F}_3\right] = 0$$

$$\mathbf{F}_3 = \frac{\mathbf{F}}{3} + \mathbf{F}\frac{1}{1-\mathbf{M}_{2,L}\mathbf{G}}\mathbf{M}_{2,L}\mathbf{F} \ , \ \mathbf{M}_{2,L} = \frac{1}{\mathbf{K}_2^{-1} - \mathbf{F}}$$



$$\mathbf{K}_{\text{df},3} \to 0 \implies \det\left[\mathbf{F}_3^{-1}\right] = 0$$

Vector Space

The space in which the QC matrices act is defined by spectator momentum k, dimer orbital angular momentum l and azimuthal component m, and the spin of each neutron*.

For a particular k and $l_{max} = 1$, this leads to 1 x 8 + 3 x 8 = 32 dimensional matrices.

Antisymmetry of the 3-neutron wavefunction requires that if l is even, the total dimer spin, s, must be as well. This gives 1 x 1 x 2 + 3 x 3 x 2 = 20 dimensional matrices.

$\Longrightarrow$ QC matrix is (# kinematically allowed spectator momenta) x 20 dimensional.

*frame dependent

Irrep Projection

The QC matrices are all invariant under the action of the little group determined by total momentum of the 3 neutrons, $\vec{P}$.

Schur's Lemma $\Longrightarrow$ block diagonalization in irreps of little group $\Longrightarrow$ solutions of QC live in definite irreps.

Projection onto a particular irrep reduces dimensionality and speeds up calculation of eigenvalues. Additionally helps distinguish different solutions.

$$\vec{P} = \frac{2\pi}{L}(0,0,1) \implies C_{4V}$$

Name	Dimension	Fermionic
A1	1	False
A2	1	False
B1	1	False
B2	1	False
Eg	2	False
G1	2	True
G2	2	True

$$\vec{P} = \frac{2\pi}{L}(0,0,0) \implies O_h^d$$

Name	Dimension	Fermionic
A1g	1	False
A2g	1	False
Eg	2	False
T1g	3	False
T2g	3	False
G1g	2	True
G2g	2	True
Hg	4	True
A1u	1	False
A2u	1	False
Eu	2	False
T1u	3	False
T2u	3	False
G1u	2	True
G2u	2	True
Hu	4	True

K_2 parameterization

In the basis of total dimer angular momentum j with $l_{max} = 1$ we have $j = 0, 1, 2$.

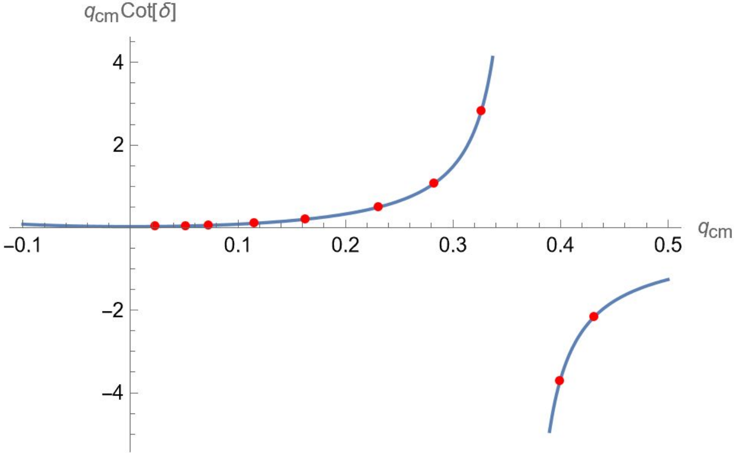
For $j = 0$, we have either $l = s = 0$ or $l = s = 1$ which do not mix due to parity and so each can be expressed in terms of a scattering phase shift.

Likewise, for $j = 1$, we have $l = s = 1$ which does not mix with any other channels.

