(Virtual) radiative Leptonic decays of charged Kaons

Updates from the RM123 collaboration

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with R. Frezzotti, G. Gagliardi, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, N. Tantalo

41st Lattice conference, Liverpool, 30/07/2024

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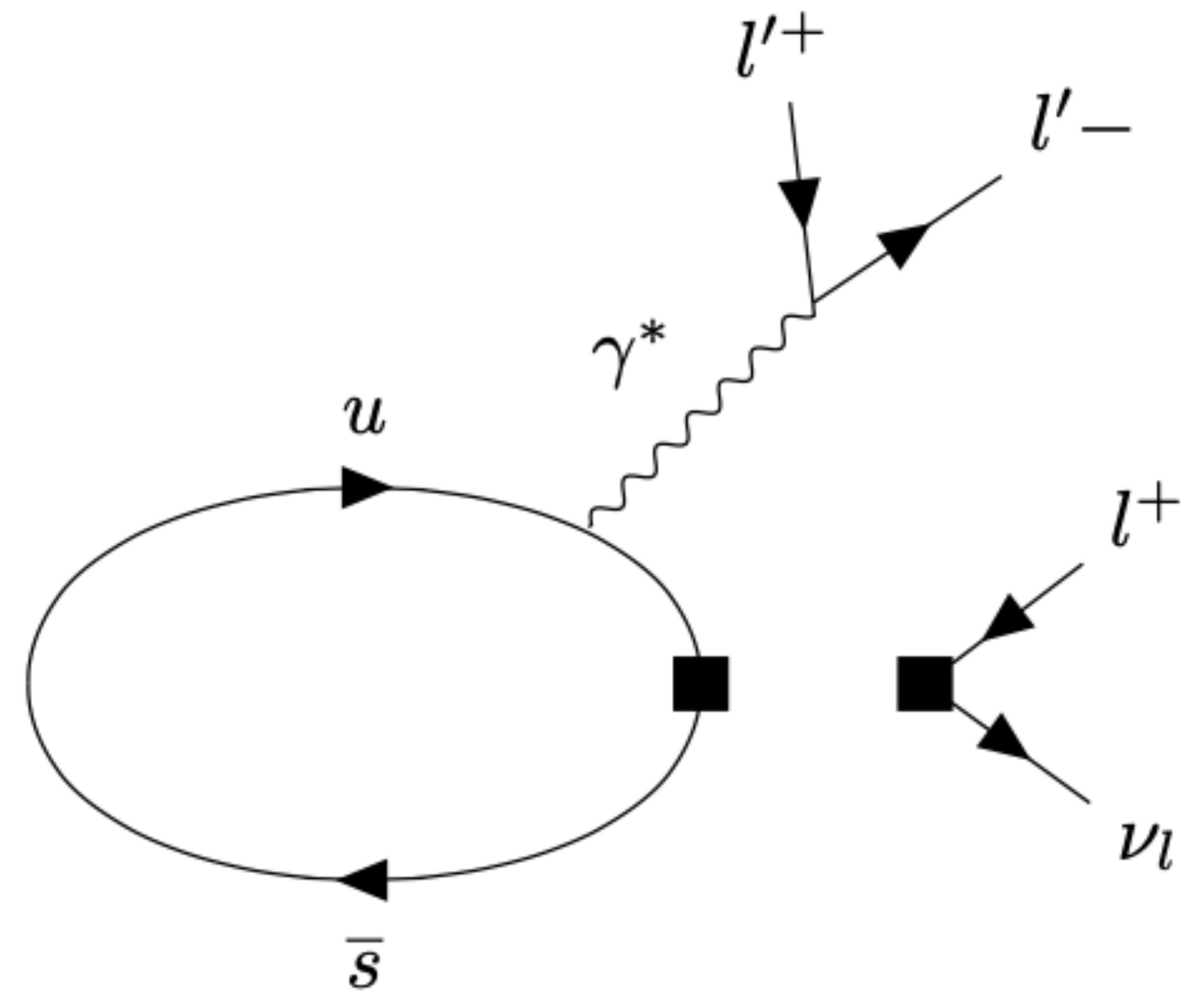
Why $K \to l\nu_l l'^+ l'^-$?

- **Starting at $\mathcal{O}(\alpha_{em}^2)$ in the SM**

Decay	$\mathcal{BR}[10^{-8}]$	$\sigma[\mathcal{BR}][10^{-8}]$	Exp.
$e^+\nu_e e^+ e^-$	2.5	0.2	[Phys.Rev.Lett. 89 (2002) 061803](#)
$\mu^+\nu_\mu e^+ e^-$	7.1	0.3	
$e^+\nu_e \mu^+ \mu^-$	1.7	0.5	[Phys.Rev.D 73 (2006) 037101](#)
$\mu^+\nu_\mu \mu^+ \mu^-$	$< 4 \times 10^{-7}$		[Phys.Rev.Lett. 63 (1989) 2177](#)



- **Building upon previous works**

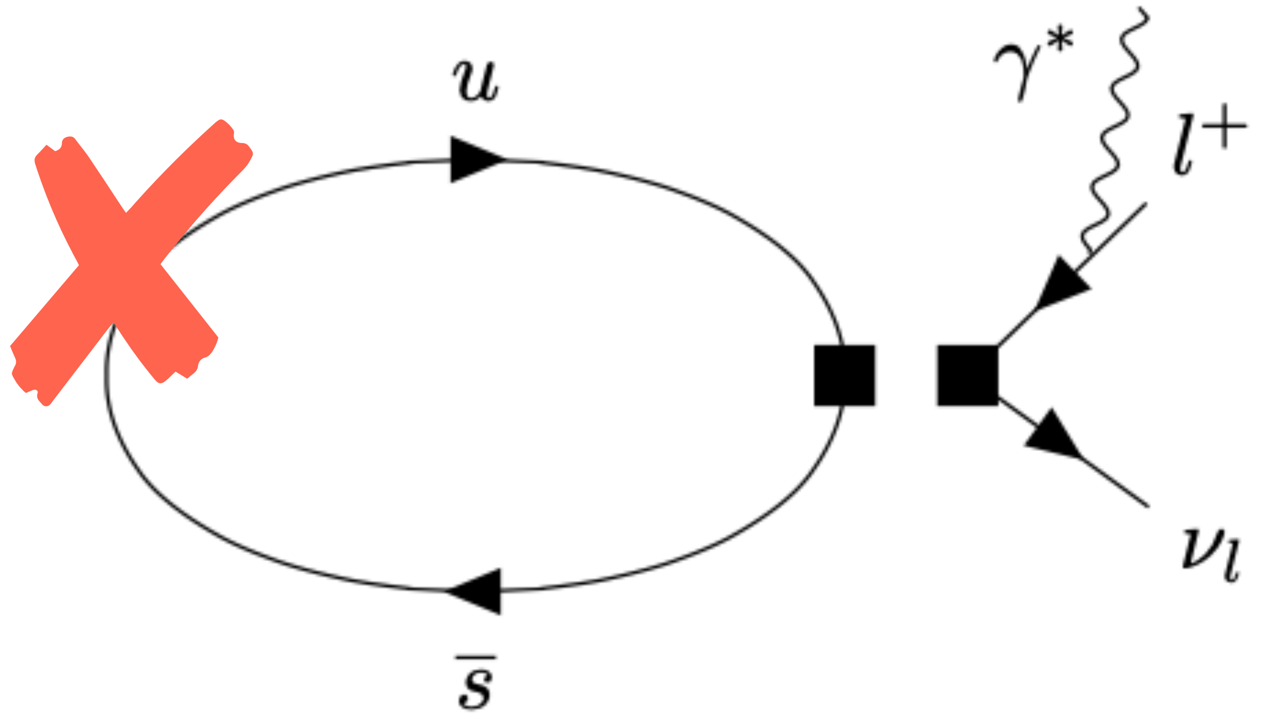
Phys.Rev.D 105 (2022) 11, 114507 **RM123**
Phys.Rev.D 105 (2022) 5, 054518 **Xu Feng et al.**

→
- Performed on a single gauge ensemble
- Heavy pions (Continuation problem)

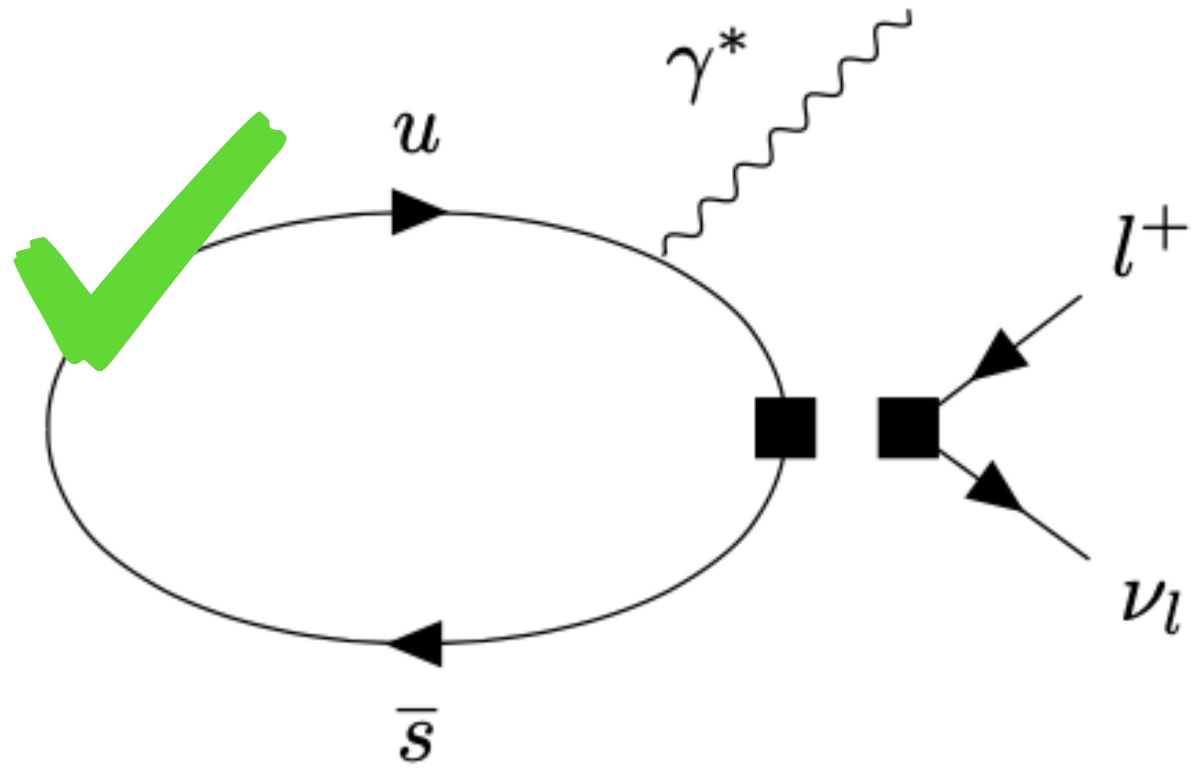
- **Testing the HLT method for hadronic amplitudes**

Phys.Rev.D 99 (2019) 9, 094508 → The method
Phys.Rev.Lett. 130 (2023) 24, 241901 → R-ratio
Phys.Rev.D 108 (2023) 7, 074513 → Inclusive τ decay
Phys.Rev.D 108 (2023) 7, 074510 → EW Amplitudes

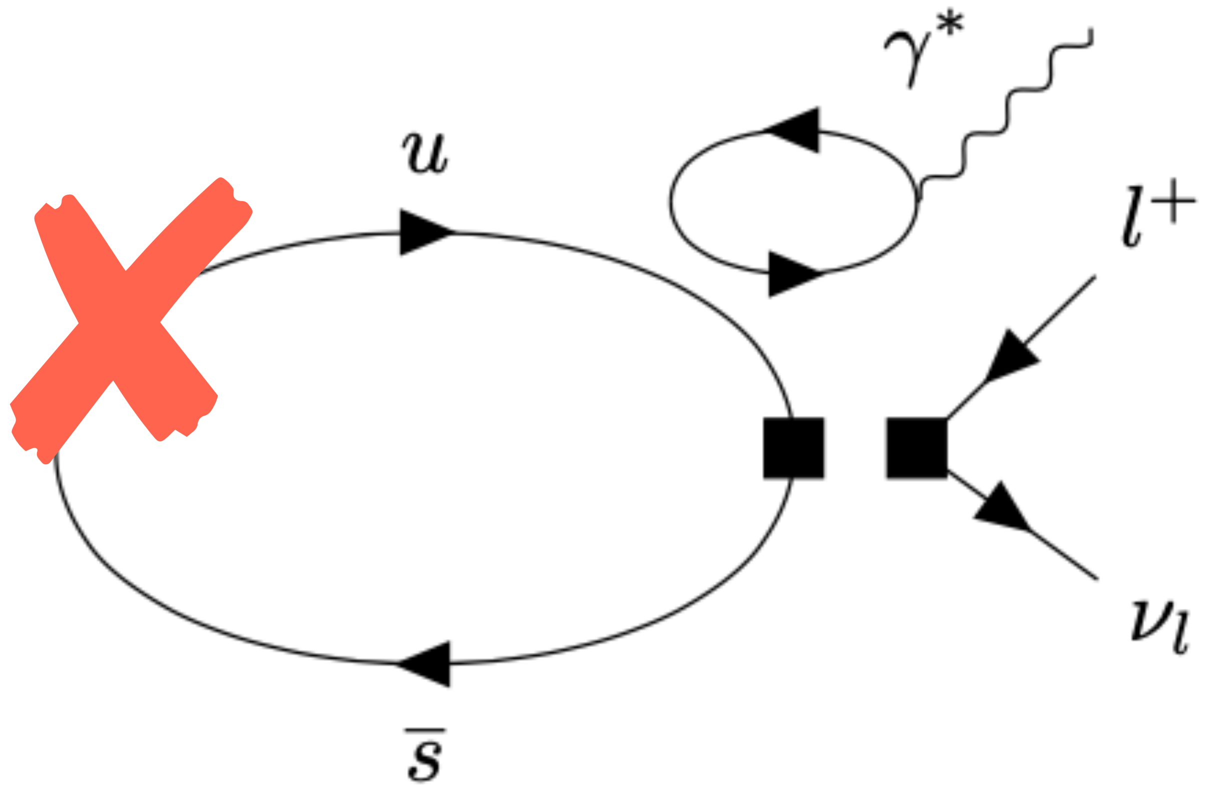
Hadronic amplitudes



Perturbative contr.



