Calculation of meson charge radii using model-independent method in the PACS10 configuration

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for PACS Collaboration

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Introduction

(mean-square) charge radius … a quantity that characterizes the structure of hadrons.

It represents the spread of the charge distribution.

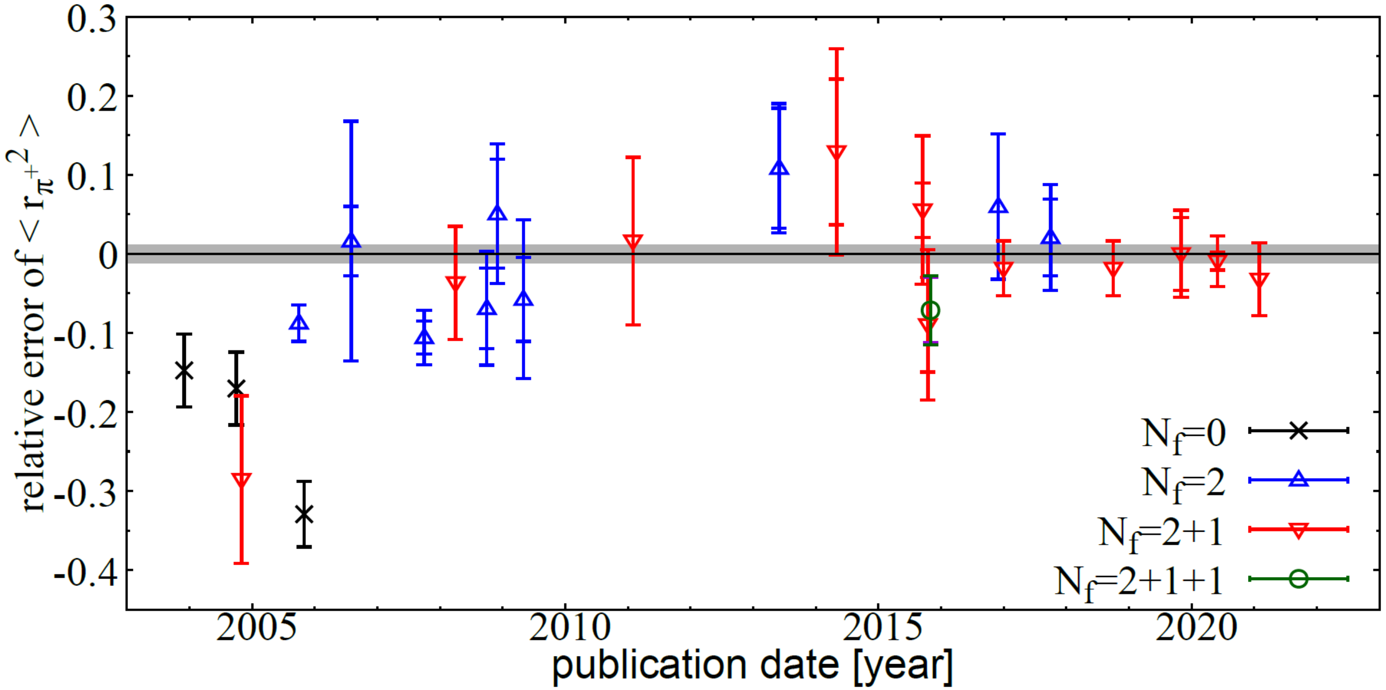
$$\langle r_\pi^2 \rangle = -6 \, \frac{\mathrm{d}}{\mathrm{d}Q^2} \underbrace{F_\pi(Q^2)}_{} \Big|_{Q^2=0}$$

electromagnetic form factor

$$\langle \pi^+(p_f) | V_\mu | \pi^+(p_i) \rangle = (p_f + p_i)_\mu F_\pi(Q^2)$$

electromagnetic current: $V_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f$

Momentum transfer : $Q^2 = -(p_f - p_i)^2 \geq 0$



- ✓ Initially : large difference
  Recently : consistent
- ✓ Error : Lattice > Experimental
  There are 4 main systematic errors
  - ➢ Chiral extrapolation
  - ➢ Continuum extrapolation
  - ➢ Finite volume effect
  - ➢ Fit ansatz

Introduction

- Traditional method for calculating the charge radius using lattice QCD

3-point function

$$\frac{\sum_{\vec{x}_{\mathrm{sink}},\vec{x}} \langle \pi^+(\vec{x}_{\mathrm{sink}},t_{\mathrm{sink}}) V_4(\vec{x},t) \pi^+(0)^\dagger \rangle e^{-ipx_1}}{\sum_{\vec{x}_{\mathrm{sink}},\vec{x}} \langle \pi^+(\vec{x}_{\mathrm{sink}},t_{\mathrm{sink}}) V_4(\vec{x},t) \pi^+(0)^\dagger \rangle} = \frac{E_\pi(p) + m_\pi}{2E_\pi(p)} e^{-(E_\pi(p)-m_\pi)t} F_\pi(Q^2)$$

Introduction

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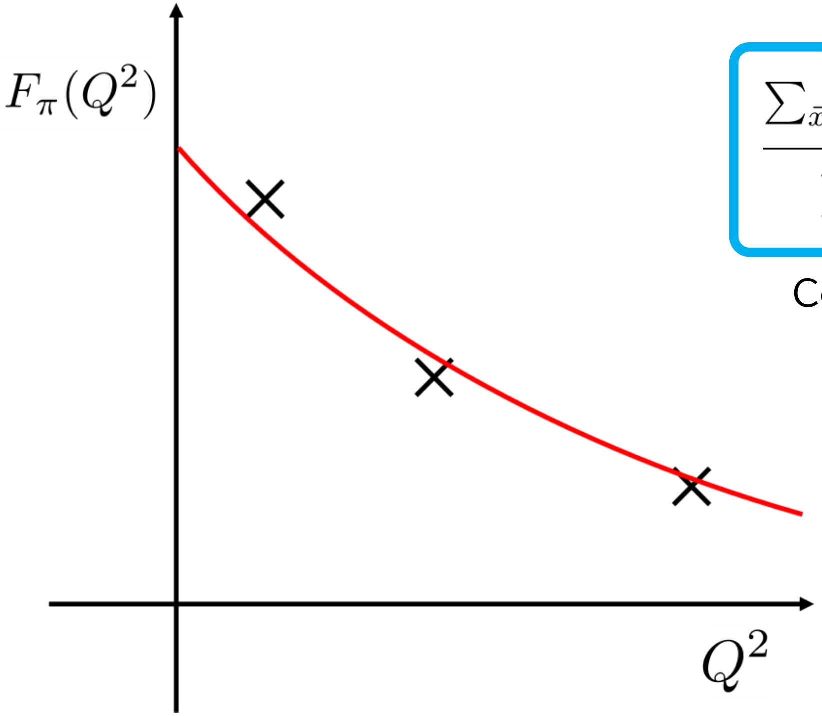

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Combining lattice calculation and known function

$$F_\pi(Q^2) = \frac{\text{Input from lattice calculation}}{\text{Known function}}$$

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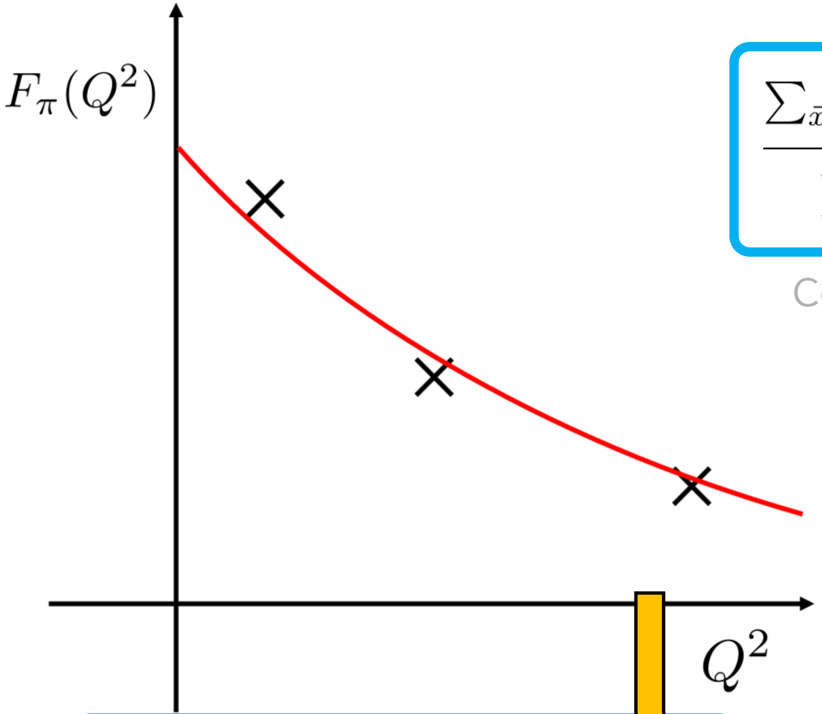

$$F_\pi(Q^2) = \begin{cases} \dfrac{1}{1+Q^2/M_{\rm pole}^2} \\ 1+\sum_k f_k (Q^2)^k \\ \sum_k a_k z^k \quad \left(z := \dfrac{\sqrt{4m_\pi^2+Q^2}-\sqrt{4m_\pi^2}}{\sqrt{4m_\pi^2+Q^2}+\sqrt{4m_\pi^2}} \right) \\ 1+\dfrac{1}{f_\pi^2}\left[2l_6^r(\mu)Q^2+4H(m_\pi^2,Q^2,\mu^2)\right] \end{cases}$$

assuming a function

and

fitting the data points

$$\left.\frac{\mathrm{d}}{\mathrm{d}Q^2} F_\pi(Q^2)\right|_{Q^2=0}$$

$$\langle r_\pi^2 \rangle = -6 \left.\frac{\mathrm{d}}{\mathrm{d}Q^2} F_\pi(Q^2)\right|_{Q^2=0}$$

$$\langle r_\pi^2 \rangle$$

✓ We extract the form factor data from the 3-point function and fit it.

