Form factor curves consistent with unitarity for semileptonic decays

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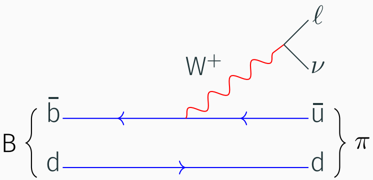
Semileptonic pseudoscalar to pseudoscalar decay - B $\to \pi\ell\nu$

We wish to find the q^2 dependence of the form factors (q is 4-momentum of the outgoing lepton pair).



$$\langle \pi(k) | \mathcal{V}^\mu(0) | B(p) \rangle = f_+(q^2)\left(p^\mu + k^\mu - \frac{M^2 - m^2}{q^2} q^\mu\right) + f_0(q^2)\frac{M^2 - m^2}{q^2} q^\mu$$

$\mathcal{V}^\mu = \bar{u}\gamma^\mu b$

B and π are pseudoscalar mesons of masses M and m

kinematic constraint $\to$ $f_+(0) = f_0(0)$

Unitarity Constraint

Unitarity constraint from subtracted dispersion relations:

$$\chi_{0^+} = \frac{1}{\pi}\int_{t_{cut}}^{\infty} dt\, \frac{\mathrm{Im}\Pi_{0^+}(t)}{t},$$

$$\chi_{1^-} = \frac{1}{\pi}\int_{t_{cut}}^{\infty} dt \frac{\mathrm{Im}\Pi_{1^-}(t)}{t^2},$$

$$\mathrm{Im}\Pi_{0^+,1^-} = \frac{1}{2}\sum_n \int d\mu(n)(2\pi)^4\delta^{(4)}(q-p_n)|\langle 0|J|n\rangle|^2.$$

A complete set of states has been inserted, restricting to a subset relating to $B \to \pi\ell\nu$, giving constraints on its form factors.

Okubo Phys. Rev. D 3, 2807 1971[1], Phys. Rev. D 4, 725 1971[2]; Okubo and Shih PRD4 2020 1971[3]; Bourrely, Machet, de Rafael NPB189 157 1981[4]; Boyd, Grinstein, Lebed PLB353 306 1995[5], NPB461 493 1996[6], PRD56 6895 1997[7]; Lellouch NPB479 353 1996[8]

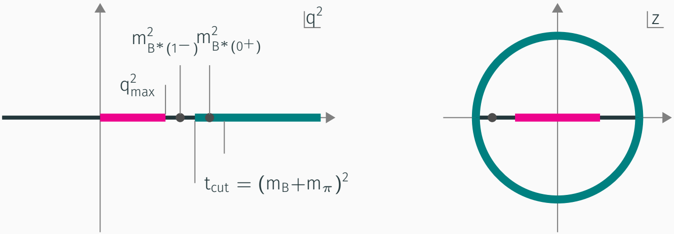
Z-transformation

Dispersion relation is now

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}\,|\phi(z)B(z)f(z)|^2 \leq \chi$$

With $q^2 = t$, we map the q^2 complex plane onto a unit disc:

$$z(t) = \frac{\sqrt{t_{cut} - t} - \sqrt{t_{cut} - t_0}}{\sqrt{t_{cut} - t} + \sqrt{t_{cut} - t_0}} \qquad t_0 = t_{cut} - \sqrt{t_{cut}(t_{cut} - (M_B - M_\pi)^2)}$$



Dispersive Matrix method

2 ways to proceed using lattice form factor information:

- Analyticity allows power series expansion of ϕBf (BGL/BCL expansion) and unitarity gives constraints on coefficients

  Boyd, Grinstein, Lebed PLB353 306 1995[5], NPB461 493 1996[6], PRD56 6895 1997[7], Bourrely, Caprini, Lellouch PRD79 013008 2009 [9]

- The Dispersive Matrix method, which finds form factor values allowed by unitarity, independent of any functional form Okubo Phys. Rev. D 3, 2807 1971[1], Phys. Rev. D 4, 725 1971[2]; Okubo and Shih PRD4 2020 1971[3]; Bourrely, Machet, de Rafael NPB189 157 1981[4]; Lellouch NPB479 353 1996[8]

  Di Carlo et al, PRD104 054502, 2021 [10]; Martinelli et al PRD105 034503, 2022 [11], JHEP 08 022 202216, 2022 [12] + More

Define an inner product:

$$\langle j \,|\, k \rangle = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \bar{j}(z) k(z),$$

Disp. rel. is now $\chi \geq \langle \phi B f \,|\, \phi B f \rangle$. We define the function, $g_t(z)$ as

$$g_t(z) \equiv \frac{1}{1 - \bar{z}(t) z},$$

so that the inner product, $\langle g_t \,|\, \phi B f \rangle = \phi\,(z(t))\, B(z(t)) f(z(t))$.