What we consider



Electroquenched approx.

- In the **Kaon rest-frame** and with $k = (E_\gamma, \mathbf{k})$, we have

$$H_w^{\mu\nu} = \int d^4x\ e^{ikx} \langle 0 \,|\, T[J_{em}^\mu(x) J_w^\nu(0)] \,|\, K(\mathbf{0}) \rangle = H_{pt}^{\mu\nu} + H_{SD}^{\mu\nu}$$

$H_{pt}^{\mu\nu}$ needs just f_K

$H_{SD}^{\mu\nu}$ needs **4 Form Factors:**

GOAL ⟶ F_V, F_A, H_1, H_2

Time orderings

$$H_w^{\mu\nu}(k,0) = \int_{-\infty}^{0} dt\ e^{iE_\gamma t} \langle 0 | J_w^\nu(0) J_{em}^\mu(t,\mathbf{k}) | K(\mathbf{0}) \rangle$$

1st TO

$$+ \int_{0}^{+\infty} dt\ e^{iE_\gamma t} \langle 0 | J_{em}^\mu(t,\mathbf{k}) J_w^\nu(0) | K(\mathbf{0}) \rangle$$

2nd TO

- Performing a **naive Wick rotation** to Euclidean times

$$t \to -it \qquad \int dt\ e^{iE_\gamma t}(\cdots)(t) \longrightarrow -i\int dt\ e^{E_\gamma t}(\cdots)(-it)$$

- Inserting a complete set of states between the currents $\sum_n |n\rangle\langle n|$

Analytic continuation

1st TO: $$-i \sum_{n_{|S|=1}} \langle 0 | J_w^\nu(0) | n_{|S|=1}(-\mathbf{k}) \rangle \langle n_{|S|=1}(-\mathbf{k}) | J_{em}^\mu(0,\mathbf{k}) | K(\mathbf{0}) \rangle \int_{-\infty}^{0} dt\ e^{t(E_\gamma + E_{n_{|S|=1}} - m_K)}$$

- **Lighter state contributing:** $n^*_{|S|=1} = K(-\mathbf{k})$
- **Convergence if** $E_\gamma + E_{n_{|S|=1}} - m_K > 0$

Analytic continuation

1st TO: $-i\sum_{n_{|S|=1}}\langle 0|J_w^\nu(0)|n_{|S|=1}(-\mathbf{k})\rangle\langle n_{|S|=1}(-\mathbf{k})|J_{em}^\mu(0,\mathbf{k})|K(\mathbf{0})\rangle \int_{-\infty}^{0} dt\ e^{t(E_\gamma+E_{n_{|S|=1}}-m_K)}$

- **Lighter state contributing:** $n^*_{|S|=1} = K(-\mathbf{k})$
- **Convergence if** $E_\gamma + E_{n_{|S|=1}} - m_K > 0$

2nd TO: $-i\sum_{n_{S=0}}\langle 0|J_{em}^\mu(0,\mathbf{k})|n_{S=0}(\mathbf{k})\rangle\langle n_{S=0}(\mathbf{k})|J_w^\nu(0)|K(\mathbf{0})\rangle \int_{0}^{\infty} dt\ e^{-t(E_{n_{S=0}}-E_\gamma)}$

- **Lighter state contributing:** $n^*_{S=0} = \pi\pi(\mathbf{k})$

  coupling with the **light e.m. current** $\bar{l}\gamma^\mu l$
- **Convergence if** $E_{n_{s=0}} - E_\gamma > 0$

Continuation problem

Starting at

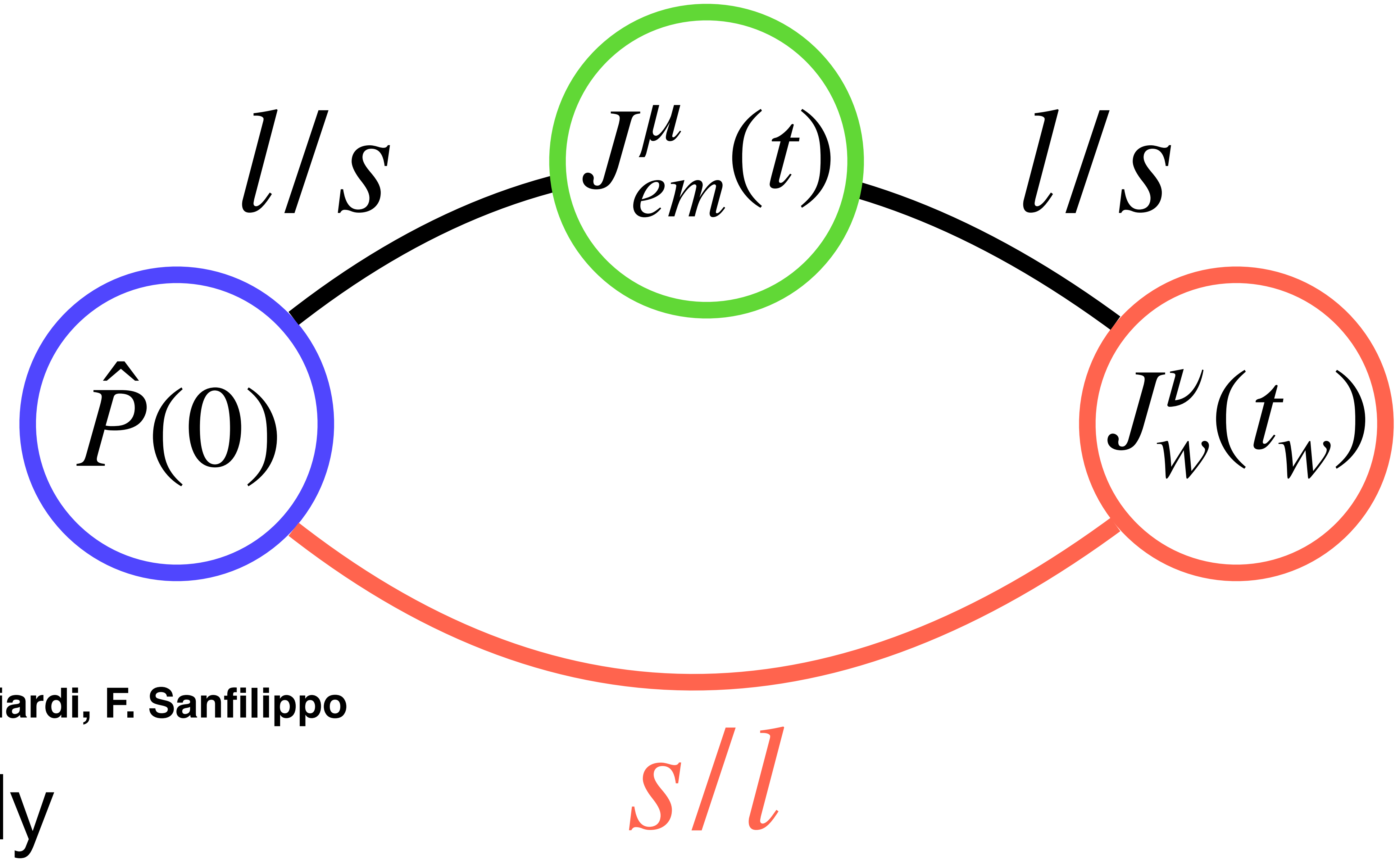
$$k^2 > 4m_\pi^2$$

Euclidean lattice correlators

- We evaluate **3-pts correlation functions** on a Euclidean lattice

$$C^{\mu\nu}_{w,E}(t,\mathbf{k}) \propto T\langle J^{\mu}_{em}(t,\mathbf{k}) J^{\nu}_{w}(t_w)\hat{P}(0)\rangle_{LT}$$

- **Interpolating operator** $\hat{P}(0)$
- **Weak current** at a fixed time t_w
- Employing exactly **conserved e.m. current**
  Nucl.Phys.B Proc.Suppl. 17 (1990) 361-364
- Optimized Gaussian smearing Lattice 2023
  R. Di Palma, G. Gagliardi, F. Sanfilippo
- Light and strange contributions separately



Lattice setup

Ensemble	a[fm]	L[fm]	T[fm]	t_w[fm]	N_{confs}	N^l_{srcs}	N^s_{srcs}
B64	0.079	5.09	10.2	2.0, 2.5	200	24	12
C80	0.068	5.45	10.9	2.2	160	24	12
B96	0.079	7.63	15.3	2.2	Ongoing		
B48	0.079	3.8	7.6	2.2			

- $N_f = 2 + 1 + 1$ Wilson-Clover twisted-mass ETMC gauge ensembles
- **Physical pions**
- **Study of the ground-state dominance** performed only for the B64
- Photon momentum covering all the values available:

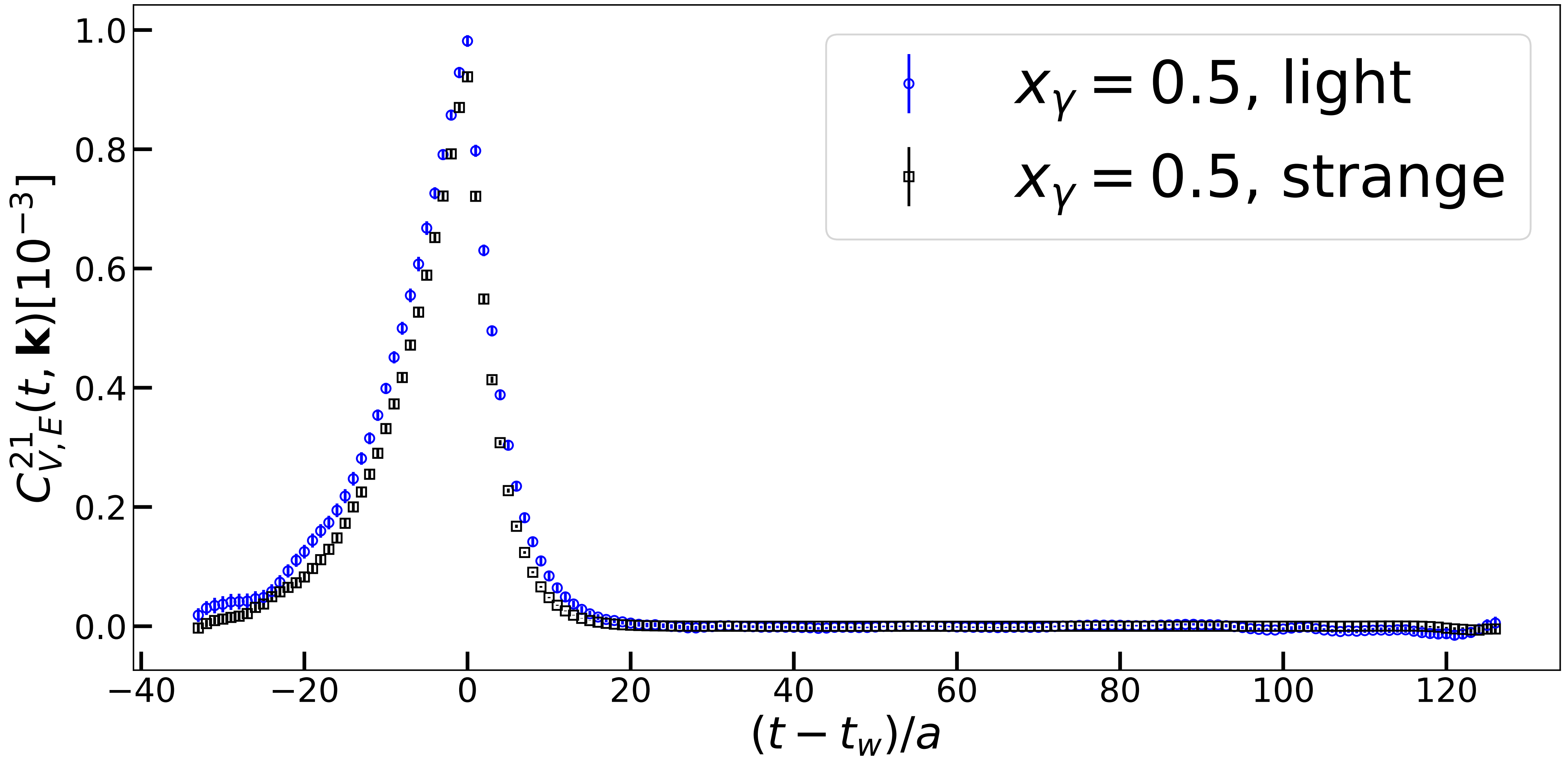
$\mathbf{k} = (0,0,k_z)$ $\quad x_\gamma = 2|\mathbf{k}|/m_K : 0.1, 0.3, 0.5, 0.7, 0.9$