Introduction

$$(\text{3-point function}) = (\text{Known function}) \times F_\pi(Q^2)$$

◆ Model-independent method for calculating the charge radius

$$\boxed{\langle {r_\pi}^2 \rangle}$$

$$\langle r_\pi^2 \rangle = -6 \frac{\mathrm{d}}{\mathrm{d}Q^2} F_\pi(Q^2)\Big|_{Q^2=0}$$

$$F_\pi(Q^2) = \frac{\text{Input from lattice calculation}}{\text{Known function}}$$

$$\boxed{\text{3-point function}} \xrightarrow{\qquad} \boxed{F_\pi(Q^2)} \xrightarrow{\text{Fit}} \boxed{\frac{d}{dQ^2} F_\pi(Q^2)\Big|_{Q^2=0}}$$

Introduction

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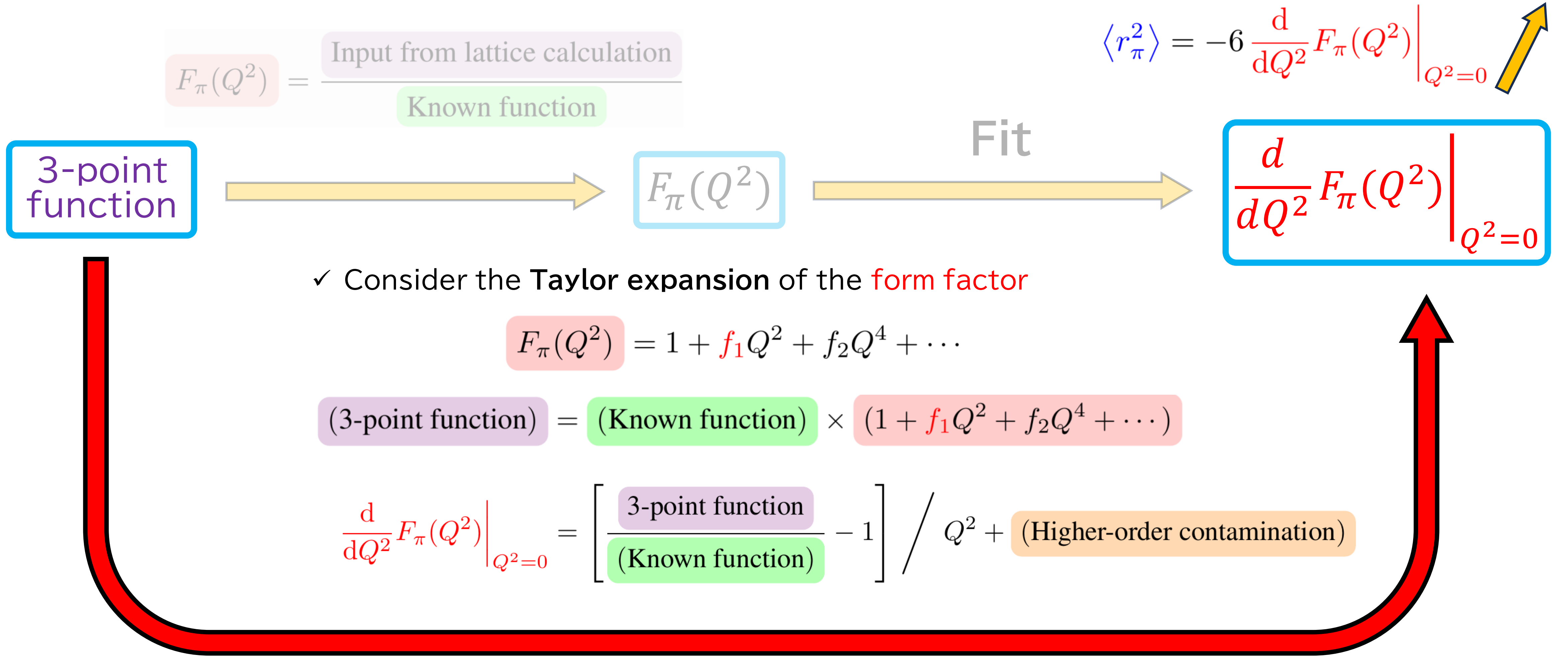
- ◆ Model-independent method for calculating the charge radius



$\langle {r_\pi}^2 \rangle$

$$\langle r_\pi^2 \rangle = -6 \frac{\mathrm{d}}{\mathrm{d}Q^2} F_\pi(Q^2) \Big|_{Q^2=0}$$

$$F_\pi(Q^2) = \frac{\text{Input from lattice calculation}}{\text{Known function}}$$

3-point function

$F_\pi(Q^2)$

Fit

$$\frac{d}{dQ^2} F_\pi(Q^2) \Big|_{Q^2=0}$$

✓ Consider the **Taylor expansion** of the form factor

$$F_\pi(Q^2) = 1 + f_1 Q^2 + f_2 Q^4 + \cdots$$

$$\text{(3-point function)} = \text{(Known function)} \times (1 + f_1 Q^2 + f_2 Q^4 + \cdots)$$

$$\frac{\mathrm{d}}{\mathrm{d}Q^2} F_\pi(Q^2) \Big|_{Q^2=0} = \left[\frac{\text{3-point function}}{\text{(Known function)}} - 1 \right] \Big/ Q^2 + \text{(Higher-order contamination)}$$

Introduction

◆ Model-independent method for calculating the charge radius

$$\left.\frac{\mathrm{d}}{\mathrm{d}Q^2}F_\pi(Q^2)\right|_{Q^2=0} = \left[\frac{\text{3-point function}}{\text{(Known function)}} - 1\right] \Big/ Q^2 + \text{(Higher-order contamination)}$$

$$\boxed{\text{3-point function}} \longrightarrow \boxed{\left.\frac{d}{dQ^2}F_\pi(Q^2)\right|_{Q^2=0}} \Rightarrow \boxed{\langle {r_\pi}^2 \rangle}$$

Important point :

- ✓ This process does not use the fit ansatz.
- ✓ This method includes contamination from higher-order terms in the Taylor expansion.

$$\text{(3-point function)} = \text{(Known function)} \times (1 + f_1 Q^2 + f_2 Q^4 + \cdots)$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}Q^2}F_\pi(Q^2)\right|_{Q^2=0} = \underbrace{\left[\frac{\text{3-point function}}{\text{(Known function)}} - 1\right] \Big/ Q^2}_{\text{can be exactly evaluated}} + \underbrace{\text{(Higher-order contamination)}}_{\text{cannot be exactly evaluated}}$$

- ✓ The charge radius may be bad affected by the higher-order term.

Outline

✓ ~~Introduction~~

- ~~charge radius~~
- ~~Traditional method for calculating the charge radius~~
- ~~Model-independent method for calculating the charge radius~~

✓ Overview of model-independent method

- Reducing contamination using spatial moment

✓ Application to PACS10 configuration (Preliminary result)

- π^+ and K^+ charge radii using traditional and model-independent method

✓ Summary

Overview of model-independent method

$$(\text{3-point function}) = (\text{Known function}) \times (1 + f_1 Q^2 + f_2 Q^4 + \cdots)$$

$$\frac{\mathrm{d}}{\mathrm{d}Q^2}F_\pi(Q^2)\Big|_{Q^2=0} = \underbrace{\left[\frac{\text{3-point function}}{(\text{Known function})} - 1\right] \Big/ Q^2}_{\text{can be exactly evaluated}} + \underbrace{(\text{Higher-order contamination})}_{\text{cannot be exactly evaluated}}$$

request：

✓ Is there a way to extract the first derivative of the form factor from the three-point function with less contamination?

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✓ Is there a way to extract the first derivative of the form factor from the three-point function with less contamination?