Dispersive Matrix method

$$M = \begin{pmatrix} \langle \phi B f | \phi B f \rangle & \langle \phi B f | g_t \rangle & \langle \phi B f | g_{t_1} \rangle & \cdots & \langle \phi B f | g_{t_n} \rangle \\ \langle g_t | \phi B f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi B f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi B f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

$\langle \phi B f | \phi B f \rangle$	Inner product constrained by disp. rel.
$\langle \phi B f | g_t \rangle$	The form factor value at $q^2 = t$ we wish to find
$\langle \phi B f | g_{t_i} \rangle$	Known form factor values

$$\vec{F} = \begin{pmatrix} \langle \phi B f | g_t \rangle & \langle \phi B f | g_{t_1} \rangle & \cdots & \langle \phi B f | g_{t_n} \rangle \end{pmatrix}$$

$$M = \begin{pmatrix} \langle \phi B f | \phi B f \rangle & \vec{F} \\ \vec{F}^T & G \end{pmatrix}$$

Dispersive Matrix method

$$\det(M) = \det(G) \times \left(\langle \phi Bf \,|\, \phi Bf \rangle - \vec{F}^T G^{-1} \vec{F} \right) \geq 0$$
$$\rightarrow \langle \phi Bf \,|\, \phi Bf \rangle - \vec{F}^T G^{-1} \vec{F} \geq 0$$

From our disp. rel. $\chi \geq \langle \phi Bf \,|\, \phi Bf \rangle$,

$\rightarrow$ Can substitute χ into our equation and the inequality still holds.

$$\chi - \vec{F}^T G^{-1} \vec{F} \geq 0$$

$\rightarrow$ Quadratic in $\phi(t)B(t)f(t)$

Dispersive Matrix method

Can easily find bounds at any q^2 using the DM method.

Want to know form factors over continuous ranges for integration.

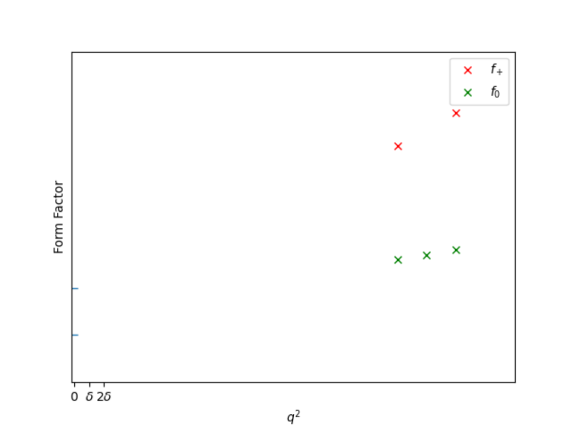
We wish to generate a family of curves, each consistent with unitarity constraints, that can be used in the same way as a set generated from a parametrized fit (e.g. a z-fit).

At the end of the process, we take into account uncertainties of our inputs through resampling.

Generating Form Factor Curves

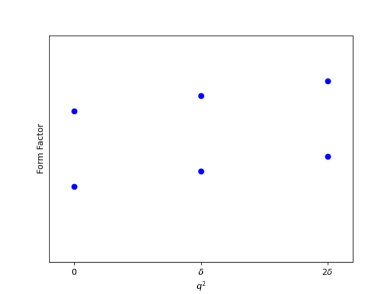
Find bounds at $q^2 = 0$ - Unitarity and Kinematic constraints

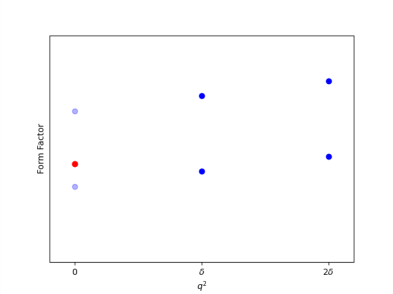
Unitarity allows an infinite number of form factor curves passing through these form factor points and through the bounds at $q^2 = 0$.