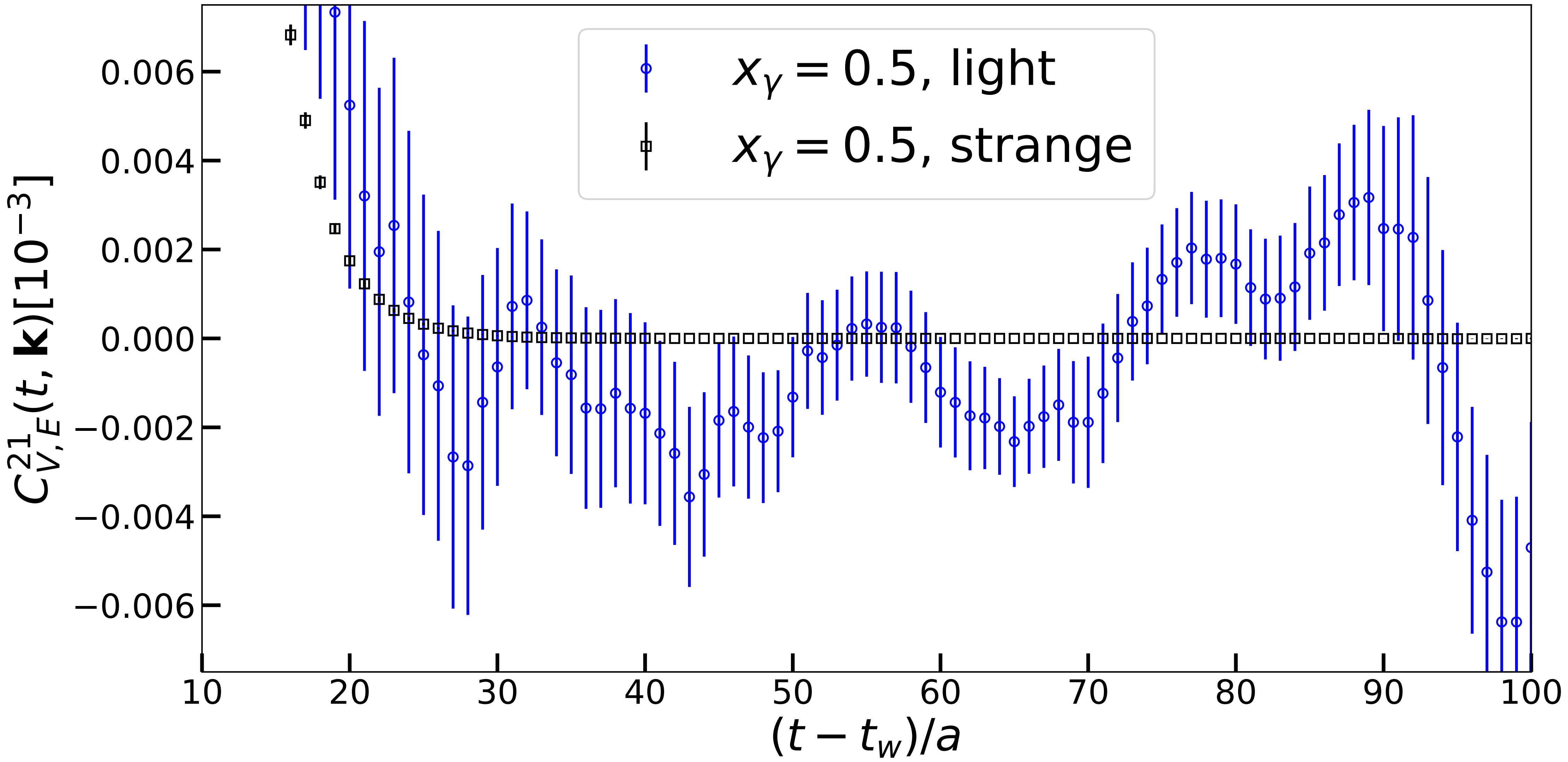
$x_\gamma = 0$ used to reduce the noise

Two strategies

Naive Wick Rotation

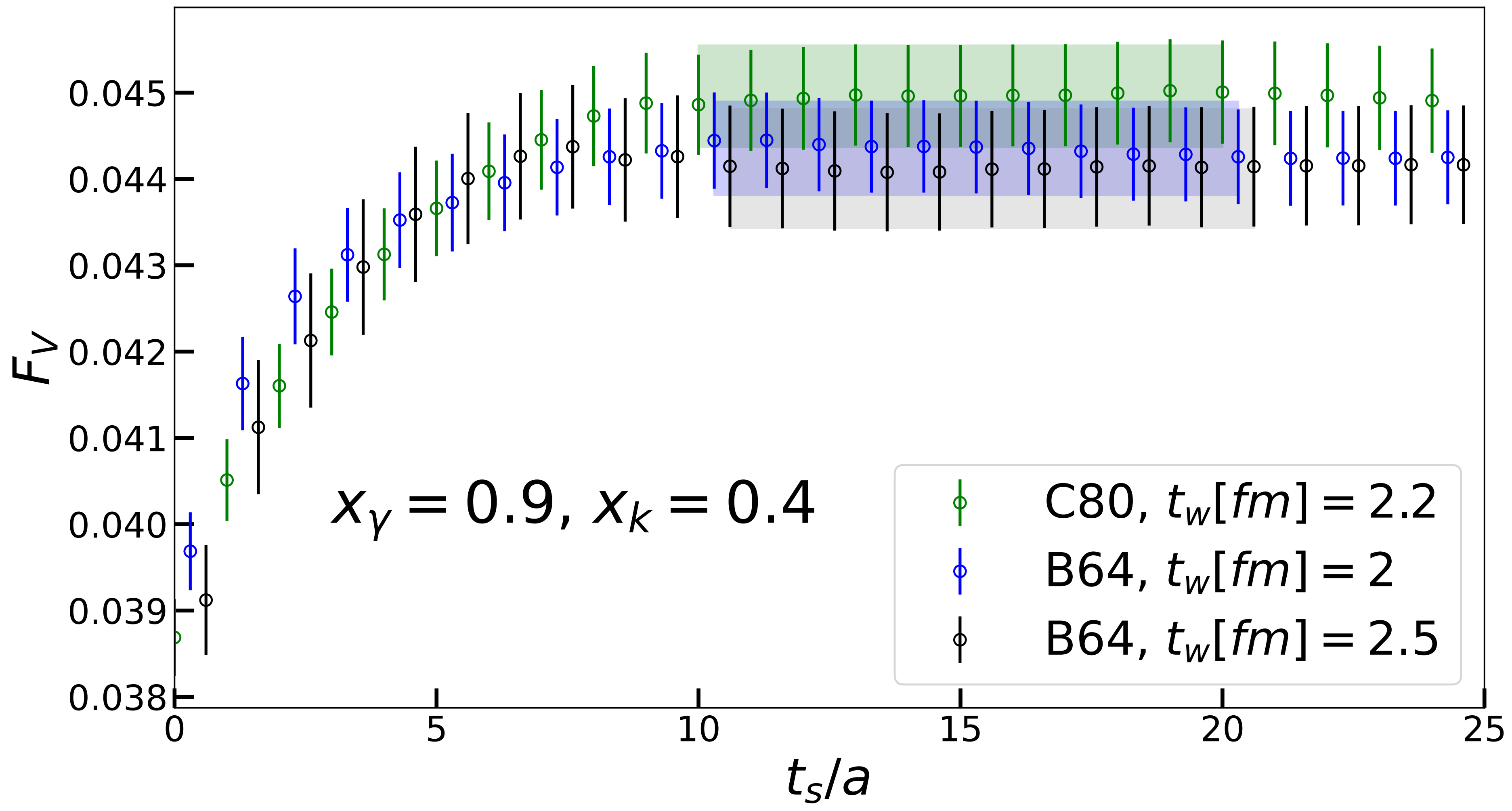
$$iH_w^{\mu\nu}(k,0) = \sum_{-\infty}^{0} e^{E_\gamma t} C_{w,E}^{\mu\nu}(t,\mathbf{k}) + \sum_{0}^{\infty} e^{E_\gamma t} C_{w,E}^{\mu\nu}(t,\mathbf{k})$$





- **Standard analysis** for 1st TO and 2nd TO for $k^2 < 4m_\pi^2$
- We rely on the so-called **HLT method for the light contribution** in the region $k^2 > 4m_\pi^2$
- **Finite-volume effects** due to the temporal truncation of the integrals

Standard approach



1st TO: $-\infty$ → $t_w - t_s$ → t_w

- **Summing the Correlator:** $\sum_{t=t_w-t_s}^{t_w} C^{\mu\nu}_{w,E}(t, \mathbf{k}) e^{E_\gamma (t-t_w)}$

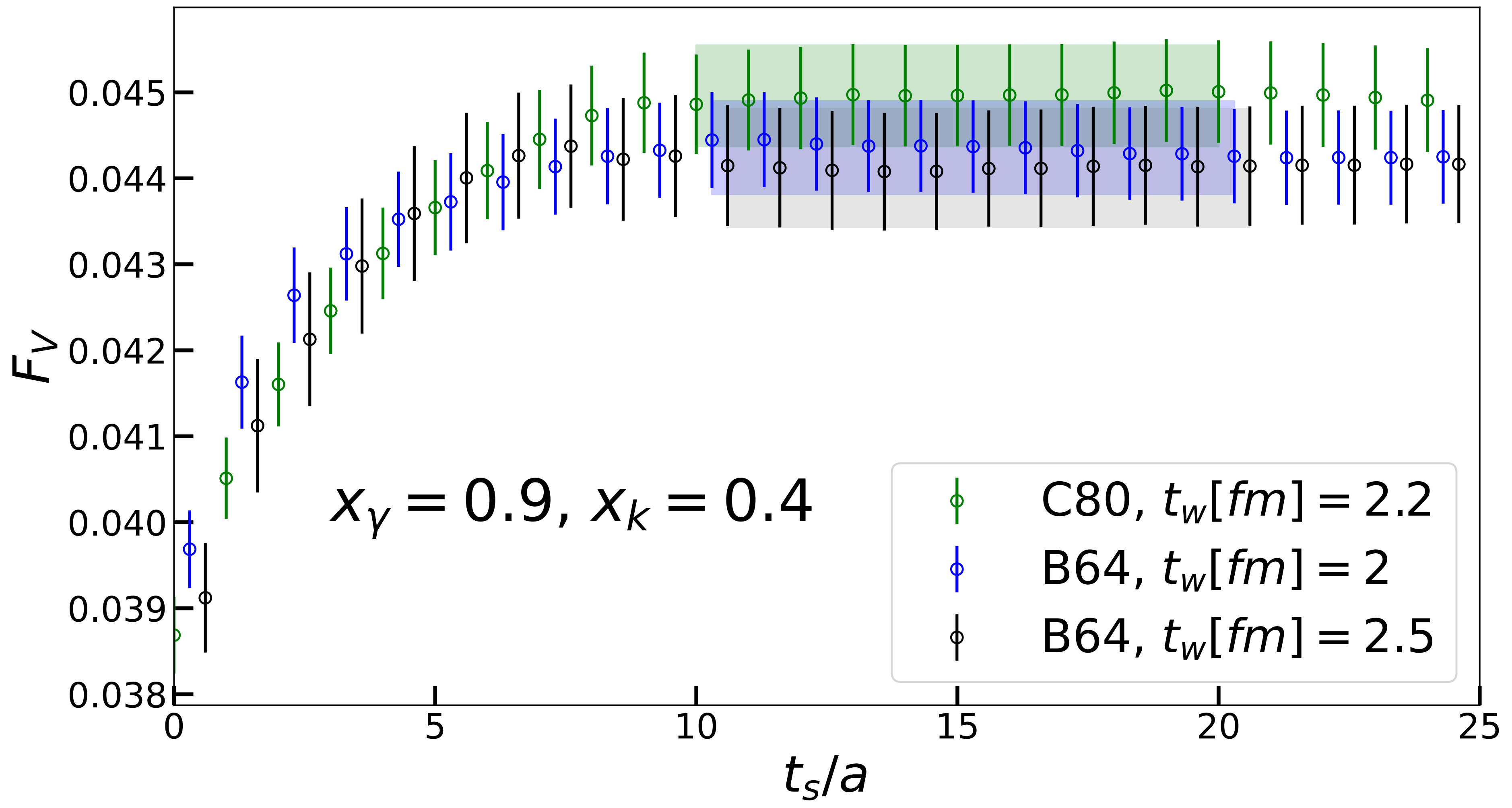
Phys.Rev.D 105 (2022) 5, 054518 Xu Feng et al.