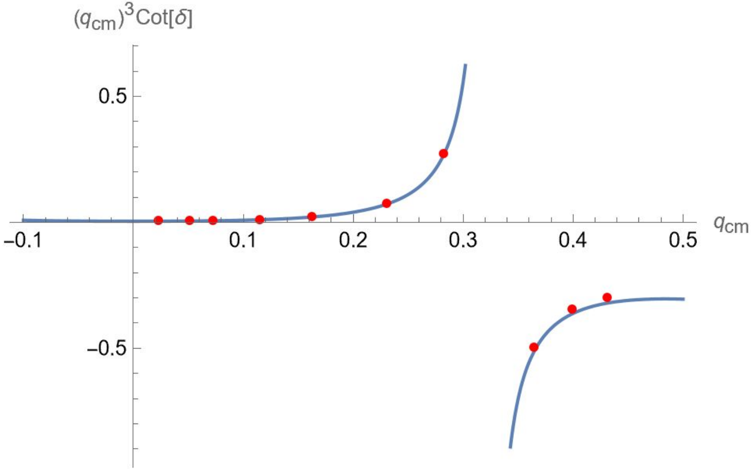
For $j = 2$, both $l = s = 1$ and $l = 3, s = 1$ are allowed and mix in general but for this analysis we assume that the $l = 3$ contribution is small and treat $j = 2, l = s = 1$ as a single channel.

We fit each of these four channels to “experimental” scattering data to determine K_2

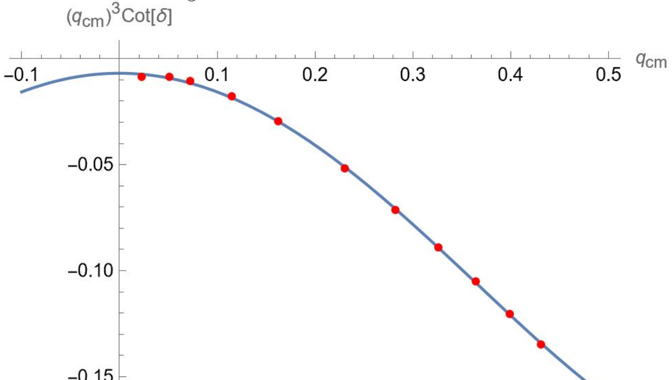
$j = l = s = 0$



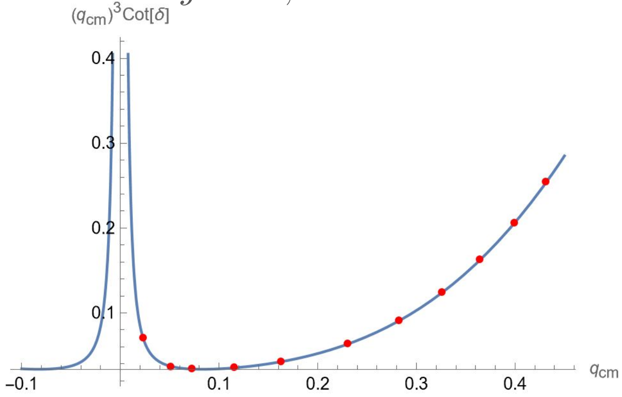
$j = 0, l = s = 1$



$j = l = s = 1$



$j = 2, l = s = 1$



"Construction of high-quality NN potential models"
Stoks, Klomp, Terheggen, de Swart

Solving the QC

Pick a frame and identify free levels

Construct the QC matrix

Project onto an irrep in little group

Identify energies where eigenvalues of the QC matrix are zero

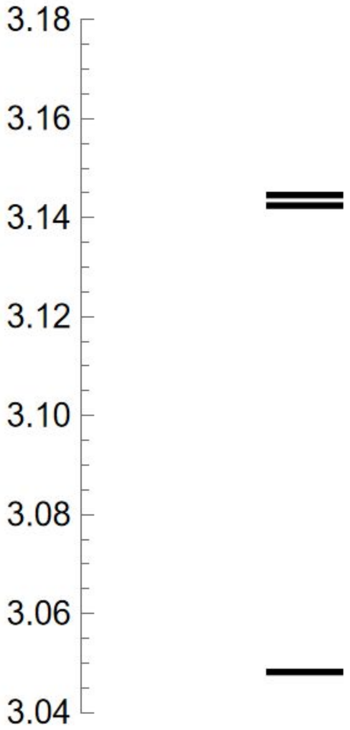
Solving the QC

Pick a frame and identify free levels

$\vec{P}$	$(\lvert\vec{k}_1\rvert, \lvert\vec{k}_2\rvert, \lvert\vec{k}_3\rvert)$	d	E	irreps
(0,0,1)	(0,0,1)	2	3.04819	G_1
(0,0,1)	(0,1,2)	32	3.14244	$8G_1 \oplus 8G_2$
(0,0,1)	(1,1,1)	18	3.14456	$5G_1 \oplus 4G_2$
(0,0,1)	(0,1,4)	8	3.2292	$3G_1 \oplus G_2$
(0,0,1)	(0,2,3)	32	3.23271	$8G_1 \oplus 8G_2$
(0,0,1)	(1,1,3)	32	3.23483	$8G_1 \oplus 8G_2$