Generating Form Factor Curves

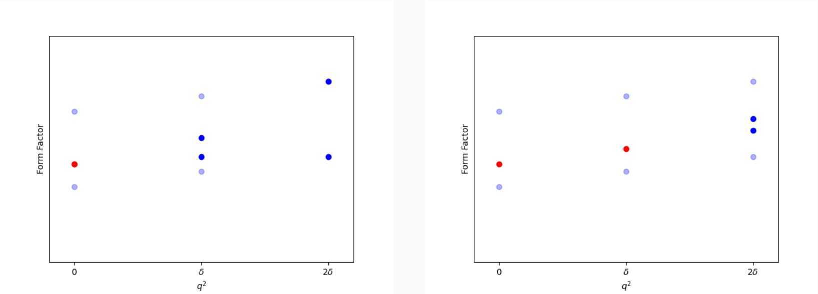
Plots for illustration





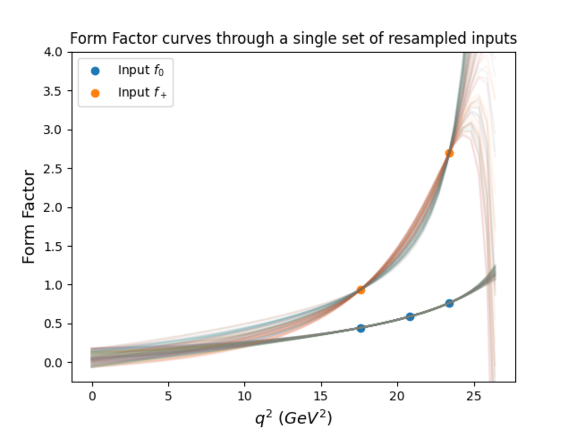
Generating Form Factor Curves

Plots for illustration



Repeat over entire range and interpolate

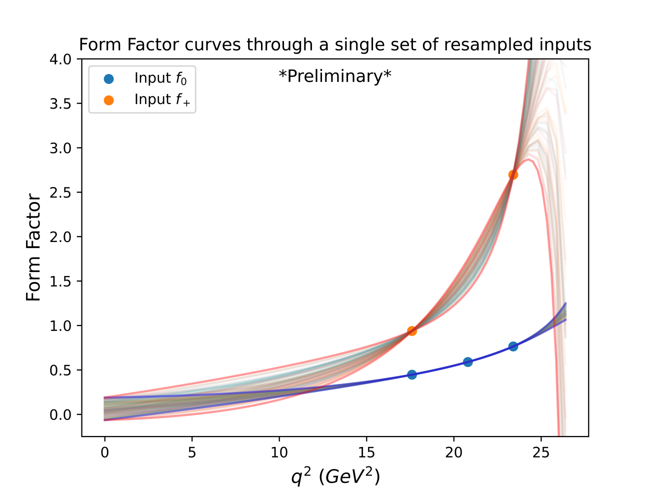
Generating Form Factor Curves



Preliminary lattice data from RBC/UKQCD collaboration

In the limit of $\delta \to 0$, repeating an infinite number of times generates all possible form factor curves through known values.

Generating Form Factor Curves



The envelope of the family of curves matches the results from the DM method

Form Factor Curves Across Bins

The described method allows us to generate form factor curves across the entire q^2 spectrum.

For comparison with experiment, we integrate over bins $\rightarrow$ only necessary to generate curves over individual bins.

Decreased range $\rightarrow$ smaller δ & more curves

Identical procedure, however the first step is from $q^2 = 0$ directly to the start of the bin.

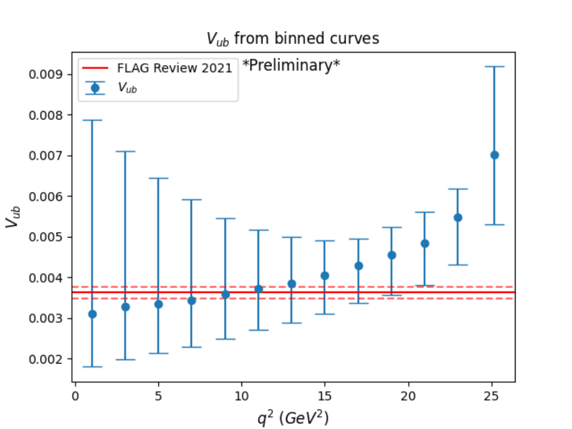
Curves passing through the start of the bin are consistent with the same uniform distribution

V_{ub} from binned curves

We repeat our curve generation method over many resamples of our lattice data and find V_{ub} for each curve.