✓ Key idea：**Calculate spatial moment**

U. Aglietti et al., Phys.Lett.B324,85(1994); UKQCD, Nucl.Phys.B444,401(1995) ; C. Bouchard et al., PoS LATTICE2016,170(2016); PACS, Phys.Rev.D104, 074514(2021)

➢ Differentiation of Fourier transform

$$\frac{\mathrm{d}\tilde{C}(p)}{\mathrm{d}p^2}\Big|_{p^2=0} = \frac{\mathrm{d}}{\mathrm{d}p^2}\sum_x C(x)e^{-ipx}\Big|_{p^2=0} = -\frac{1}{2}\sum_x x^2 C(x)$$

($N_{space} \to \infty$, a :finite ; $L = N_{space}a \to \infty$)

infinite volume limit

We can suppress the contamination at large volume.

$$\sum_x x^2 C_{3pt}(x,t) \sim f_1 + (f_2 + f_3 + \cdots)$$

Overview of model-independent method

$$\text{(3-point function)} = \text{(Known function)} \times (1 + f_1 Q^2 + f_2 Q^4 + \cdots)$$

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infinite volume limit

We can suppress the contamination at large volume.

$$\sum_x x^2 C_{3pt}(x,t) \sim f_1 + (f_2 + f_3 + \cdots)$$

✓ Method：**Combining x^2 and x^4 moments**

Xu Feng et al., Phys.Rev.D101,051502(R)(2020)

$$\sum_x x^4 C_{3pt}(x,t) \sim f_1 + (f_2 + f_3 + \cdots)$$

We can cleverly add these higher-order moments to reduce the contamination.

$$R(t) = \alpha_1\,(x^2\text{moment}) + \alpha_2\,(x^4\text{moment})$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}Q^2}F_\pi(Q^2)\right|_{Q^2=0} + (f_3 + f_4 + \cdots)$$

α_1, α_2: parameters which set to cancel out the contamination

Our Improvement : K. S. et al., PoS LAT22,122(2022) ; PoS LAT23,312(2023)

-> Effective at small lattice size

Simulation parameters

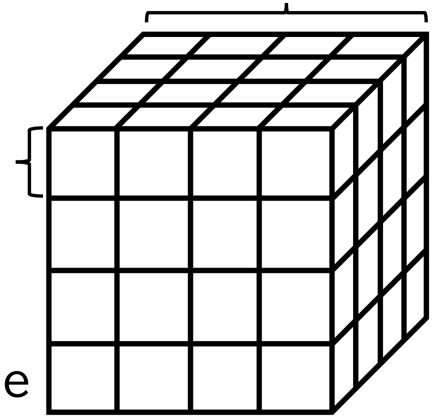
✓ Gauge configuration (PACS, PRD 99, 014504 (2019))

PACS10 configuration

$N_f = 2+1$ six-stout-smeared non-perturbative
$O(a)$-improved Wilson action+ Iwasaki gauge action

β	$L^3 \cdot T$	L[fm]	a[fm]	a^{-1}[GeV]	m_π[MeV]	m_K[MeV]	N_{conf}
2.20	256^4	10.5	0.041	4.792	142	514	20
2.00	160^4	10.2	0.063	3.111	137	501	20
1.82	128^4	10.9	0.085	2.316	135	497	20

All preliminary results are obtained on the coarsest lattice.



Lattice size
128^4

- ➢ Chiral extrapolation
  - → Physical point
- ➢ Continuum extrapolation
  - → 3 lattice spacings
- ➢ Finite volume effects
  - → Large volume

✓ Measurement parameter

- 16 sources × 4 driections (t,x,y,z) × 3 random sources = 192 meas. Per config.
- $|t_{sink} - t_{source}| = 36$

Details of 3-point functions and calculation methods : K. S. et al., PoS LAT22,122(2022) ; PoS LAT23,312(2023)

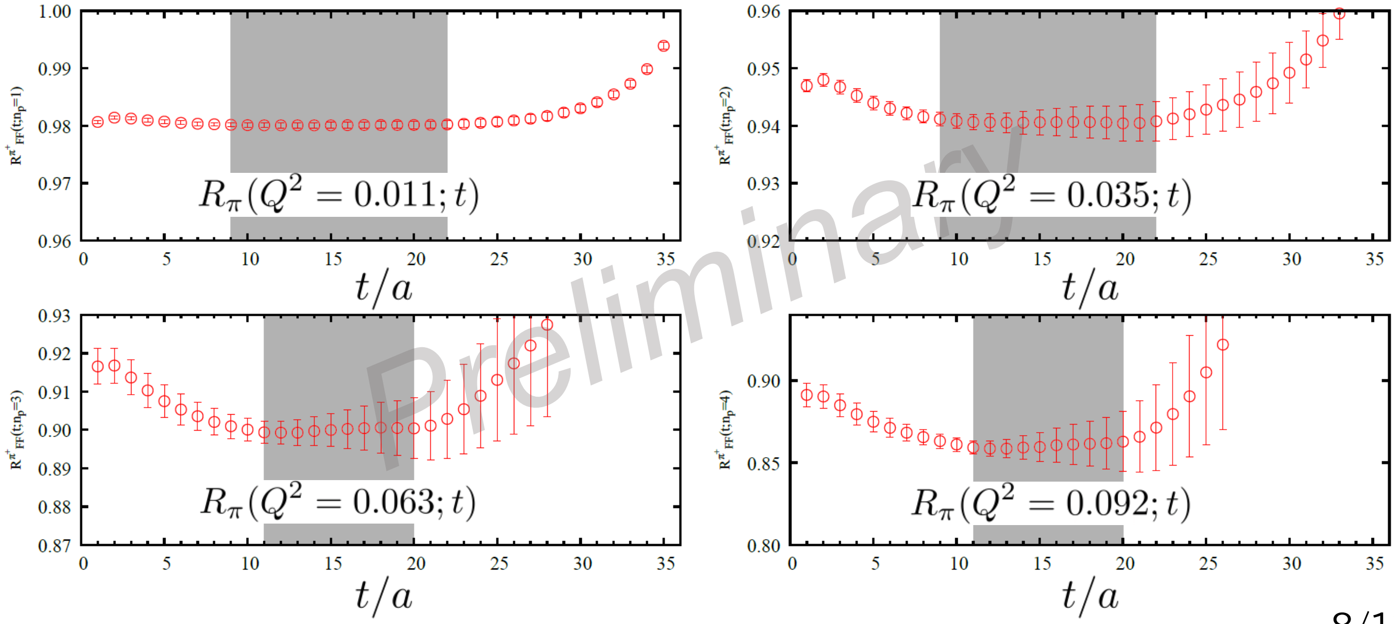
Preliminary result

- ✓ **Traditional method**
  (with fit ansatz)

3-point function

$$\frac{\sum_{\vec{x}_{\rm sink},\vec{x}} \langle \pi^+(\vec{x}_{\rm sink},t_{\rm sink}) V_4(\vec{x},t)\pi^+(0)^\dagger\rangle \, e^{-ipx_1}}{\sum_{\vec{x}_{\rm sink},\vec{x}}\langle \pi^+(\vec{x}_{\rm sink},t_{\rm sink}) V_4(\vec{x},t)\pi^+(0)^\dagger\rangle} = \frac{E_\pi(p)+m_\pi}{2E_\pi(p)} e^{-(E_\pi(p)-m_\pi)t} \, F_\pi(Q^2)$$

Extract form factor $R_\pi(Q^2;t) = \dfrac{\text{Input from lattice calculation}}{\text{Known function}}$



Preliminary result

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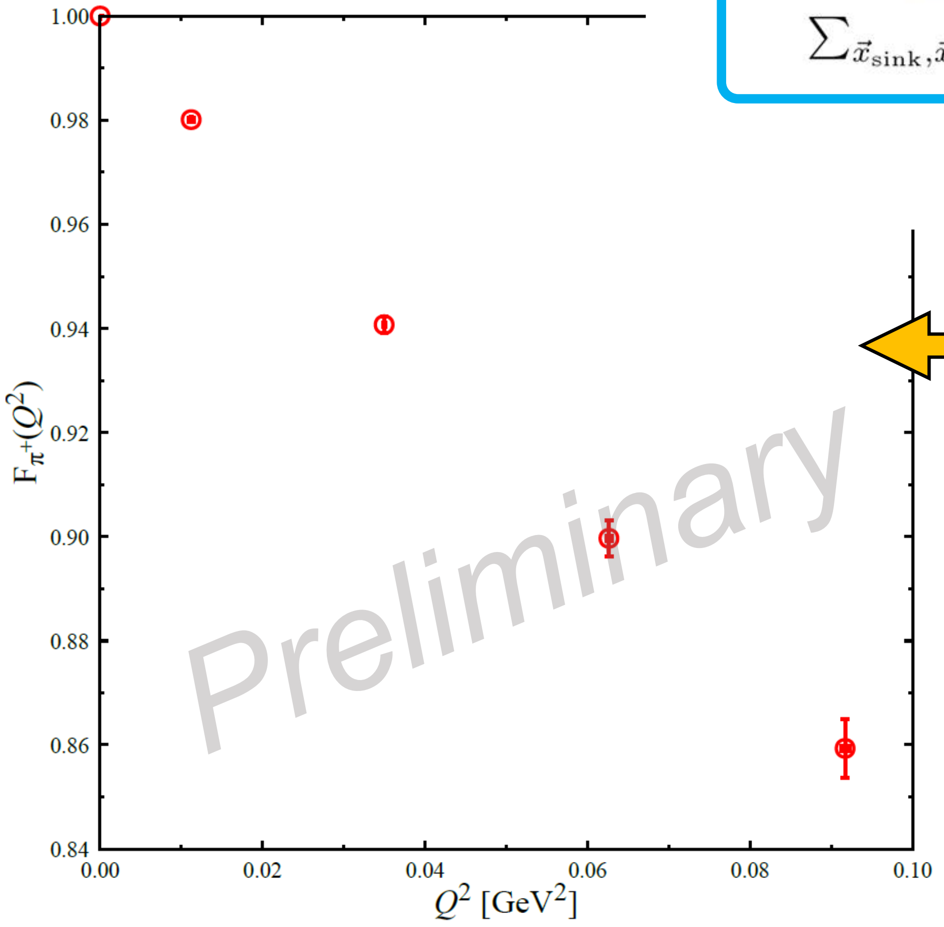