$$m_N L = 20, m_\pi = .15 m_N$$

Solving the QC



Pick a frame and identify free levels

$\vec{P}$	$(\lvert\vec{k}_1\rvert, \lvert\vec{k}_2\rvert, \lvert\vec{k}_3\rvert)$	d	E	irreps
(0,0,1)	(0,0,1)	2	3.04819	G_1
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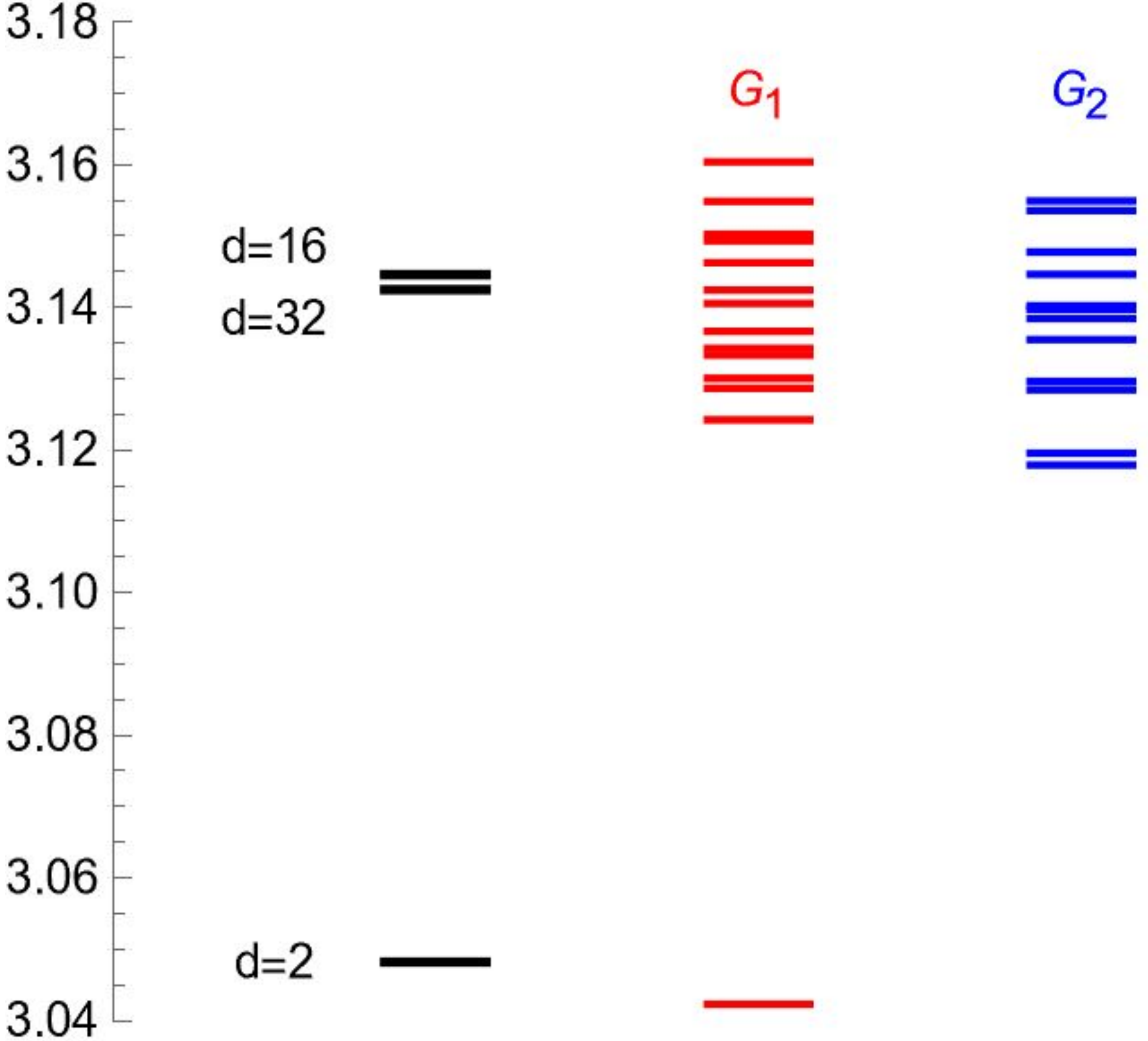
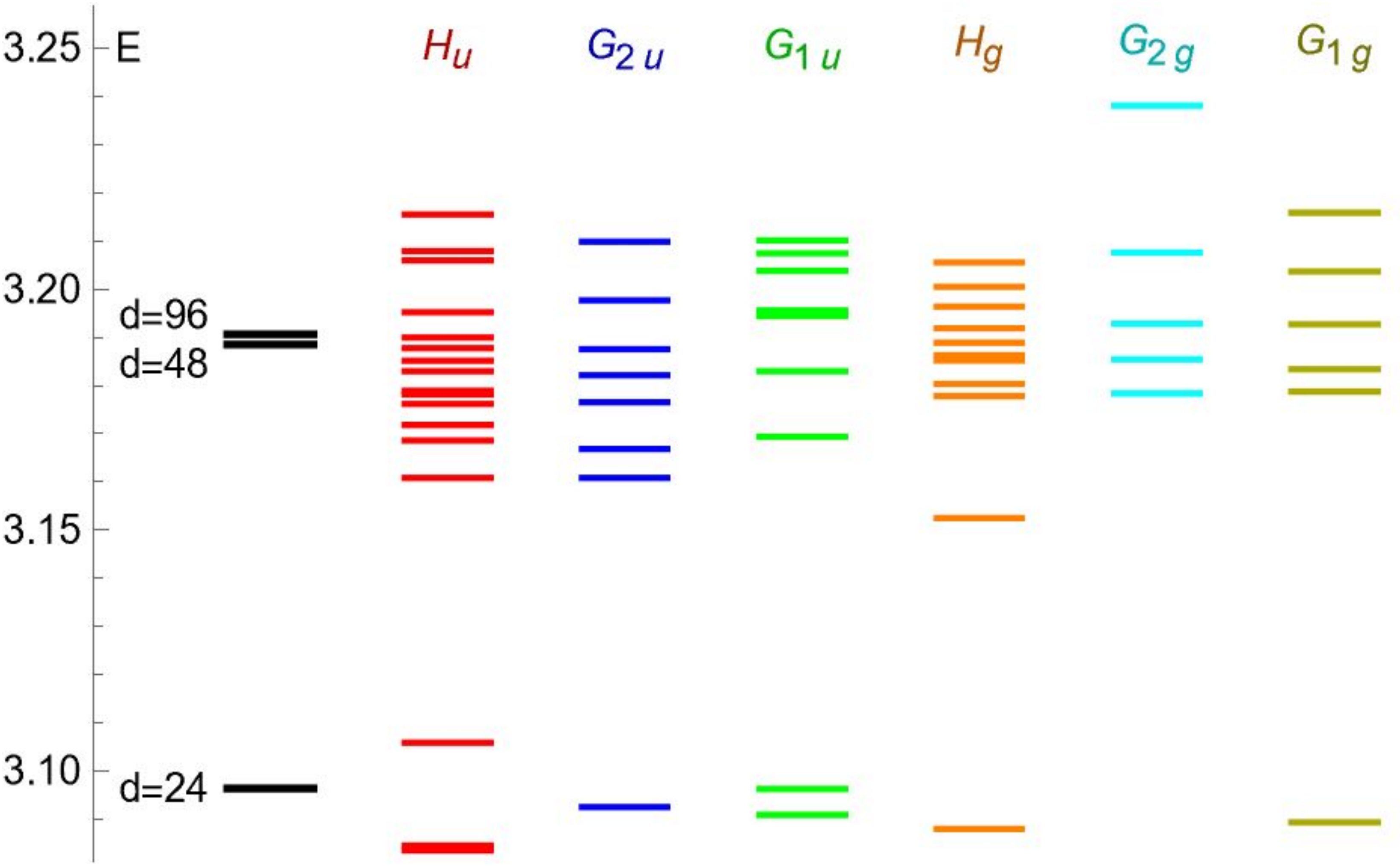