$$\int_{q_1^2}^{q_2^2} dq^2 \frac{d\Gamma}{dq^2} = |V_{ub}|^2 \int_{q_1^2}^{q_2^2} dq^2 \frac{G_F^2}{24\pi^3} \frac{\left(q^2 - m_\ell^2\right)^2 \sqrt{E_\pi^2 - M_\pi^2}}{q^4 M_B^2} \times \left[\left(1 + \frac{m_\ell^2}{2q^2}\right) M_B^2 \left(E_\pi^2 - M_\pi^2\right) \left|f_+\left(q^2\right)\right|^2 + \frac{3m_\ell^2}{8q^2} \left(M_B^2 - M_\pi^2\right)^2 \left|f_0\left(q^2\right)\right|^2\right]$$

V_{ub} from binned curves



[1]

FLAG 2021 [14]

[1] Experimental data used is from the Belle collaboration [13]

Exclusion of points

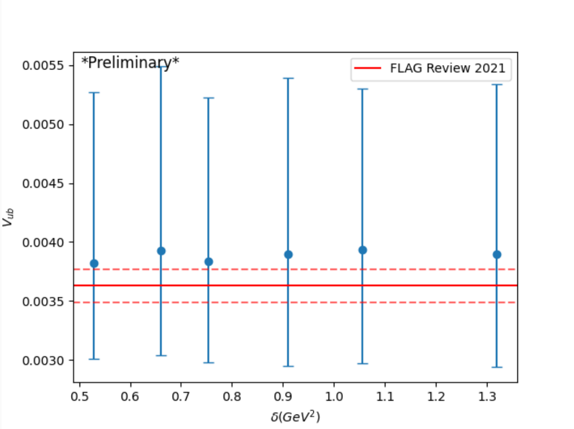
After a certain number of steps, the upper and lower bounds calculated are effectively equal (the width of the bound is near 0).

With an increased number of points, the width of these bounds remains below threshold.

We may remove previous points from our Dispersive Matrix.

This speeds up the generation of curves, as our Dispersive Matrix remains a manageable size.

Results should not depend on δ if it is small enough



[2]

Here we have integrated over the whole kinematic range.

[2] Experimental data used is from the Belle collaboration [13]

Summary

This method allows us to generate form factor curves using the DM method framework.

These curves allow us to easily integrate our form factor results (in the same way as parametrized fits).

δ can be made small enough that its effect (and the effects of interpolation method/width threshold) is not visible compared to all other error sources.

Both the generation of curves over individual bins and the ”width threshold” method result in decreases in computational cost, making this method practical to use.

References I

[1] Susumu Okubo. **Exact bounds for K_{l3} decay parameters.** Phys. Rev. D, 3:2807–2813, Jun 1971.

[2] Susumu Okubo. **New improved bounds for K_{l3} parameters.** Phys. Rev. D, 4:725–733, Aug 1971.

[3] Susumu Okubo and I-Fu Shih. **Exact inequality and test of chiral sw(3) theory in K_{l3} decay problem.** Phys. Rev. D, 4:2020–2029, Oct 1971.

[4] C. Bourrely, B. Machet, and E. de Rafael. **Semileptonic decays of pseudoscalar particles (m→m?+ℓ+νℓ) and short-distance behaviour of quantum chromodynamics.** Nuclear Physics B, 189(1):157–181, 1981.

References II

[5] C. Glenn Boyd, Benjamin Grinstein, and Richard F. Lebed. **Model-independent determinations of b -> d l nu , d* l nu form factors, 1995.**

[6] C. Glenn Boyd, Benjamín Grinstein, and Richard F. Lebed. **Model-independent determinations of b → dlv, d?lv form factors.** Nuclear Physics B, 461(3):493–511, 1996.

[7] C. Glenn Boyd, Benjamín Grinstein, and Richard F. Lebed. **Precision corrections to dispersive bounds on form factors.** Physical Review D, 56(11):6895–6911, December 1997.

[8] Laurent Lellouch. **Lattice-constrained unitarity bounds for decays.** Nuclear Physics B, 479(1–2):353–391, November 1996.

References III

[9] Claude Bourrely, Laurent Lellouch, and Irinel Caprini. **Model-independent description of $b \to \pi l \nu$ decays and a determination of $|V_{ub}|$.** Phys. Rev. D, 79:013008, Jan 2009.

[10] M. Di Carlo, G. Martinelli, M. Naviglio, F. Sanfilippo, S. Simula, and L. Vittorio. **Unitarity bounds for semileptonic decays in lattice qcd.** Phys. Rev. D, 104:054502, Sep 2021.

[11] G. Martinelli, S. Simula, and L. Vittorio. **$|V_{cb}|$ and $r(D^{(*)})$ using lattice qcd and unitarity.** Phys. Rev. D, 105:034503, Feb 2022.