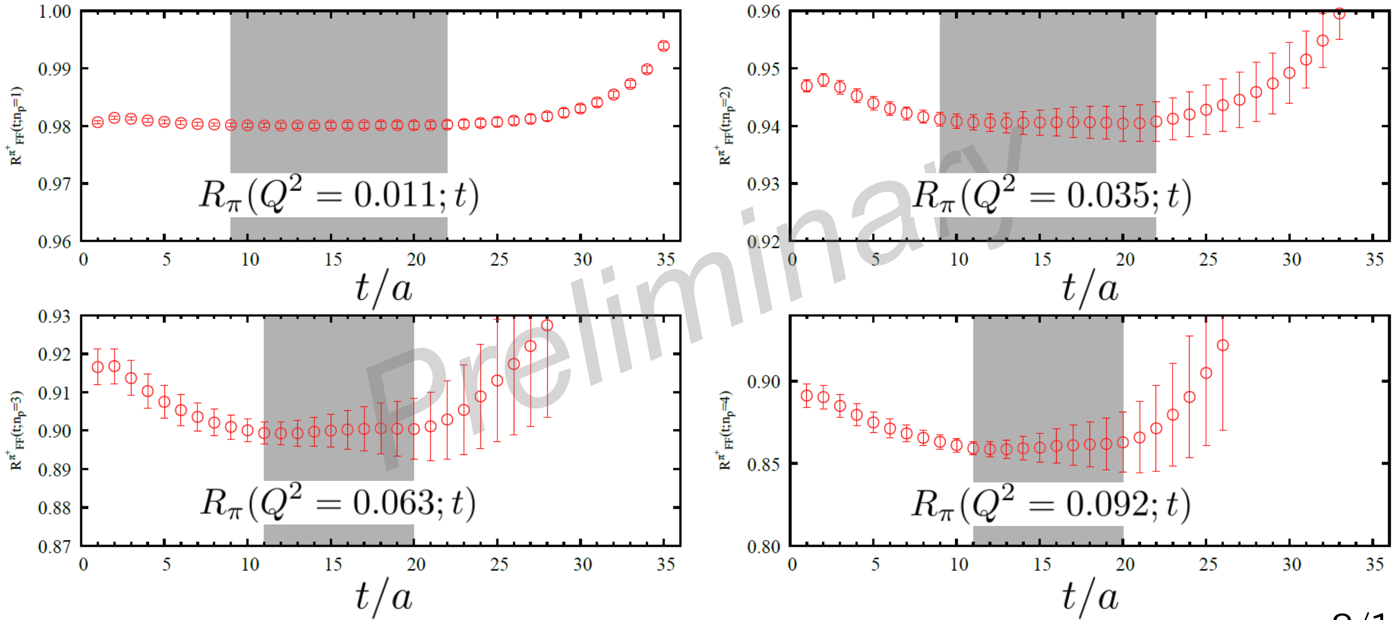

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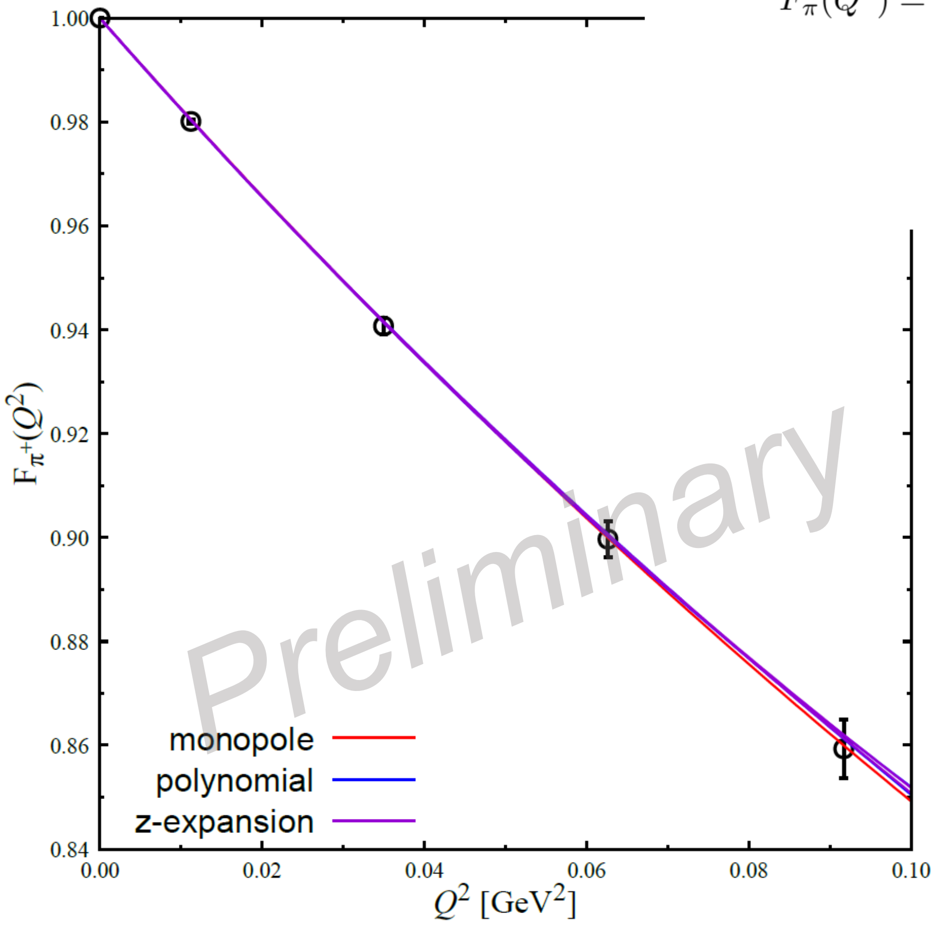
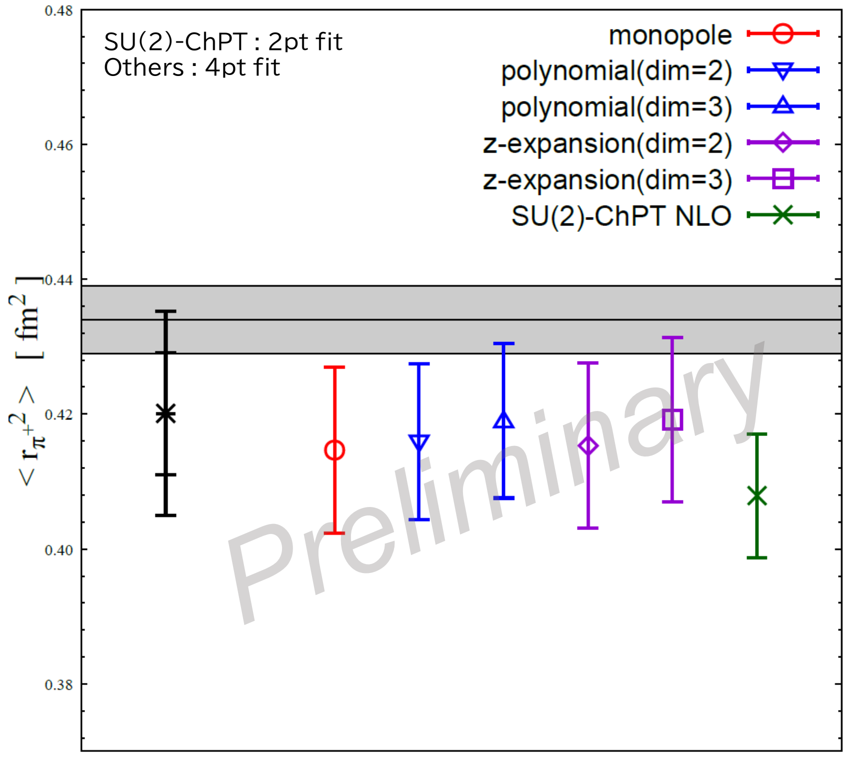


$$F_\pi(Q^2) = \begin{cases} \dfrac{1}{1+Q^2/M_{\text{pole}}^2} \\ 1+\sum_k f_k (Q^2)^k \\ \sum_k a_k z^k \quad \left(z := \dfrac{\sqrt{4m_\pi^2+Q^2}-\sqrt{4m_\pi^2}}{\sqrt{4m_\pi^2+Q^2}+\sqrt{4m_\pi^2}}\right) \\ 1+\dfrac{1}{f_\pi^2}\left[2l_6^r(\mu)Q^2+4H(m_\pi^2,Q^2,\mu^2)\right] \end{cases}$$