- **One particle state dominance:** $\dfrac{C^{\mu\nu}_{w,E}(t_w - t_s, \mathbf{k}) e^{-E_\gamma t_s}}{E_\gamma + E^\infty_w - m_k}$

  $V : K^*(\mathbf{k}),\ A : K(\mathbf{k}),\ K_1(\mathbf{k})$

Standard approach

1st TO: $-\infty \quad t_w - t_s \quad t_w$



- **Summing the Correlator:** $\sum_{t=t_w-t_s}^{t_w} C^{\mu\nu}_{w,E}(t,\mathbf{k})e^{E_\gamma(t-t_w)}$

Phys.Rev.D 105 (2022) 5, 054518 **Xu Feng et al.**

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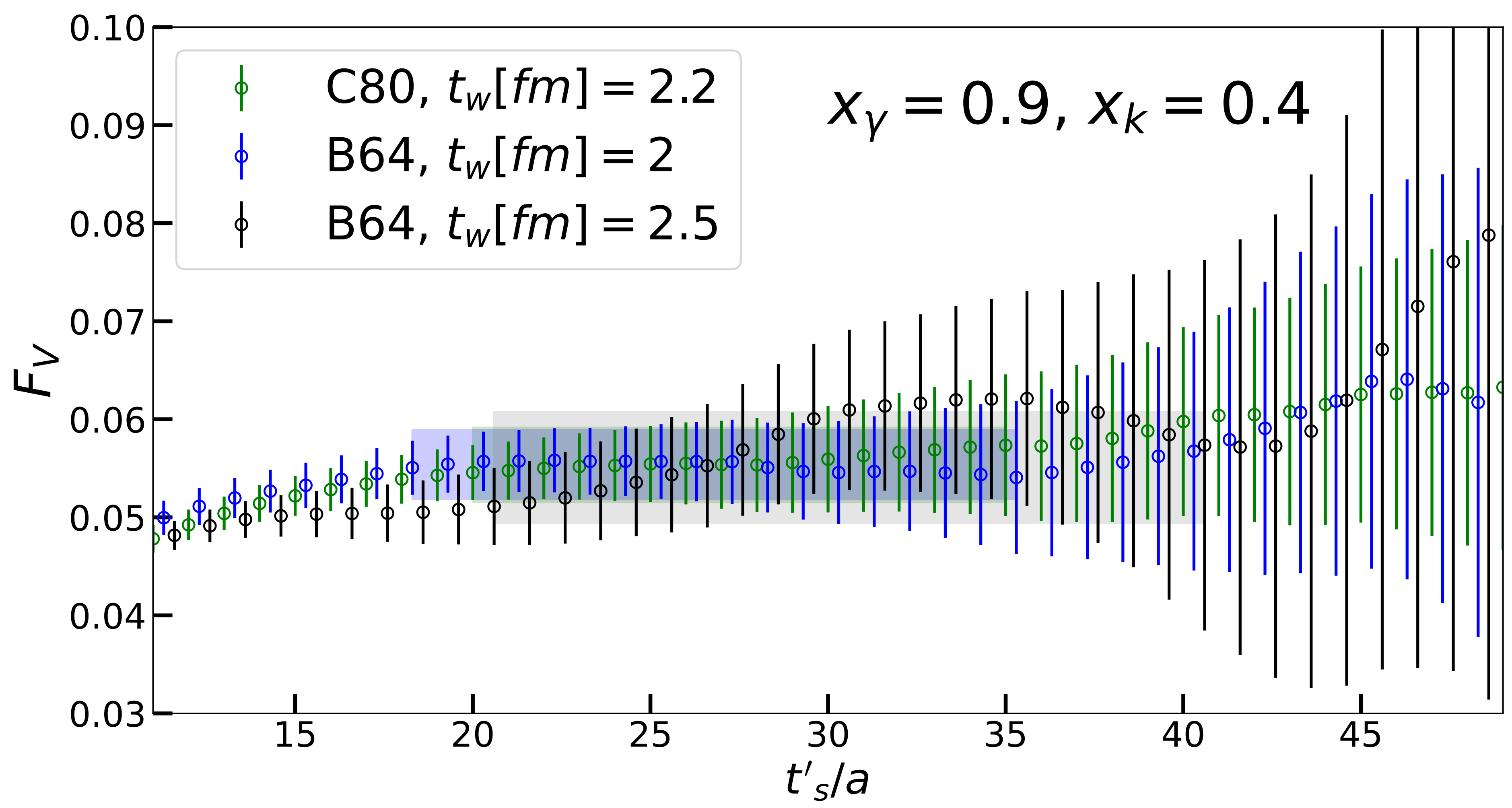
  $V : K^*(\mathbf{k}),\ A : K(\mathbf{k}),\ K_1(\mathbf{k})$

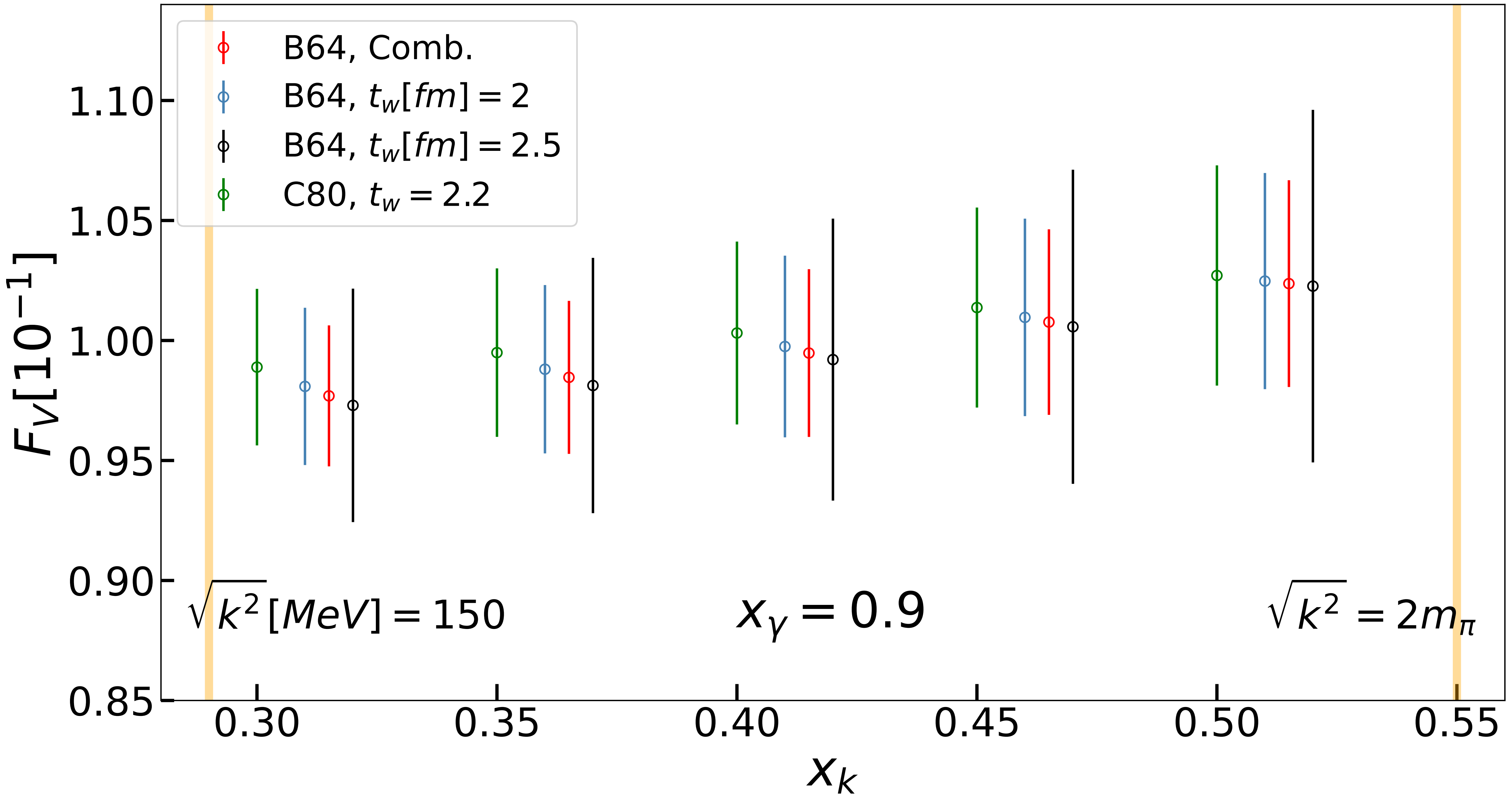
2nd TO: $t_w \quad t_w + t'_s \quad +\infty$

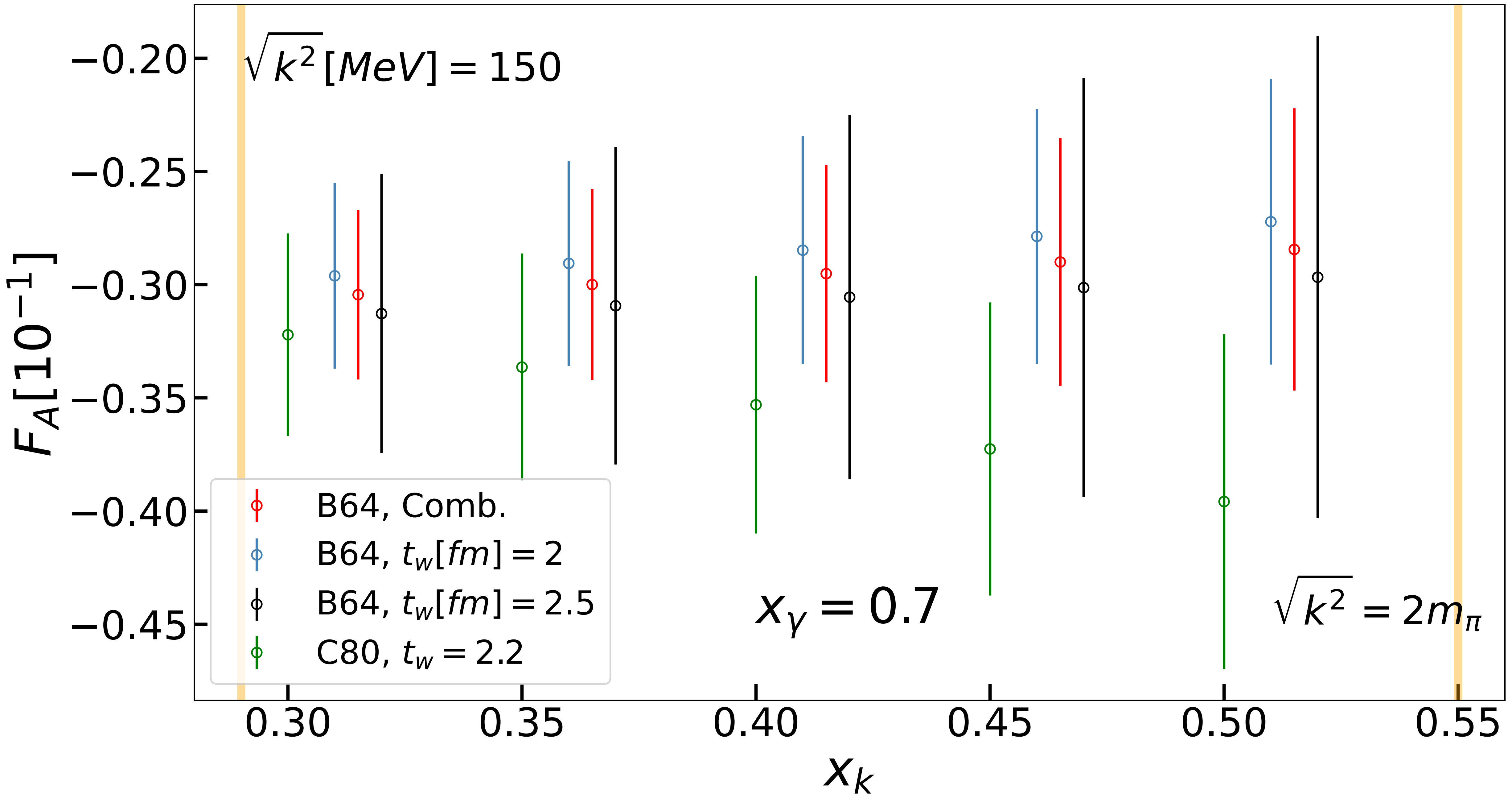
- **Summing the Correlator:** $\sum_{t=t_w+1}^{t_w+t'_s} C^{\mu\nu}_{w,E}(t,\mathbf{k})e^{E_\gamma(t-t_w)}$