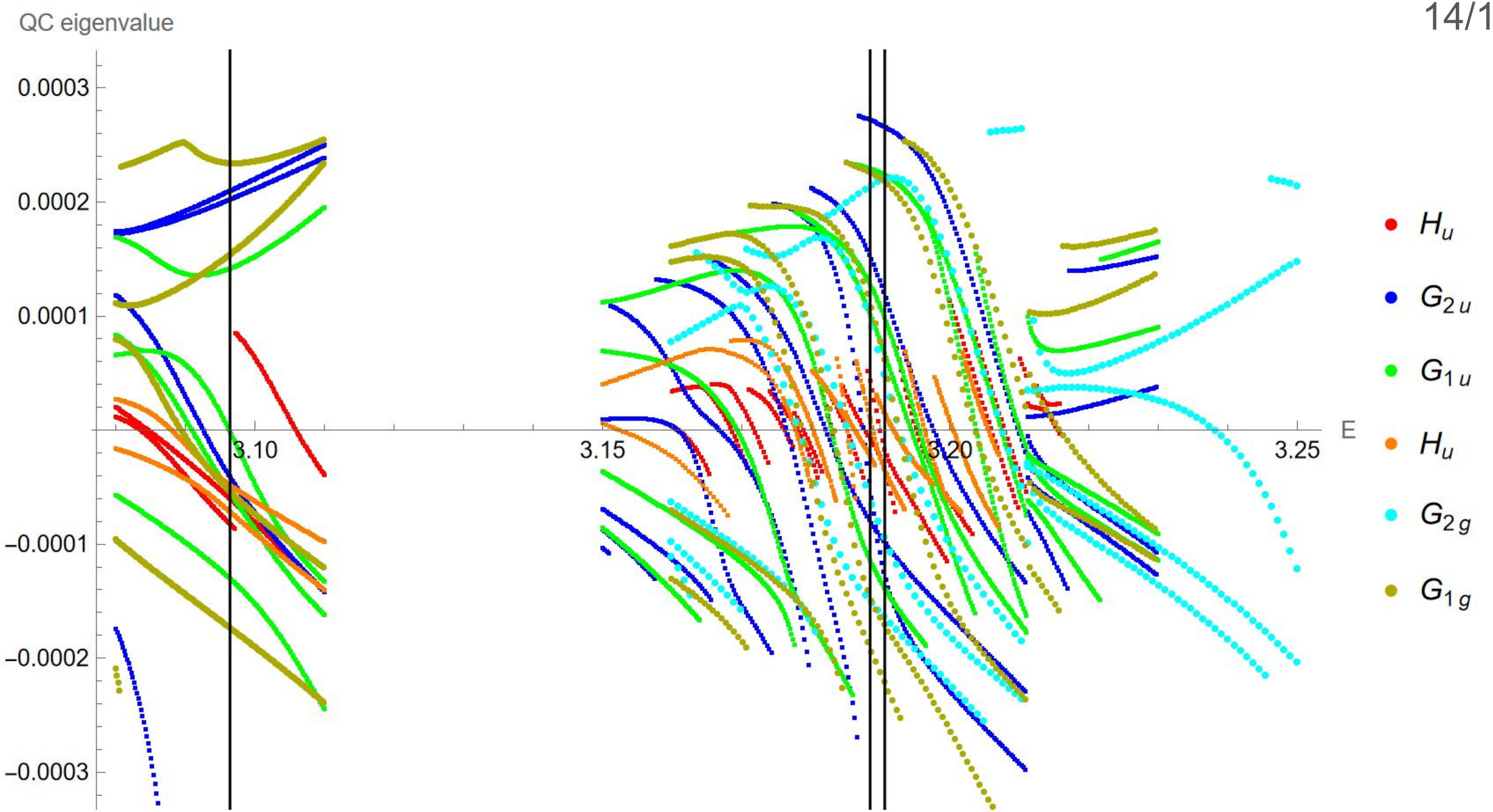
3.18
3.16
3.14
3.12
3.10
3.08
3.06
3.04
d=16
d=32
d=2
G_1
G_2