using various fit ansatzes

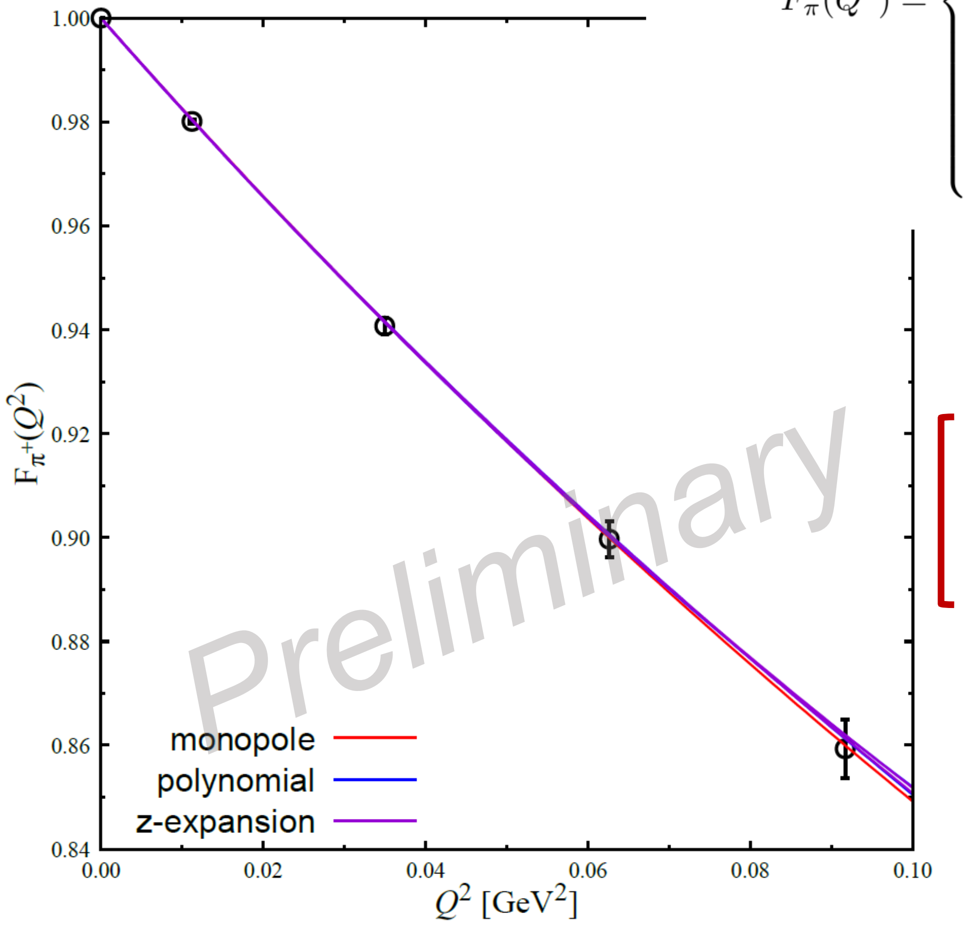


- ✓ These results are consistent with the PDG within the margin of error.
- ✓ There is variation in the central value depending on the fit ansatz.

Preliminary result

✓ **Traditional method**

(with fit ansatz)



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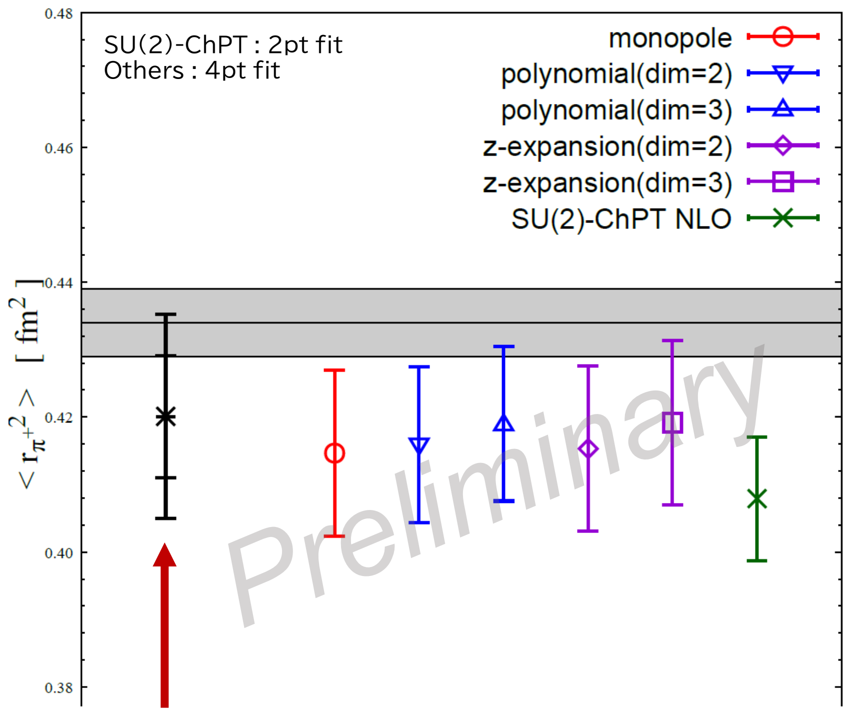


- Central value : Weighted mean of these results
- Statistic error : Jackknife error of the central value
- Systematic error : Maximum difference between the central value and value on each form

✓ These results are consistent with the PDG within the margin of error.

✓ There is variation in the central value depending on the fit ansatz.

Preliminary result

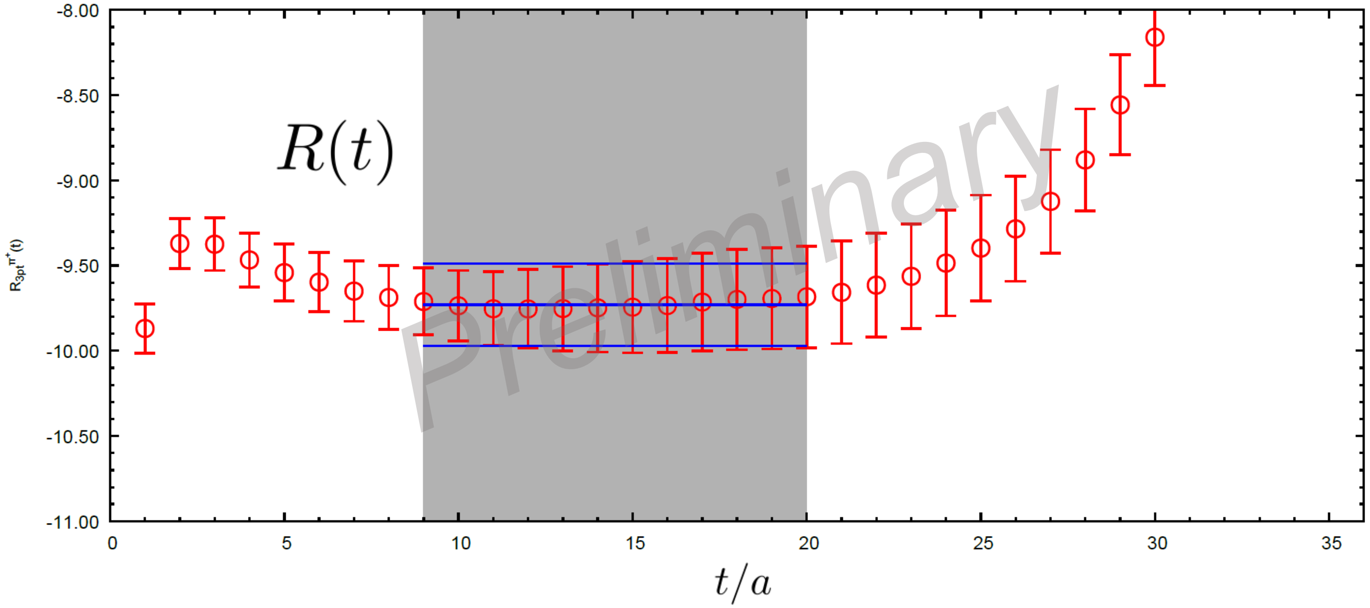
✓ Model-independent method

$$\sum_x x^2 C_{3pt}(x,t) \sim f_1 + (f_2 + f_3 + \cdots)$$

$$\sum_x x^4 C_{3pt}(x,t) \sim f_1 + (f_2 + f_3 + \cdots)$$

$$R(t) = \alpha_1 \, (x^2\text{moment}) + \alpha_2 \, (x^4\text{moment})$$

$$= \frac{\mathrm{d}}{\mathrm{d}Q^2} F_\pi(Q^2)\Big|_{Q^2=0} + (f_3 + f_4 + \cdots)$$