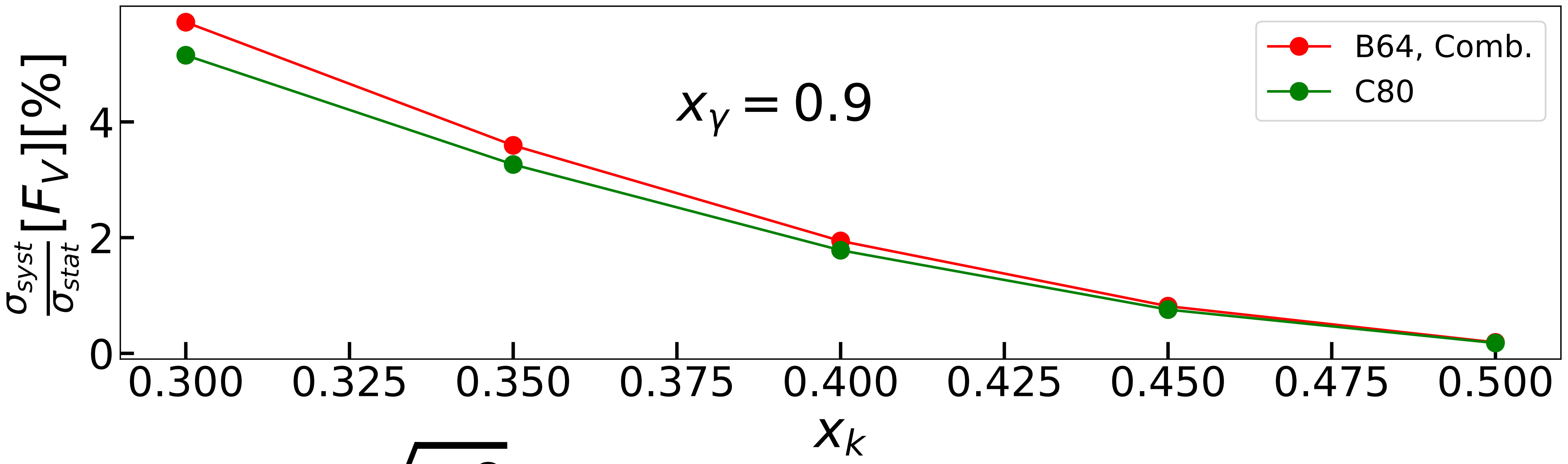
- $t > t_w + t'_s : C^{\mu\nu}_{w,E}(t,\mathbf{k}) = 0$