[12] Guido Martinelli, Silvano Simula, and Ludovico Vittorio. **Exclusive semileptonic $b \to \pi\ell\nu\ell$ and $bs \to k\ell\nu\ell$ decays through unitarity and lattice qcd.** Journal of High Energy Physics, 2022(8):22, August 2022.

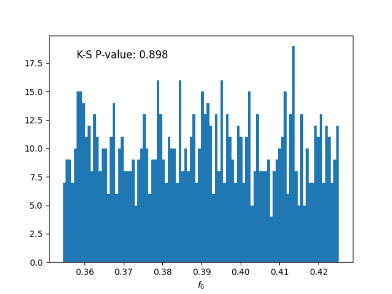
[13] H. Ha et al. **Measurement of the decay $B^0 \to \pi^- \ell^+ \nu$ and determination of $|V_{ub}|$.** Phys. Rev. D, 83:071101, Apr 2011.

[14] Y. Aoki, T. Blum, G. Colangelo, et al. **Flag review 2021.** Eur. Phys. J. C, 82:869, 2022.

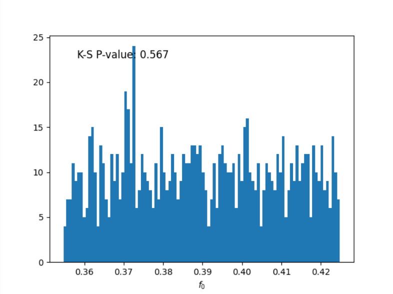
Backup slides

Direction testing

Below are the distributions of curves passing through $q^2 = 10\text{GeV}^2$ for low-to-high and high-to-low q^2



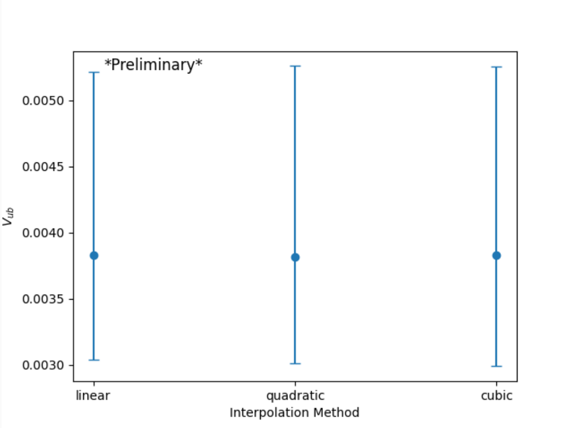
(a) Distribution of low-to-high q^2 curves



(b) Distribution of high-to-low q^2 curves

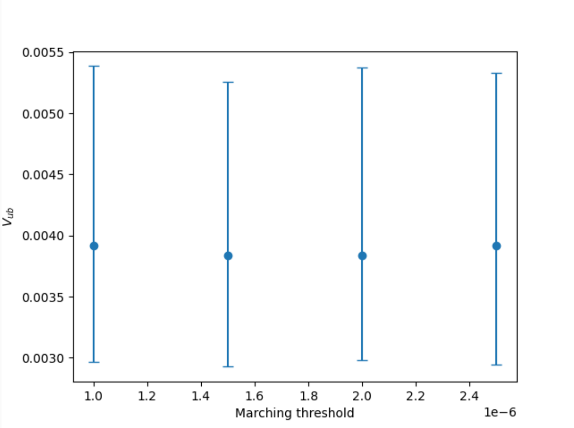
Both distributions are consistent with the same uniform distribution (Kolmogorov–Smirnov test).

Interpolation method



Our interpolation method also does not have a significant effect on the value of V_{ub} we calculate.

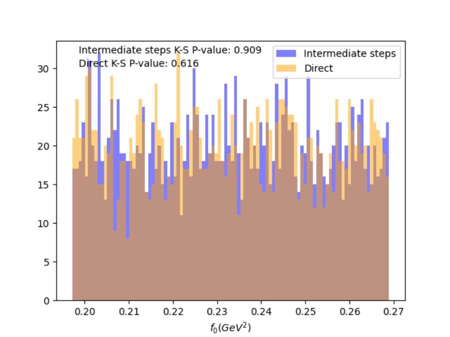
Marching threshold



Marching threshold has no visible effect when δ is small enough.

Generating Form Factor Curves

We test if these segments of form factor curves are equivalent when starting at the bin vs reaching the bin after a series of small steps.



The presence of intermediate steps does not change the curves generated $\rightarrow$ Both f_0 are consistent with the same uniform distribution (Kolmogorov–Smirnov test).