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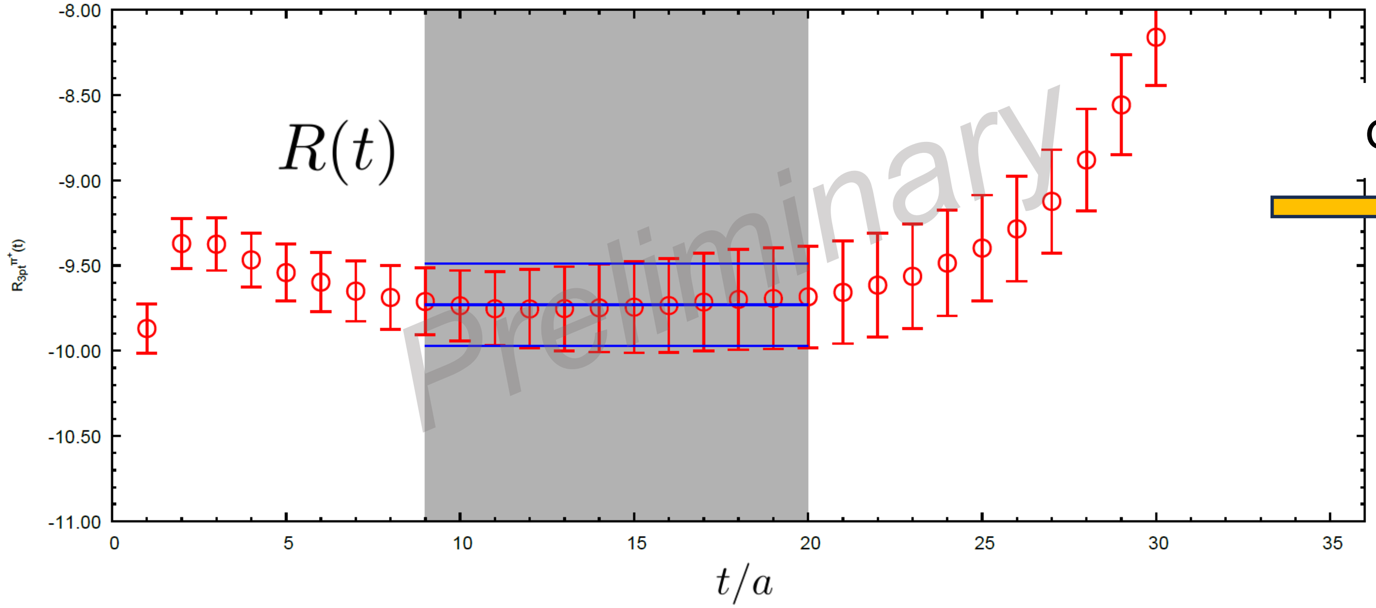
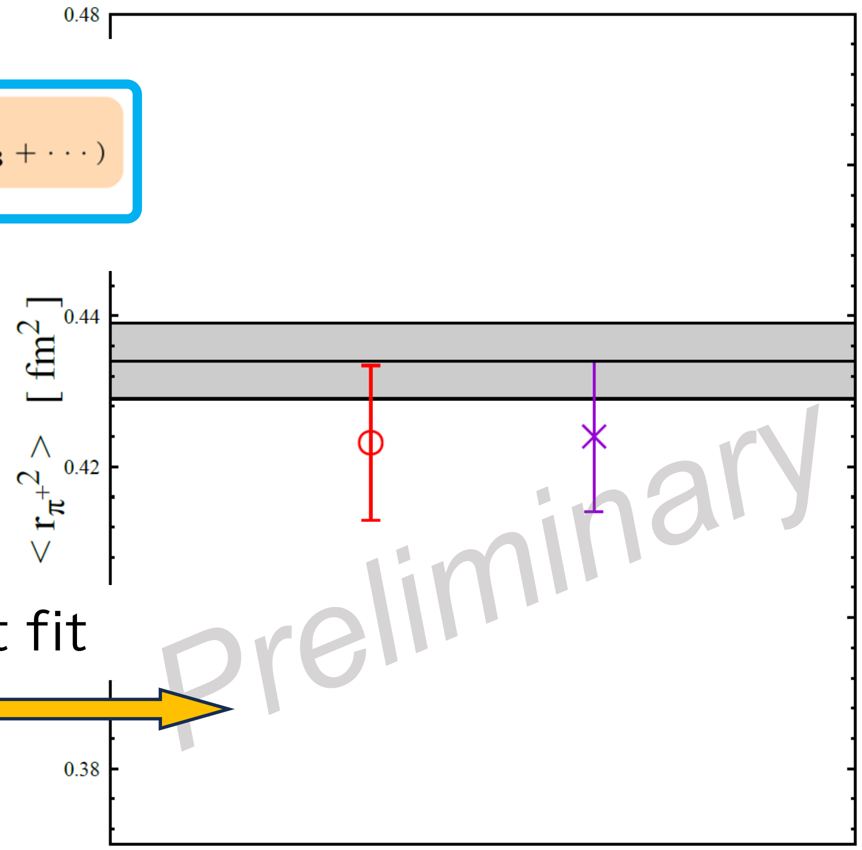


constant fit



✓ We perform similar calculations for K meson.

Preliminary result

- ✓ Our improved method is consistent with original model-independent method.
- ✓ Due to the large volume configuration, contamination is already well suppressed.

✓ **Model-independent method**

$$\sum_x x^2 C_{3pt}(x,t) \sim f_1 + (f_2 + f_3 + \cdots)$$

$$\sum_x x^4 C_{3pt}(x,t) \sim f_1 + (f_2 + f_3 + \cdots)$$

$$R(t) = \alpha_1 \, (x^2\text{moment}) + \alpha_2 \, (x^4\text{moment})$$

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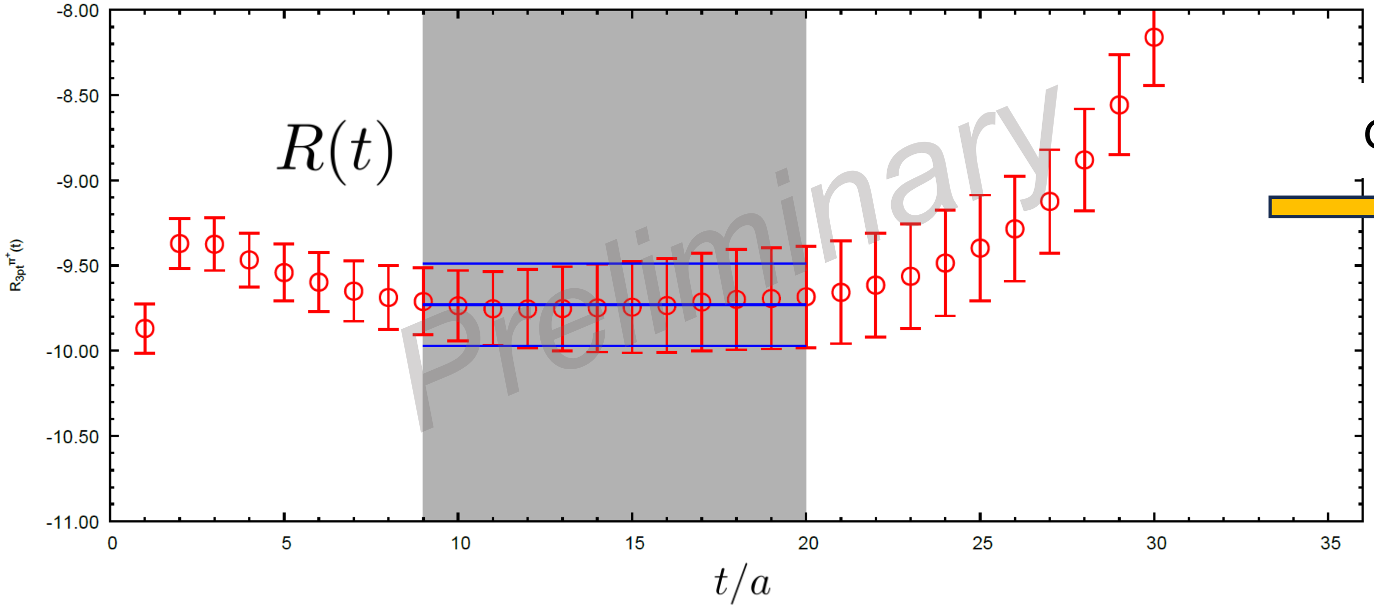