$\vec{P} = (0,0,0)$:

$\vec{P}$	$(\lvert\vec{k}_1\rvert, \lvert\vec{k}_2\rvert, \lvert\vec{k}_3\rvert)$	d	E	irreps
(0,0,0)	(0,1,1)	24	3.09637	$G_{1g} \oplus H_g \oplus 2G_{1u} \oplus G_{2u} \oplus 3H_u$
(0,0,0)	(0,2,2)	48	3.18851	$G_{1g} \oplus G_{2g} \oplus 2H_g \oplus 3G_{1u} \oplus 3G_{2u} \oplus 6H_u$
(0,0,0)	(1,1,2)	96	3.19063	$4G_{1g} \oplus 4G_{2g} \oplus 8H_g \oplus 4G_{1u} \oplus 4G_{2u} \oplus 8H_u$
(0,0,0)	(0,3,3)	32	3.27692	$G_{1g} \oplus G_{2g} \oplus H_g \oplus 2G_{1u} \oplus 2G_{2u} \oplus 4H_u$
(0,0,0)	(1,1,4)	12	3.27738	$G_{1g} \oplus H_g \oplus G_{1u} \oplus H_u$

$$m_N L = 20, m_\pi = .15 m_N$$


E
3.25
3.20
3.15
3.10
d=96
d=48
d=24
H_u
G_{2u}
G_{1u}
H_g
G_{2g}
G_{1g}


QC eigenvalue
0.0003
0.0002
0.0001
−0.0001
−0.0002
−0.0003
3.10
3.15
3.20
3.25
E
H_u
G_{2u}
G_{1u}
H_u
G_{2g}
G_{1g}

Nonphysical Solutions

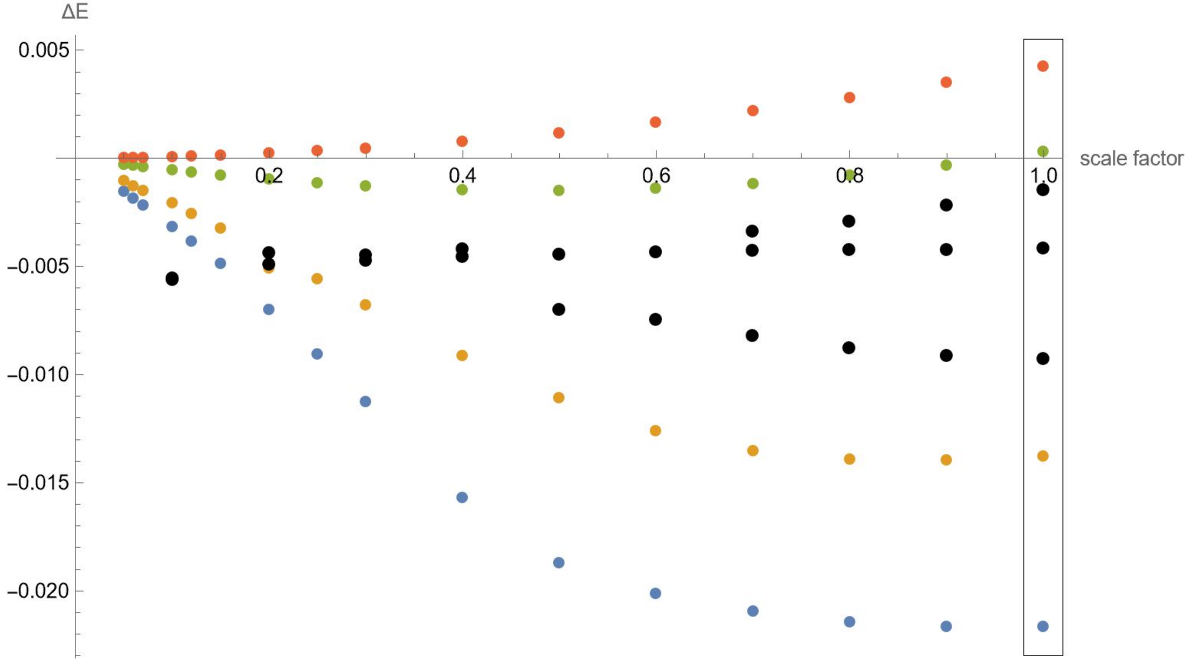
$\vec{P}$	$(\lvert\vec{k}_1\rvert, \lvert\vec{k}_2\rvert, \lvert\vec{k}_3\rvert)$	d	E	irreps
(1,1,0)	(0,1,1)	8	3.09637	$4G$

Nonphysical solutions can arise from features of K_2.

We can trace them by scaling down the K_2 interactions.

Introducing a cutoff in K_2 for negative CM momentum removes most nonphysical solutions.

Physical+nonphysical zero shifts from free energy for P={1,1,0}, nk={0,1,1}



Conclusion

We have produced the finite-energy spectra for the case of two-particle interactions only, separated by irrep.

Future work includes investigating the spectral dependence on K_2 and extending the QC to include $K_{df,3}$.

In the end: use lattice data to fix parameterization of K_2 and $K_{df,3}$ and extract infinite volume amplitudes.