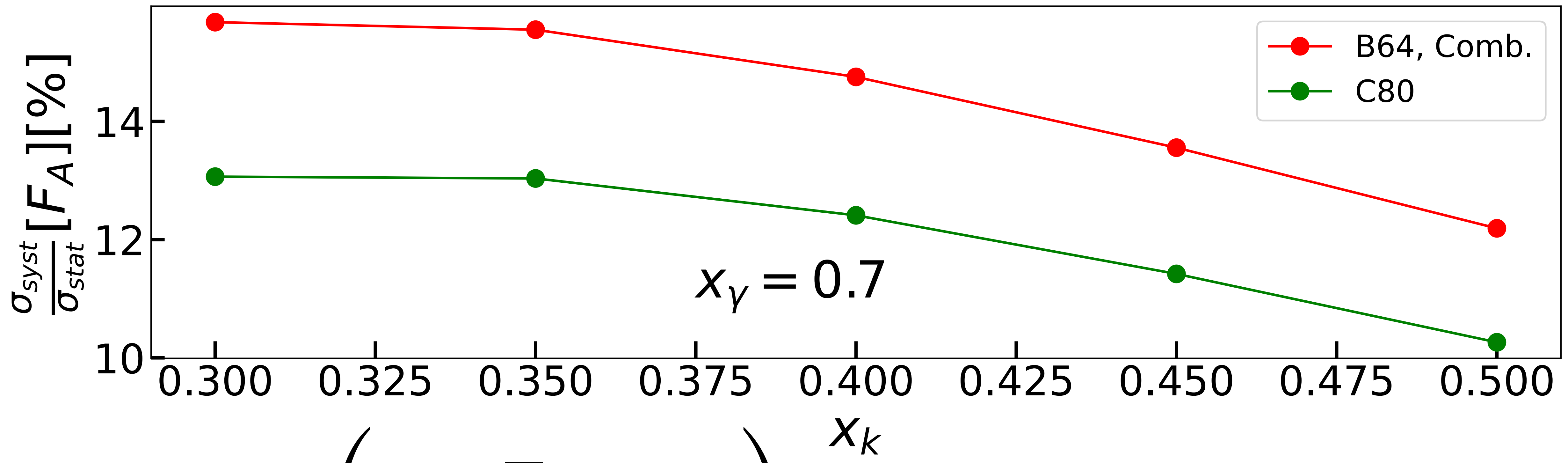


Form factors: F_V and F_A





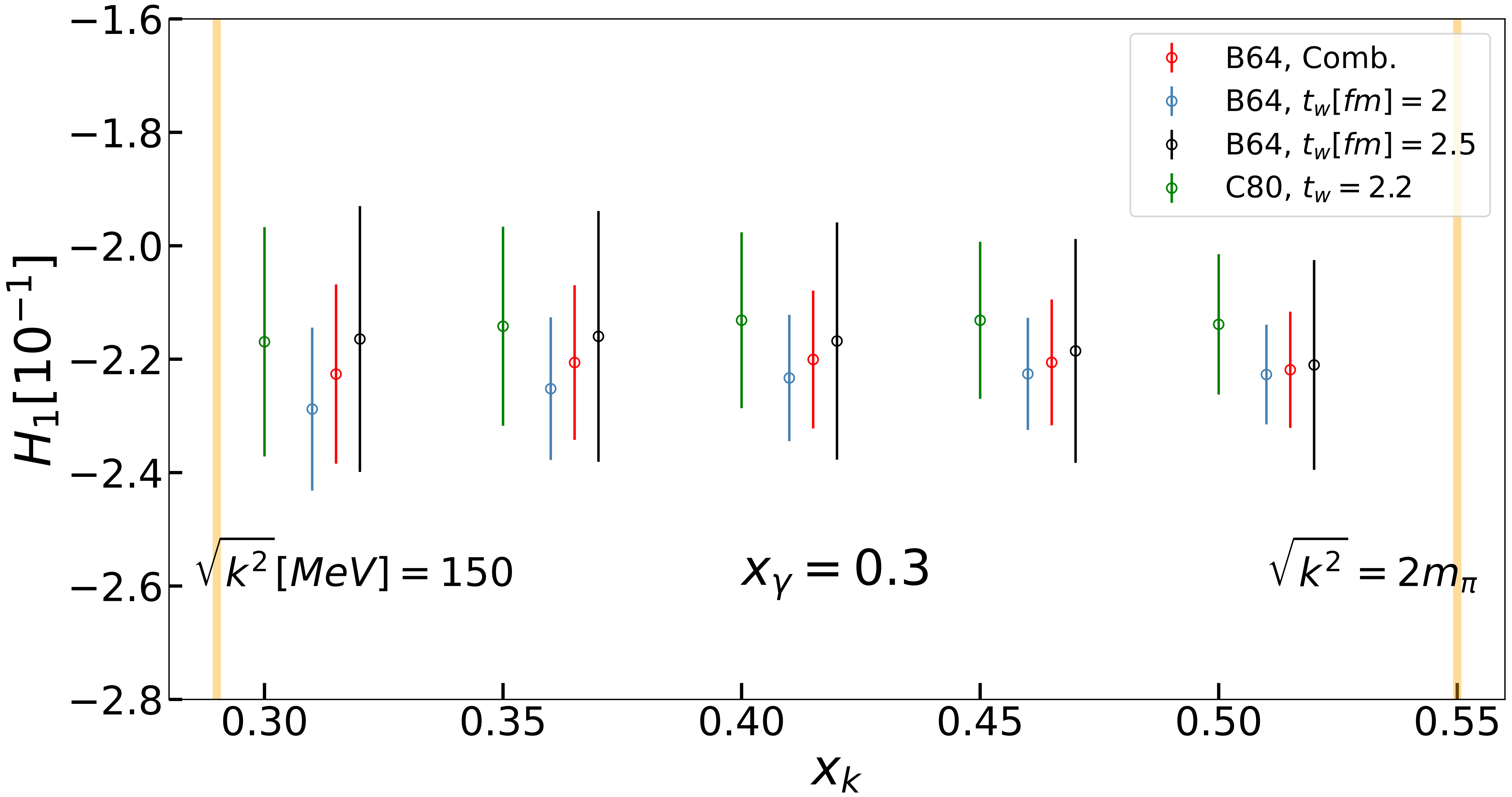


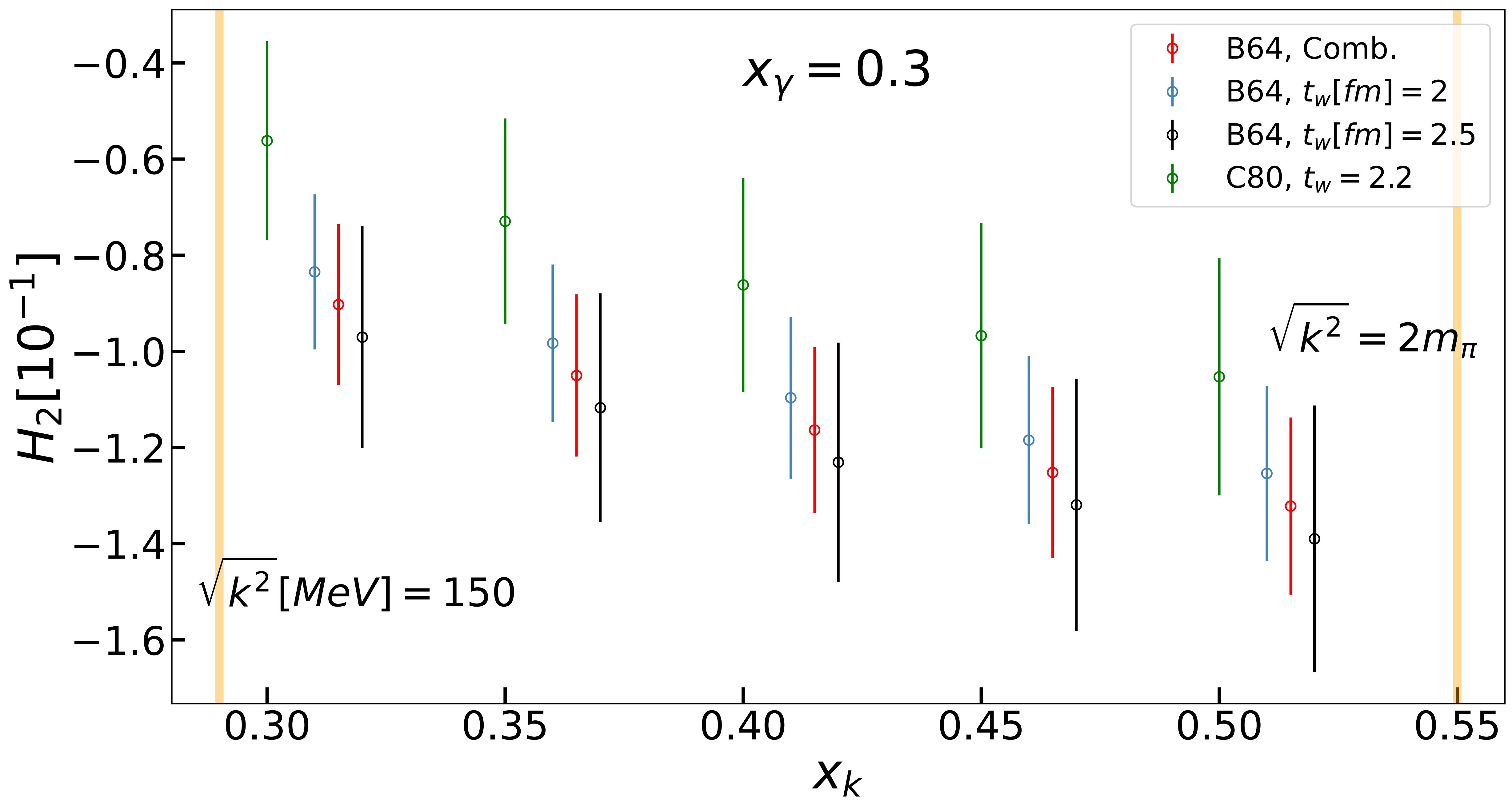


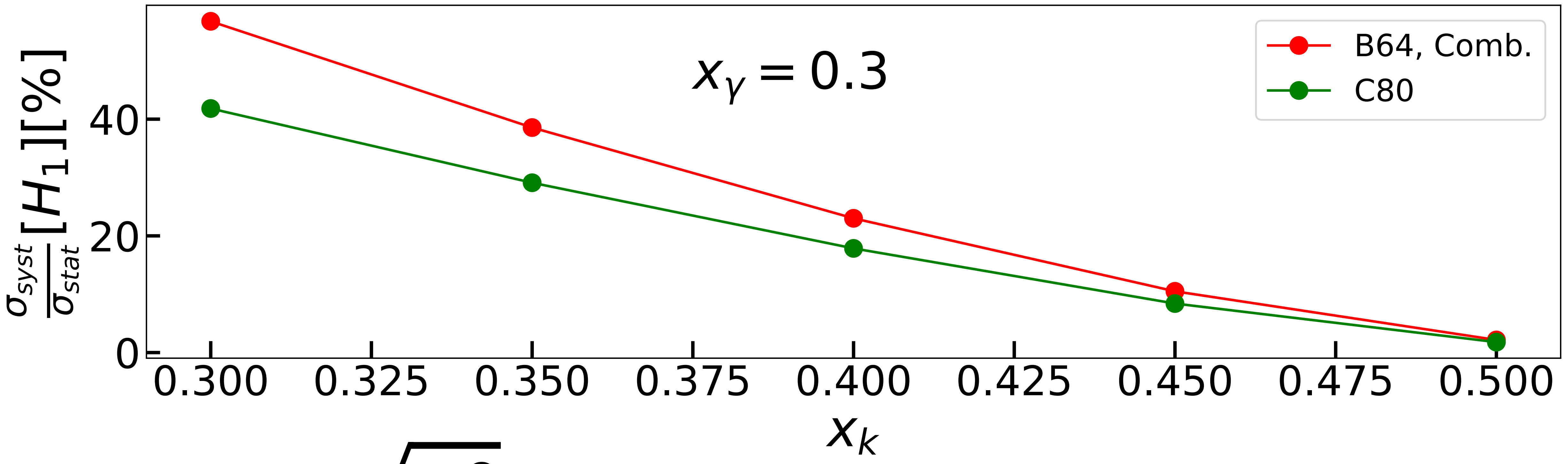
$$x_k = \sqrt{k^2}/m_K \qquad x_\gamma = 2|\mathbf{k}|/m_K$$

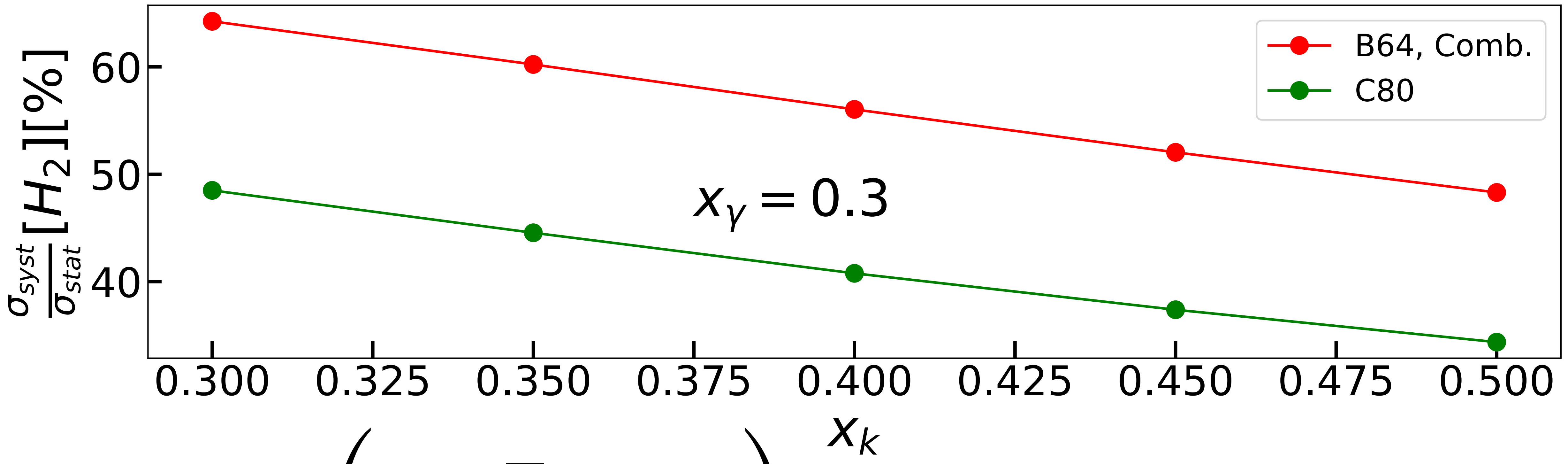
$$\sigma_{syst} = \Delta \mathbf{erf}\left(\Delta/\sqrt{2}\sigma_{stat}[F]\right) \qquad \Delta = |F(t_w^1) - F(t_w^2)|$$

Form factors: H_1 and H_2









$x_k = \sqrt{k^2}/m_K$ $\quad x_\gamma = 2|\mathbf{k}|/m_K$ $\quad \sigma_{syst} = \Delta \mathbf{erf}\left(\Delta/\sqrt{2}\sigma_{stat}[F]\right)$ $\quad \Delta = |F(t_w^1) - F(t_w^2)|$

Spectral representation of $H_w^{\mu\nu}$

- The correlator can be written in terms of its **spectral density** $\rho_w^{\mu\nu}(E')$

$$C_w^{\mu\nu}(t,\mathbf{k}) = \int_{E^*}^{\infty} \frac{dE'}{2\pi}\, e^{-iE't}\rho_w^{\mu\nu}(E',\mathbf{k})$$

$$\rho_w^{\mu\nu}(E',\mathbf{k}) = \langle 0 \,|\, J_{em}^{\mu}(0)(2\pi)^4\delta^3(p-\mathbf{k})\delta(H-E')J_w^{\nu}(0)\,|\, K\rangle$$

- Trading time with energy Phys.Rev.D 108 (2023) 7, 074510

$$H_w^{\mu\nu}(k,0) = \int dt\; e^{iE_\gamma t}C_w^{\mu\nu}(t,\mathbf{k}) \to \lim_{\epsilon\to 0}\int_{E^*}^{\infty}\frac{dE'}{2\pi}\,\frac{\rho_w^{\mu\nu}(E',\mathbf{k})}{E'-E_\gamma - i\varepsilon}$$

$i\varepsilon$ prescription

Relevant for $E^* < E_\gamma$, since it acts as a regulator and provides an **Imaginary part**

- We can use any kernel that does the job

$$H_w^{\mu\nu}(k,0) = \lim_{\epsilon\to 0}\int_{E^*}^{\infty}\frac{dE'}{2\pi}\,\rho_w^{\mu\nu}(E',\mathbf{k})K(E'-E_\gamma;\varepsilon)$$

The HLT method

- The determination of the spectral density from Lattice is an **ill-posed problem.** Evaluating convolution of $\rho_w^{\mu\nu}(E', \mathbf{k})$ with a Kernel at $\epsilon \neq 0$ is **well-posed**

Geophys.J.Int. 16 (1968) 2, 169-205 **G. Backus & F. Gilbert**

$$K(E' - E; \epsilon) = \sum_{t=t_w}^{T/2} g_t(E, \epsilon) e^{-E't}$$

Complex coefficients to determine

$$g_t(E, \epsilon) = g_t^R(E, \epsilon) + i g_t^I(E, \epsilon)$$

$$iH_w^{\mu\nu}(k,0) \simeq \sum_{t=t_w+1}^{T/2} g_t(E, \epsilon) C_{w,E}^{\mu\nu}(t, \mathbf{k})$$

- **The HLT method finds the best choice** for $g_t(E, \epsilon)$ by minimizing

$$W[\mathbf{g}] = A[\mathbf{g}]/A[\mathbf{0}] + \lambda B[\mathbf{g}]$$

Trade-off — **Kernel reconstruction error** — **Statistical error**

More on the subject @ Lattice 24

A. De Santis' talk

G. Gagliardi's talk

F_V from HLT

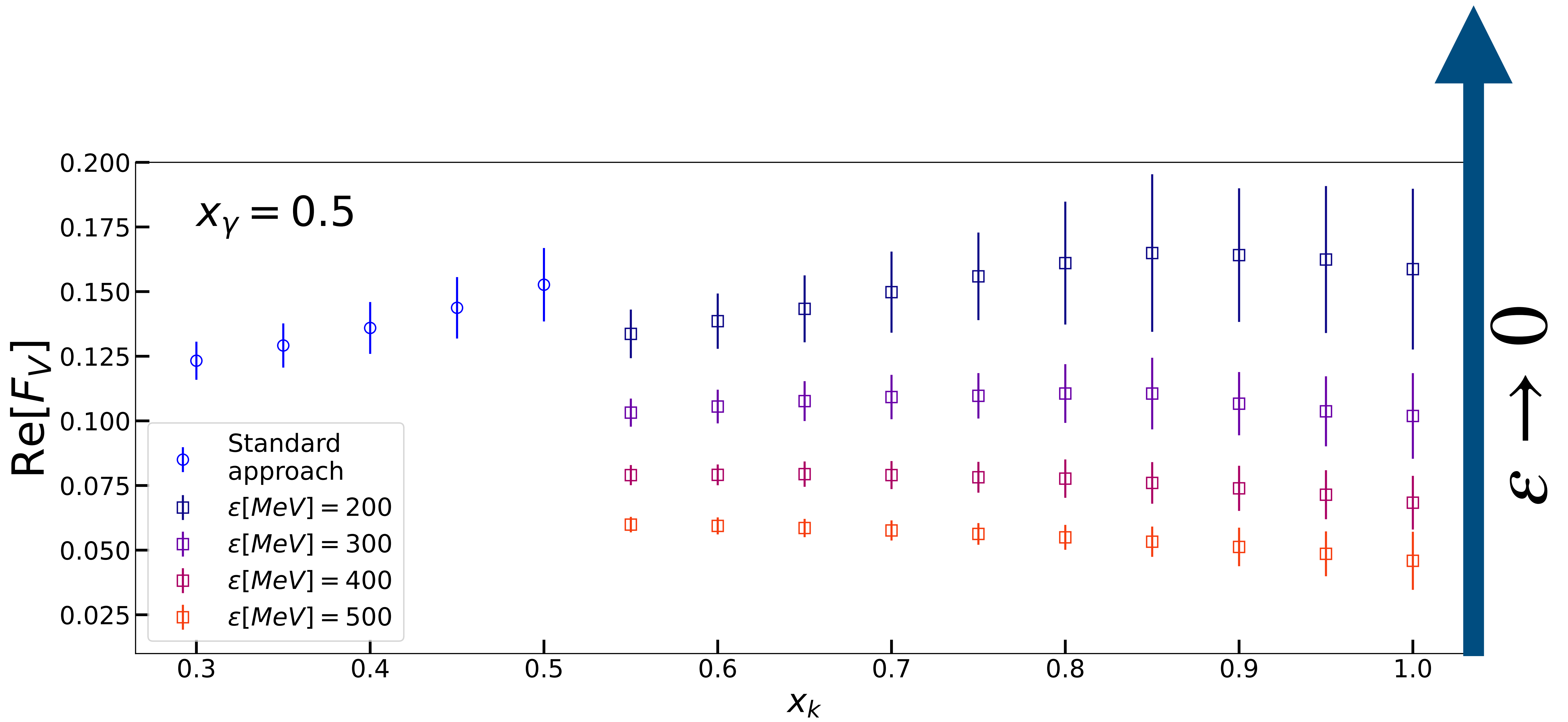
- **2nd TO,** photon emission from the **light quark**. B64 gauge ensemble