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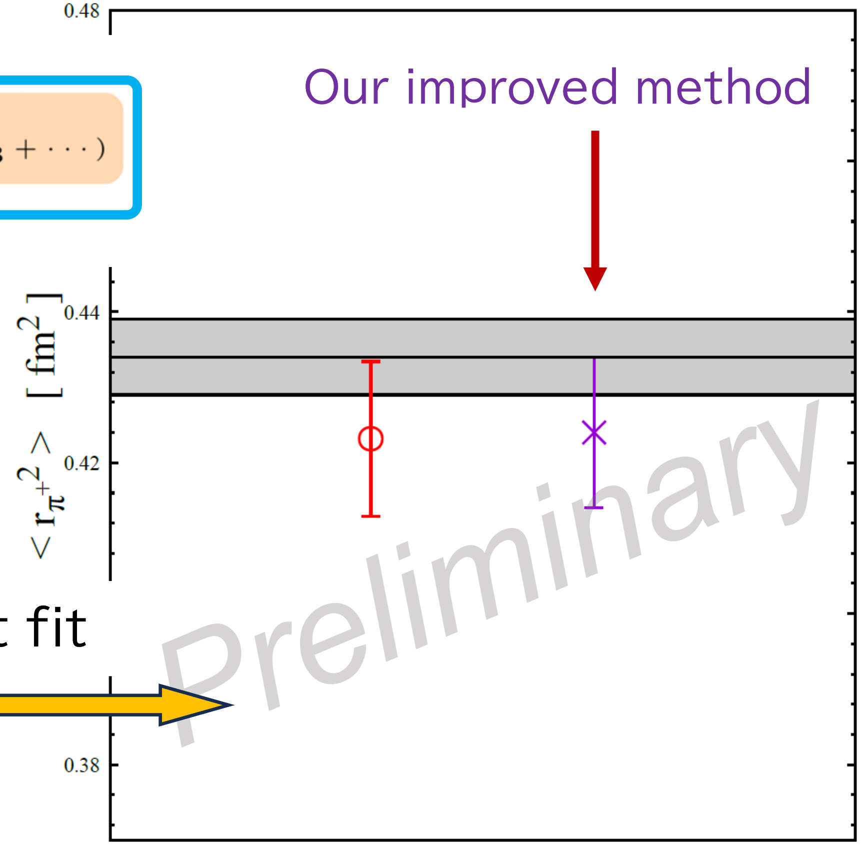
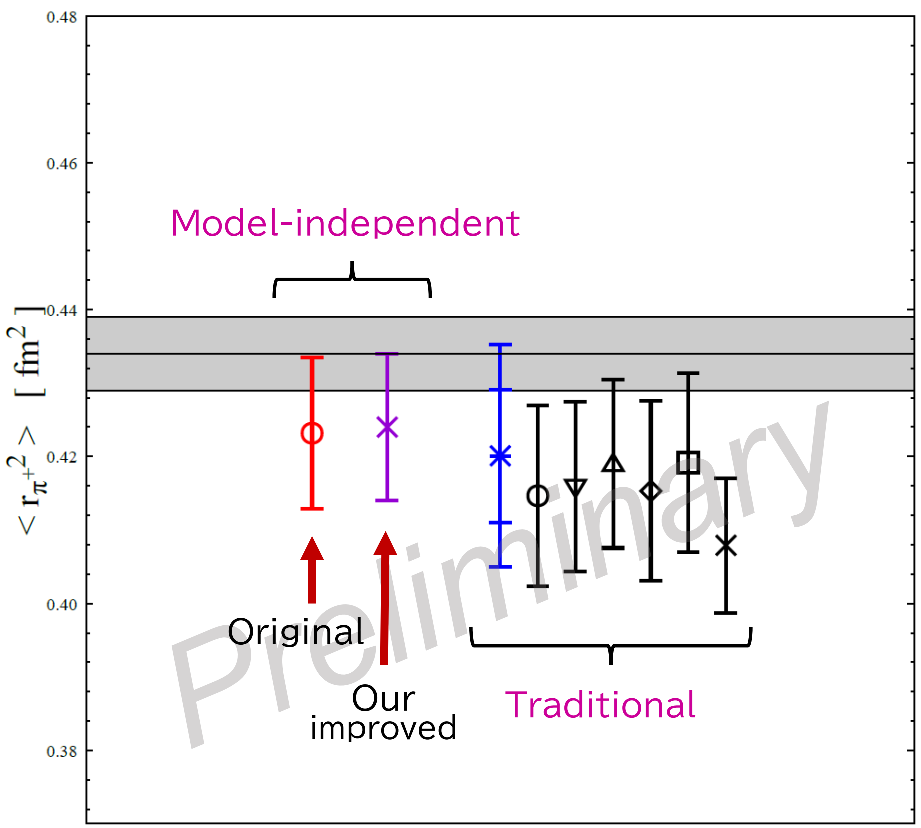


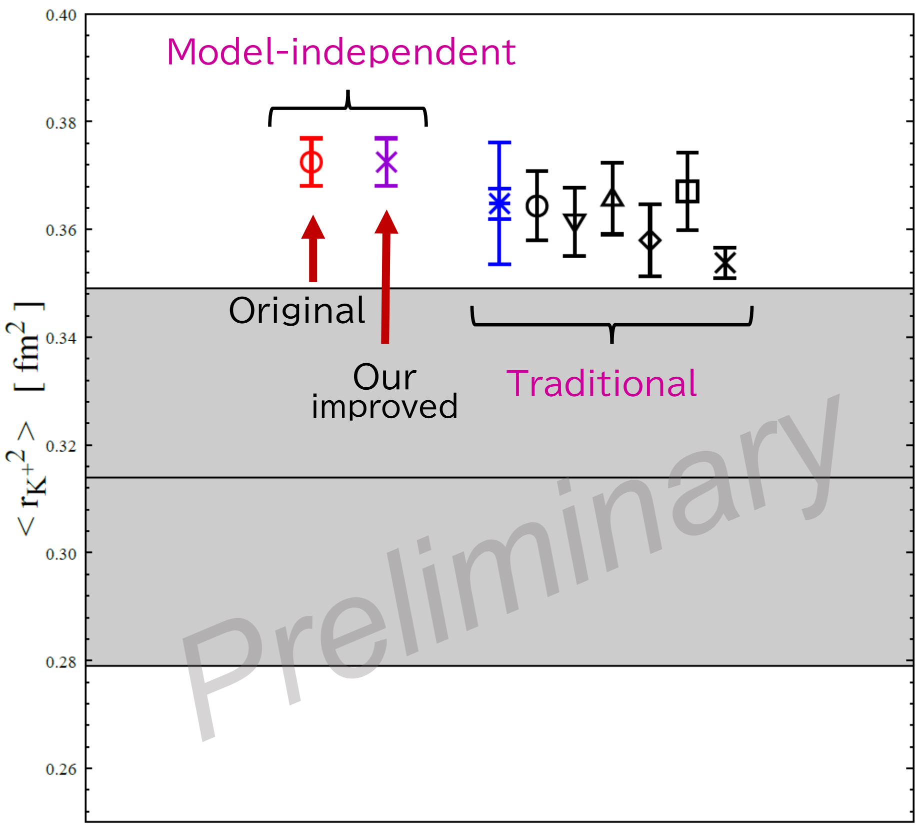
✓ We perform similar calculations for K meson.

Preliminary result

✓ π^+charge radius



✓ K^+charge radius



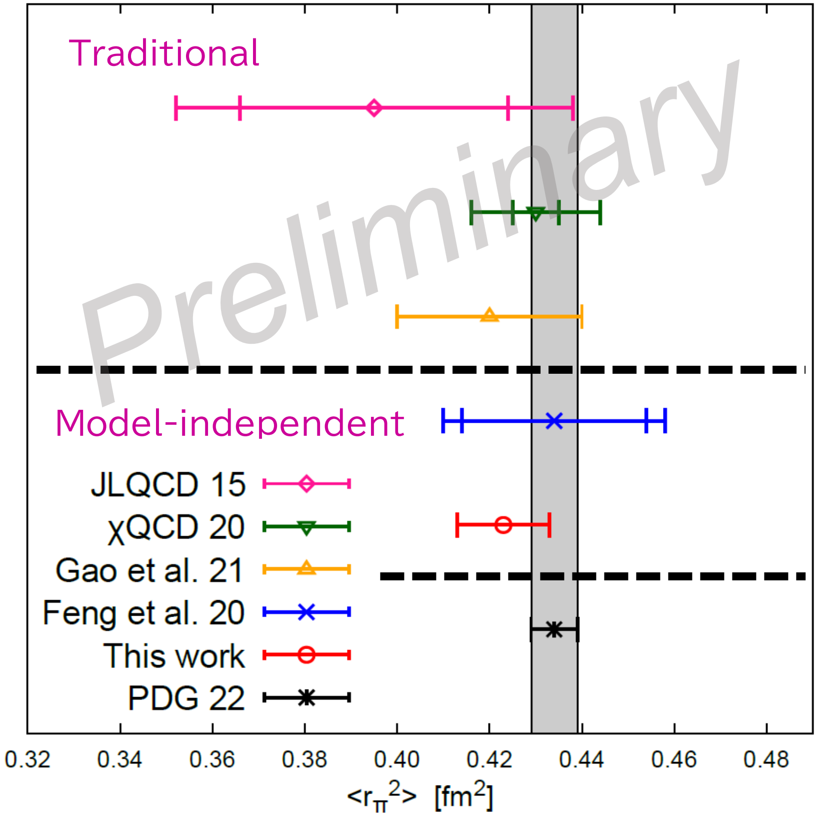
- ✓ Calculated results agree with PDG within the margin of error.
- ✓ The model-independent and traditional methods agree, but the model-independent method has smaller error.
- ✓ We obtained K^+ charge radius with less error than PDG.