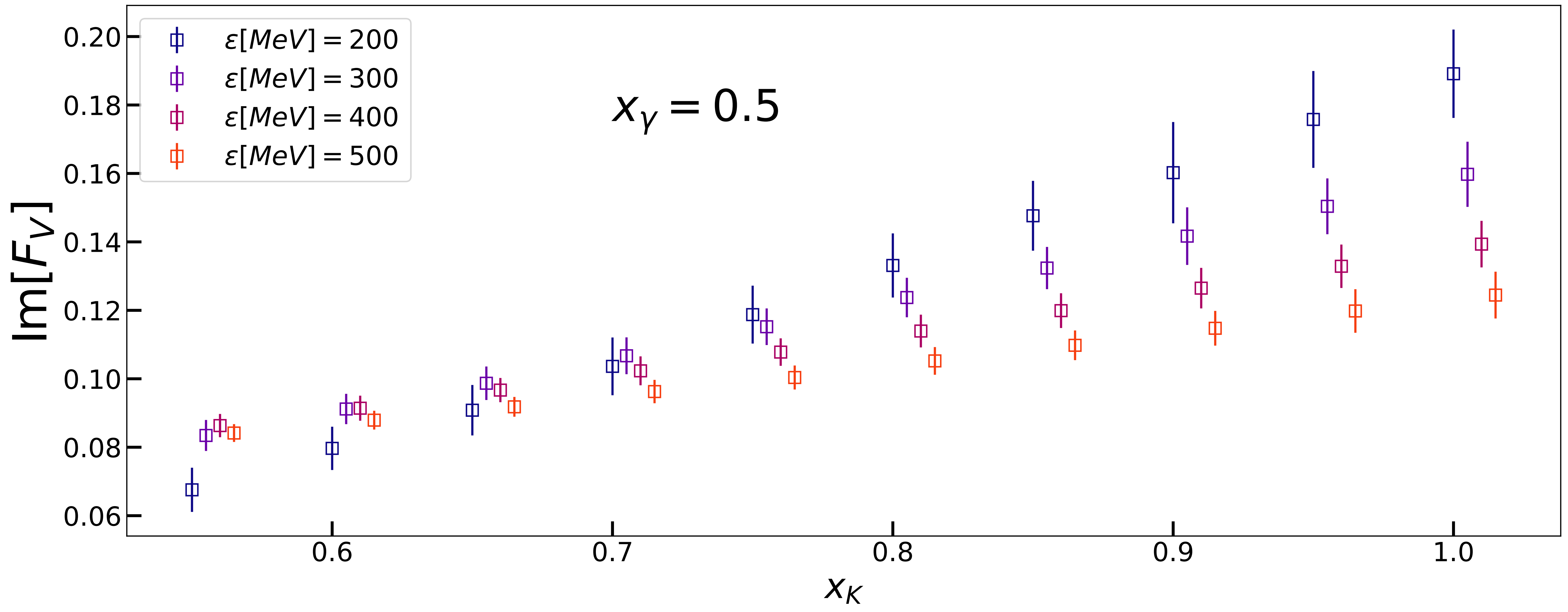
- Real part connects to the standard analysis

  as ε decreases

  (from the bottom up)

- Reconstructing the imaginary part

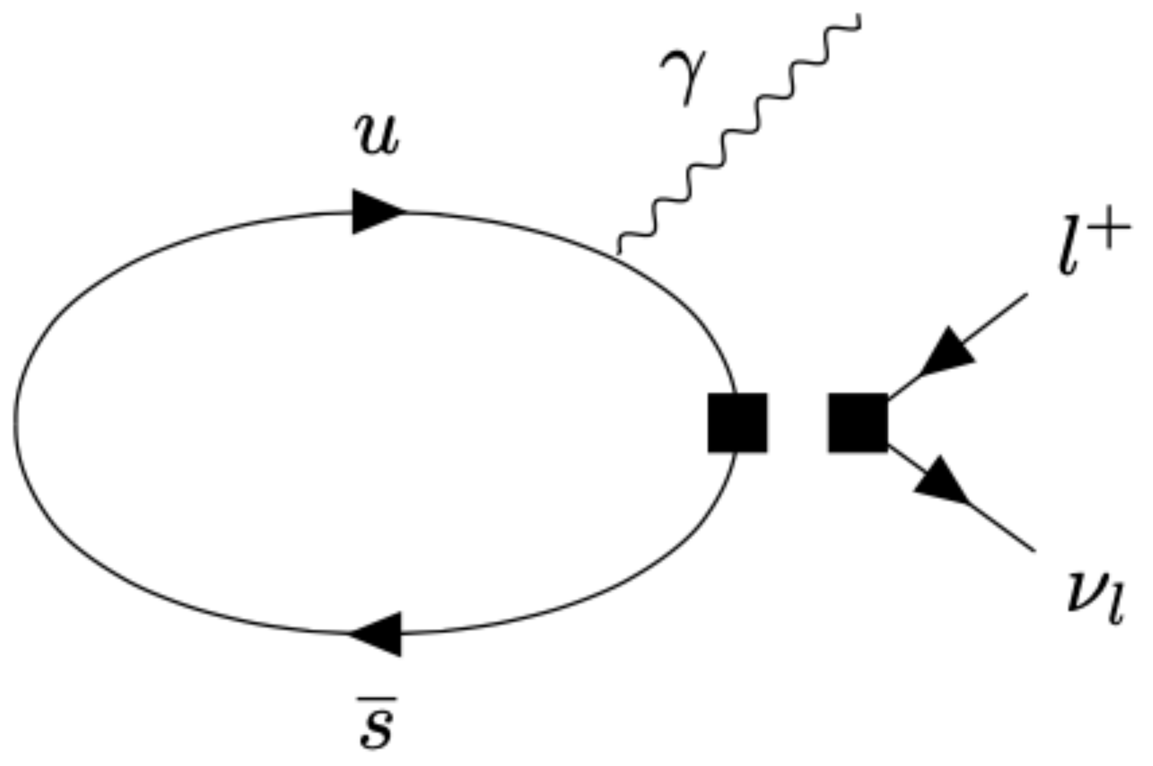




A quick look at the real photon

$$K \to l\nu_l\gamma$$

Real photon emission



- Starting at $\mathcal{O}(\alpha_{em})$ in the SM and removes the helicity suppression affecting pure leptonic decays

- Standard approach: No analytic continuation problem, $k^2 = 0$

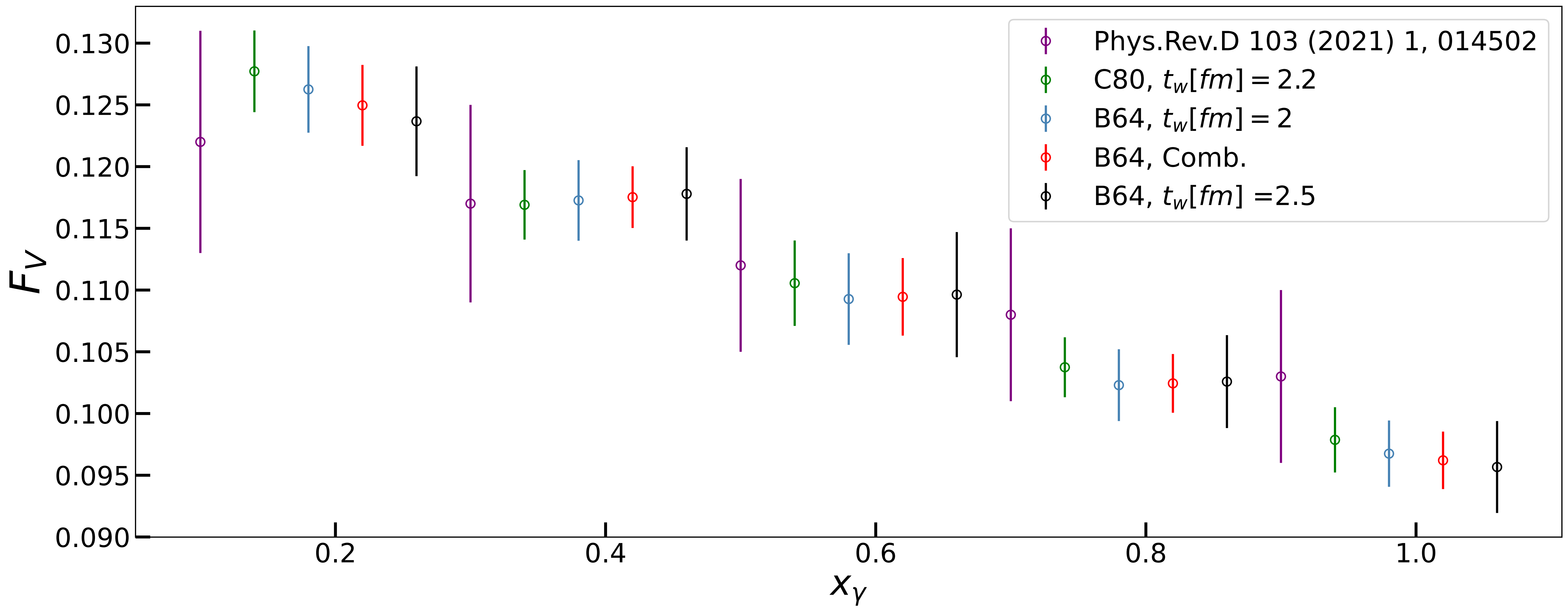
- Just two form factors relevant for $H^{\mu\nu}_{SD}$: F_V and F_A

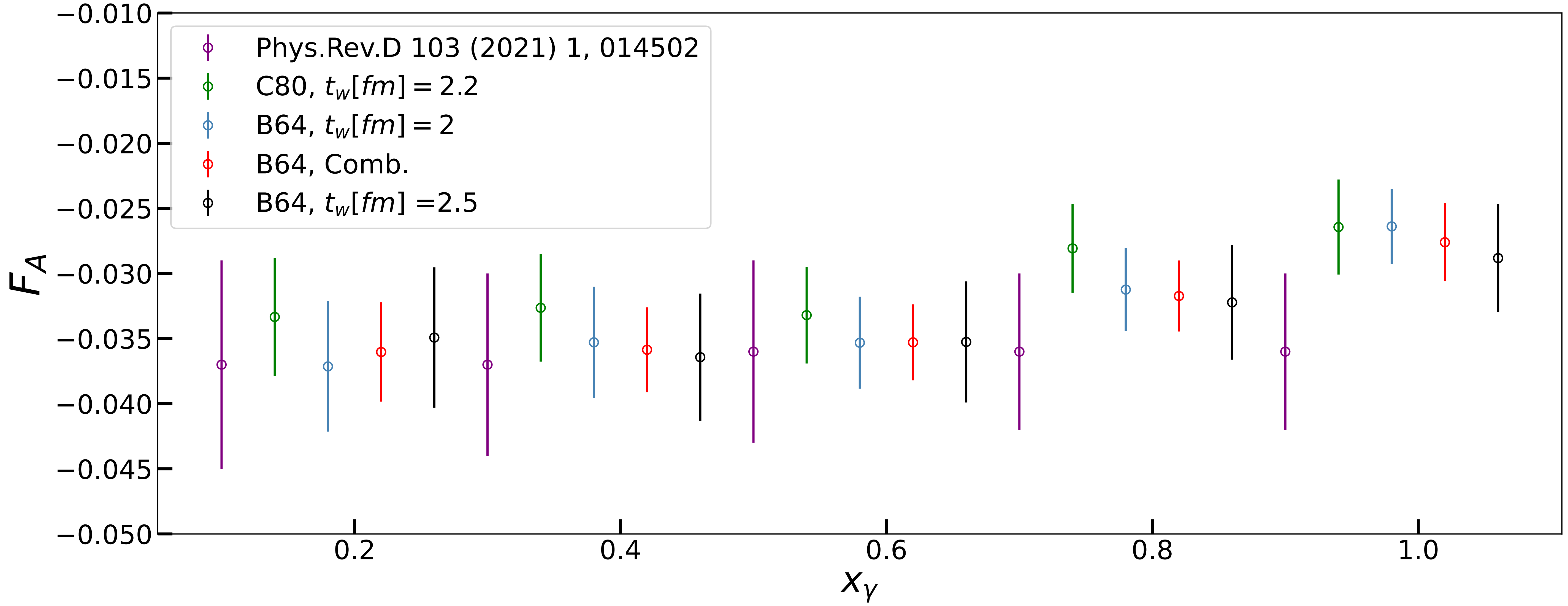
- **Full computation** available → Phys.Rev.D 103 (2021) 1, 014502
  - Heavy pions
  - Continuum and chiral limit

Form factors

- Compatible with the previous results **(purple points)**

- Uncertainty reduced by a factor of 2 (**systematics yet to be included**)