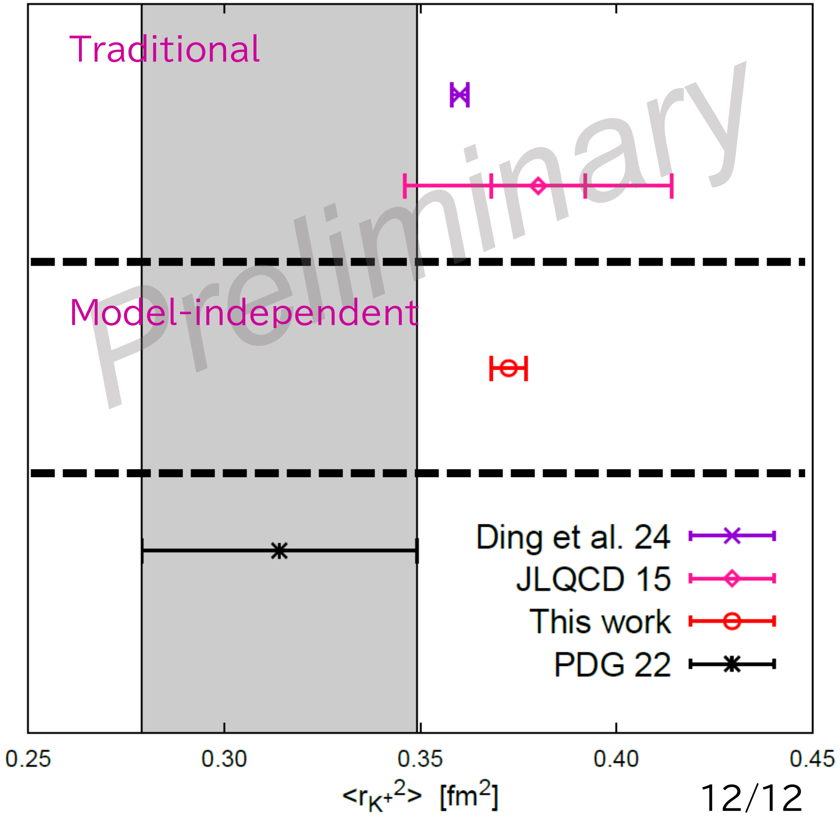
Summary

- ✓ We calculated the charge radii of π^+ and K^+ on the coarsest PACS10 configuration.
- ✓ We use the model-independent method to obtain it.
- ✓ Although preliminary, the results are consistent with the experimental value(PDG) and the results of the previous lattice calculations.

Future works

- ✓ Analysis at various source-sink time separations
- ✓ Analysis with other particles such as K^0
- ✓ Other PACS10 configuration





backup

model-independent method

Phys.Lett.B324,85(1994) ; Nucl.Phys.B444,401(1995) ; PoS LATTICE2016,170(2016)

$$\tilde{C}_{\pi V\pi}(t,t_{\rm sink};p) = Z_V Z_\pi(0) Z_\pi(p) L^2 \frac{(E_\pi(p)+m_\pi)}{2m_\pi 2E_\pi(p)} F_\pi(q^2) e^{-E_\pi(p)t} e^{-m_\pi(t_{\rm sink}-t)}$$

<u>For $a \to 0$ and $V \to \infty$</u>

$$F(\vec{x}) = F(|\vec{x}|)$$

$$\left.\frac{\mathrm{d}\tilde{F}(\vec{p})}{\mathrm{d}|\vec{p}|^2}\right|_{|\vec{p}|^2=0} = \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}^3x\, F(\vec{x})e^{-i\vec{p}\cdot\vec{x}}\right|_{|\vec{p}|^2=0} = -\frac{1}{3!}\int \mathrm{d}^3x\, |\vec{x}|^2 F(\vec{x})$$

n-th order momentum-derivative at $|\vec{p}|^2 = 0$ ⟷ $2n$-th order spatial moment ($|\vec{x}|^{2n}$)

<u>For finite V</u>, the higher-order contaminations appear.

1-dimension 3-point function

$$C_{\pi V\pi}(t,t_{\rm sink};r) := Z_V \sum_{\vec{z}} \sum_{y_2,y_3} \sum_{x_2,x_3} \langle 0|\pi^+(\vec{z},t_{\rm sink})V_4(\vec{y},t)\pi^{+\dagger}(\vec{x},0)|0\rangle$$

$$(r := |x_1 - y_1|\,;\ 0 \le r \le L/2;\ \text{Periodic B.C.})$$

$$\begin{aligned} C^{(n)}(t) \quad &:= \sum_r r^{2n} C_{\pi V\pi}(t,t_{\rm sink};r) = \sum_r r^{2n}\frac{1}{L}\sum_p \tilde{C}_{\pi V\pi}(t,t_{\rm sink};p)e^{ipr} \\ &= \sum_p \Delta(t,t_{\rm sink},p)T_n(p)F_\pi(q^2) \qquad \left(\tilde{C}_{\pi V\pi}(t,t_{\rm sink};p) = \Delta(t,t_{\rm sink},p)F_\pi(q^2),\ \ T_n(p) := \frac{1}{L}\sum_r r^{2n}e^{ipr}\right) \\ &= f_0\beta_{0,n}(t) + \underbrace{f_1\beta_{1,n}(t)} + \underbrace{f_2\beta_{2,n}(t) + \cdots}_{\text{higher-order contamination}} \qquad \left(F_\pi(q^2) = \sum_{m=0}^{\infty} f_m q^{2m},\ \ \underbrace{\beta_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t,t_{\rm sink},p)T_n(p)q^{2m}\right) \end{aligned}$$

$$\langle r_\pi^2\rangle = -6\boxed{\left.\frac{\mathrm{d}}{\mathrm{d}q^2}F_\pi(q^2)\right|_{q^2=0}}$$

model-independent method

Phys.Rev.D101,051502(R)(2020)

$$(C^{(0)}(t) := 1, \quad f_0 = 1)$$

$$\underbrace{C^{(n)}(t)}_{\text{moment function}} := \sum_r r^{2n} C_{\pi V \pi}(t, t_{\rm sink}; r) = \sum_{m=0}^{\infty} f_m \beta_{m,n}(t)$$

$$\underbrace{\beta_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t, t_{\rm sink}, p) T_n(p) q^{2m}$$

To reduce the higher-order contamination

$$\begin{aligned} R(t) \quad &:= \quad \alpha_1 C^{(1)}(t) + \alpha_2 C^{(2)}(t) + h \\ &= \quad (\alpha_1 \beta_{0,1} + \alpha_2 \beta_{0,2} + h) + (\alpha_1 \beta_{1,1} + \alpha_2 \beta_{1,2}) \underline{f_1} + (\alpha_1 \beta_{2,1} + \alpha_2 \beta_{2,2}) f_2 + \cdots \end{aligned}$$

Define parameters α_1, α_2, h to satisfy the following

$$\langle r_\pi^2 \rangle = -6 \boxed{\frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2) \Big|_{q^2=0}}$$

$$\alpha_1 \beta_{0,1} + \alpha_2 \beta_{0,2} + h = 0 \qquad \alpha_1 \beta_{1,1} + \alpha_2 \beta_{1,2} = 1 \qquad \alpha_1 \beta_{2,1} + \alpha_2 \beta_{2,2} = 0$$

$$\Rightarrow \quad R(t) = \underbrace{f_1}_{\text{constant}} + \sum_{m=3}^{\infty} \underbrace{\left(\sum_{k=1}^{2} \alpha_k \beta_{m,k}(t) \right)}_{\text{time-dependent}} f_m \qquad \langle r_\pi^2 \rangle = -6 \underbrace{\frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2) \Big|_{q^2=0}}_{f_1} \sim -6 \times R(t)$$

If the high-order contamination terms is small, we get the charge radius

Our improved model-independent method

PoS LATTICE2022,122(2023)

$$R(t) = f_1 + \sum_{m=3}^{\infty}\left(\sum_{k=1}^{2}\alpha_k\beta_{m,k}(t)\right)f_m$$

Original method remains the contamination from high-order at small M_{pole}^2 and volume.

Improve the convergence of f_m
and reduce the contamination

-- Fact --

Pion form factor is well represented by

$$F_\pi(q^2) = \frac{1}{1+q^2/M_{\mathrm{pole}}^2}$$

from phenomenology.

$$\begin{aligned}
C^{(n)}(t) &= \sum_p \Delta(t, t_{\mathrm{sink}}, p)T_n(p)F_\pi(q^2) = \sum_p \Delta(t, t_{\mathrm{sink}}, p)T_n(p)F_\pi(q^2)\frac{G(q^2)}{G(q^2)} \\
&= \sum_p \Delta(t, t_{\mathrm{sink}}, p)T_n(p)S(q^2)\frac{1}{G(q^2)} \qquad (S(q^2) := F_\pi(q^2)G(q^2)) \\
&= \sum_m s_m\tilde{\beta}_{m,n}(t) \qquad \left(S(q^2) = \sum_m s_m q^{2m}, \quad \underbrace{\tilde{\beta}_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t, t_{\mathrm{sink}}, p)T_n(p)q^{2m}/G(q^2)\right)
\end{aligned}$$

Original model-independent method changes to $R(t) = s_1 + \sum_{m=3}^{\infty}\left(\sum_{k=1}^{2}\alpha_k\tilde{\beta}_{m,k}(t)\right)s_m$

Change $F_\pi(q^2)$ to $S(q^2)$ and choose $G(q^2)$ with good convergence s_m