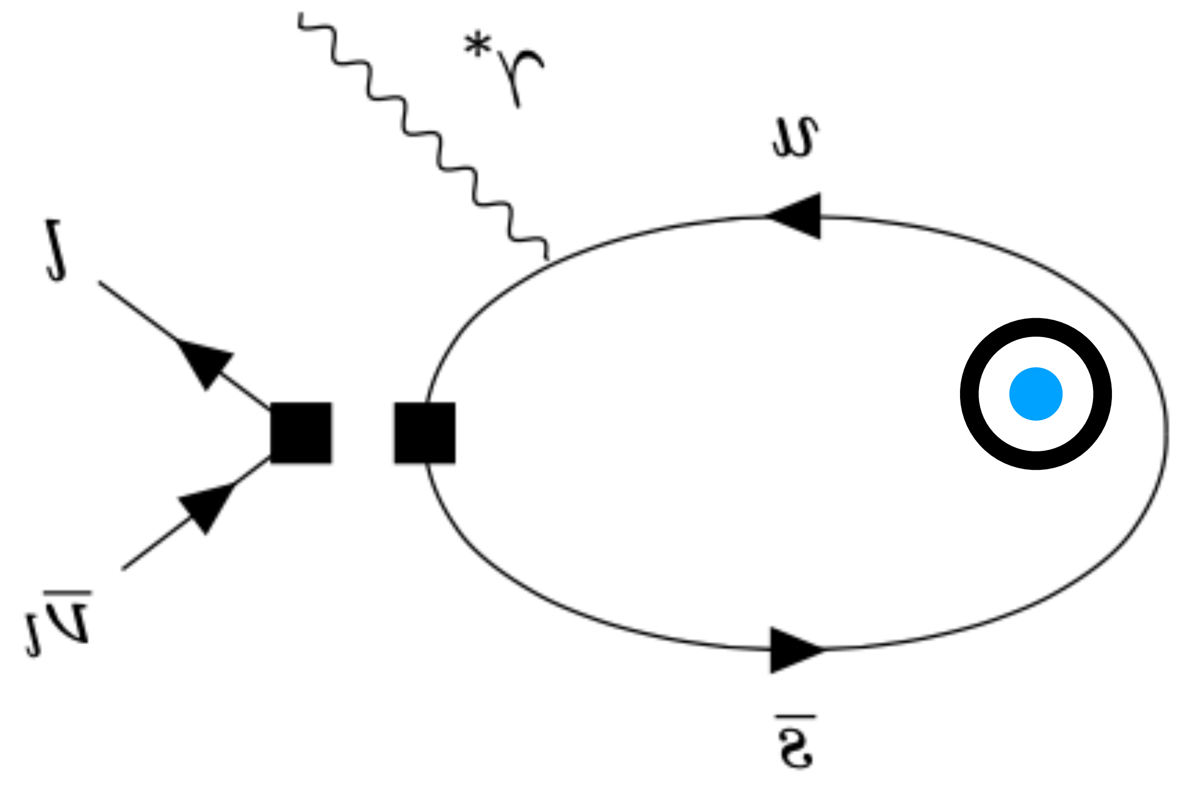


Summary

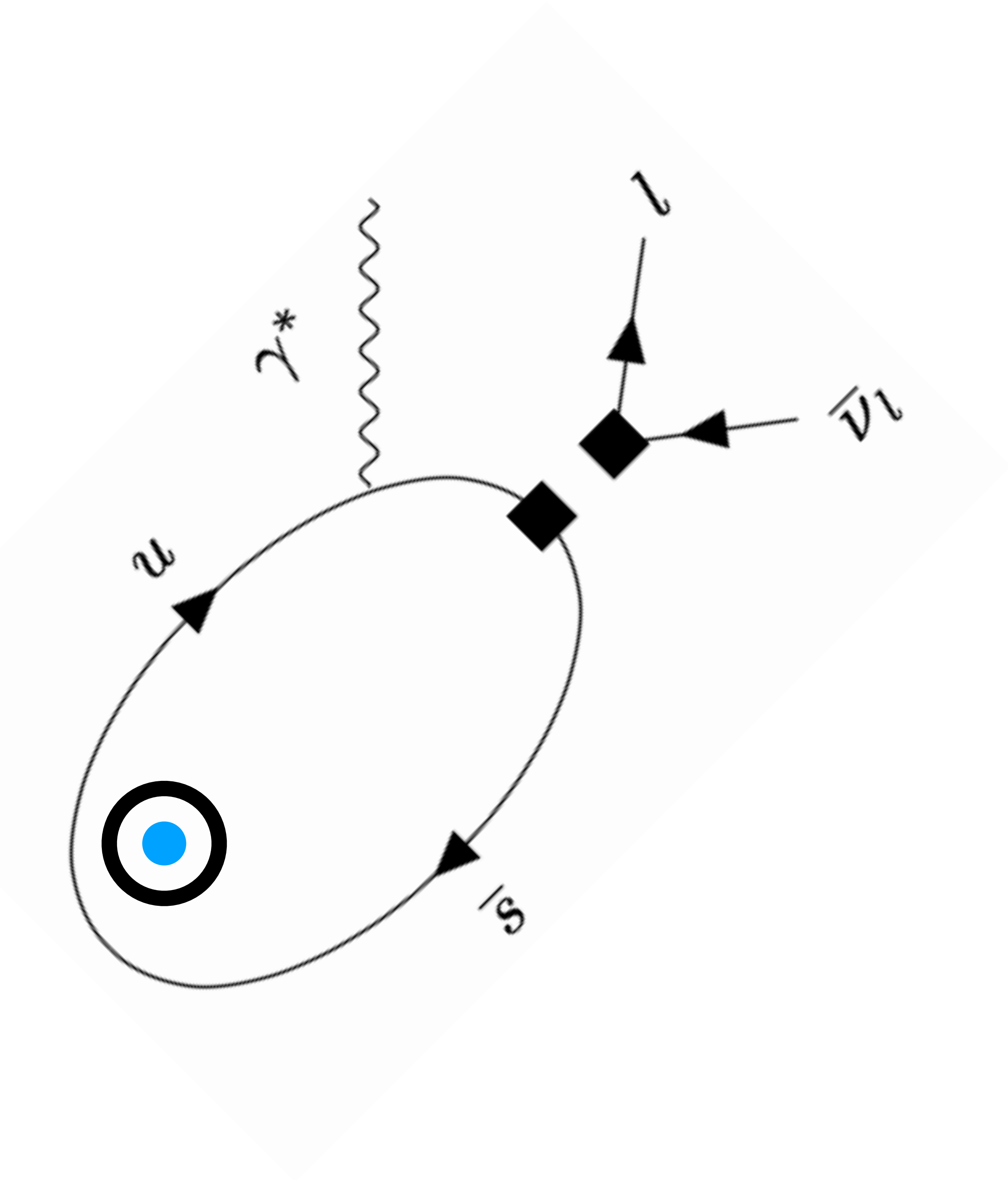
- Computed the **four form factors** describing the SD part of $K \to l\nu_l\gamma*$ **with physical pions**
- Using the **HLT method as a powerful tool** to overcome the analytic continuation problem
- Update of $K \to l\nu_l\gamma$ results with physical pions

To-do list

- Finishing all the gauge ensembles and estimating the systematics
- Computation of the decay rates
- Performing the $\varepsilon \to 0$ extrapolation



Thank you!



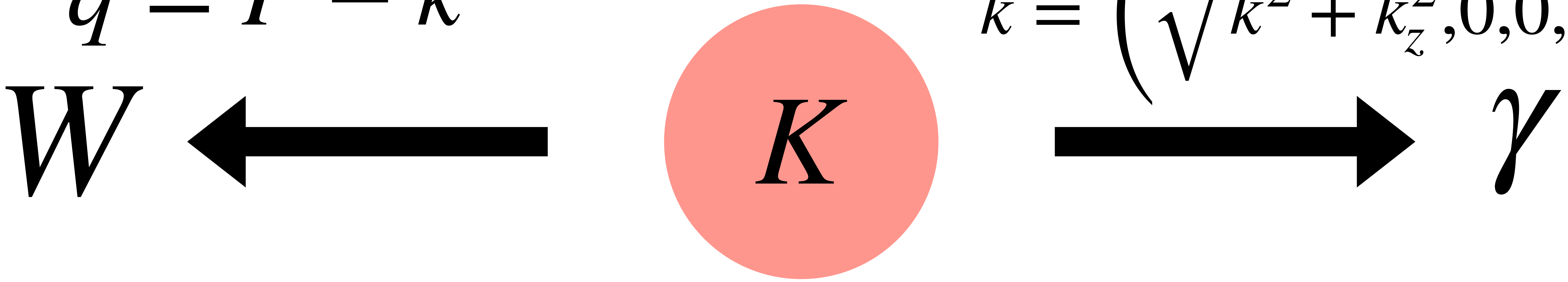
Backup

Kaon rest frame

$$P = (m_K, \overrightarrow{0})$$

$$q = P - k$$

$$k = \left(\sqrt{k^2 + k_z^2}, 0, 0, k_z\right)$$



$$x_q = \frac{\sqrt{q^2}}{m_K}$$

$$x_\gamma = \frac{2\left|k_z\right|}{m_K}$$

$$x_k = \frac{\sqrt{k^2}}{m_K}$$

Hadronic amplitude

$$H_W^{\mu\nu} = H_{\rm pt}^{\mu\nu} + H_{\rm SD}^{\mu\nu}\,,$$
$$H_{\rm pt}^{\mu\nu} = f_P\left[g^{\mu\nu} - \frac{(2p-k)^\mu(p-k)^\nu}{(p-k)^2 - m_P^2}\right]\,,$$
$$H_{\rm SD}^{\mu\nu} = \frac{H_1}{m_P}\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) + \frac{H_2}{m_P}\frac{\left[(k\cdot p - k^2)k^\mu - k^2\,(p-k)^\mu\right]}{(p-k)^2 - m_P^2}(p-k)^\nu + \frac{F_A}{m_P}\left[(k\cdot p - k^2)g^{\mu\nu} - (p-k)^\mu k^\nu\right]$$
$$-i\frac{F_V}{m_P}\epsilon^{\mu\nu\alpha\beta}k_\alpha p_\beta\,.$$

Point-like contribution

Needs the Kaon decay constant

Structure-dependent contribution

1 Vector FF: F_V

3 Axial FFs: F_A, H_1, H_2

Removing the point-like contribution

$$\tilde{H}_A^{33}(k_z,k^2) \equiv H_A^{33}(k_z,k^2) - H_A^{33}(0,0)\frac{E_\gamma\left(2m_P - E_\gamma\right)}{2m_P E_\gamma - k^2} = -H_1\frac{E_\gamma^2}{m_P} + H_2\frac{E_\gamma k_z^2}{2m_P E_\gamma - k^2} - F_A\frac{E_\gamma\left(m_P - E_\gamma\right)}{m_P}$$

$$\tilde{H}_A^{11}(k_z,k^2) \equiv H_A^{11}(k_z,k^2) - H_A^{11}(0,0) = -H_1\frac{k^2}{m_P} - F_A\frac{\left(m_P E_\gamma - k^2\right)}{m_P}\,.$$

$$H_A^{[3,0]}(k_z,k^2) \equiv H_A^{30}(k_z,k^2) - H_A^{03}(k_z,k^2)\left(\frac{m_P - E_\gamma}{2m_P - E_\gamma}\right) = -H_1\frac{E_\gamma k_z}{2m_P - E_\gamma} - H_2\frac{k_z(m_P - E_\gamma)}{2m_P - E_\gamma} + F_A\frac{k_z m_P}{2m_P - E_\gamma}\,.$$

Form factors

Vector

$$F_V = iH_V^{21}/k_z$$

$$C_{V,E}^{21}(t,\mathbf{k}) = C_{V,E}^{21}(t,\mathbf{k}) - C_{V,E}^{21}(t,\mathbf{0})$$

$$2|\mathbf{k}|/m_K = 0.1,\ 0.3$$

Axial

$$C_{A,E}^{30}(t,\mathbf{k}) = C_{A,E}^{30}(t,\mathbf{k}) - C_{A,E}^{30}(t,\mathbf{0}) \qquad C_{A,E}^{03}(t,\mathbf{k}) = C_{A,E}^{03}(t,\mathbf{k}) - C_{A,E}^{03}(t,\mathbf{0})$$

$$\begin{pmatrix} F_A \\ H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} -\frac{E_\gamma k_z}{2m_P - E_\gamma} & -\frac{k_z(m_P - E_\gamma)}{2m_P - E_\gamma} & \frac{k_z m_P}{2m_P - E_\gamma} \\ -\frac{E_\gamma^2 + k^2}{m_P} & \frac{E_\gamma k_z^2}{2E_\gamma m_P - k^2} & \frac{E_\gamma^2 - 2E_\gamma m_P + k^2}{m_P} \\ \frac{k^2 - E_\gamma^2}{m_P} & \frac{E_\gamma k_z^2}{2E_\gamma m_P - k^2} & \frac{E_\gamma^2 - k^2}{m_P} \end{pmatrix}^{-1} \begin{pmatrix} H_A^{30} - H_A^{03} \\ \tilde{H}_A^{11} + \tilde{H}_A^{33} \\ \tilde{H}_A^{33} - \tilde{H}_A^{11} - (\tilde{H}_A^{33} - \tilde{H}_A^{11})(k_z = 0) \end{pmatrix}$$

0

The HLT method

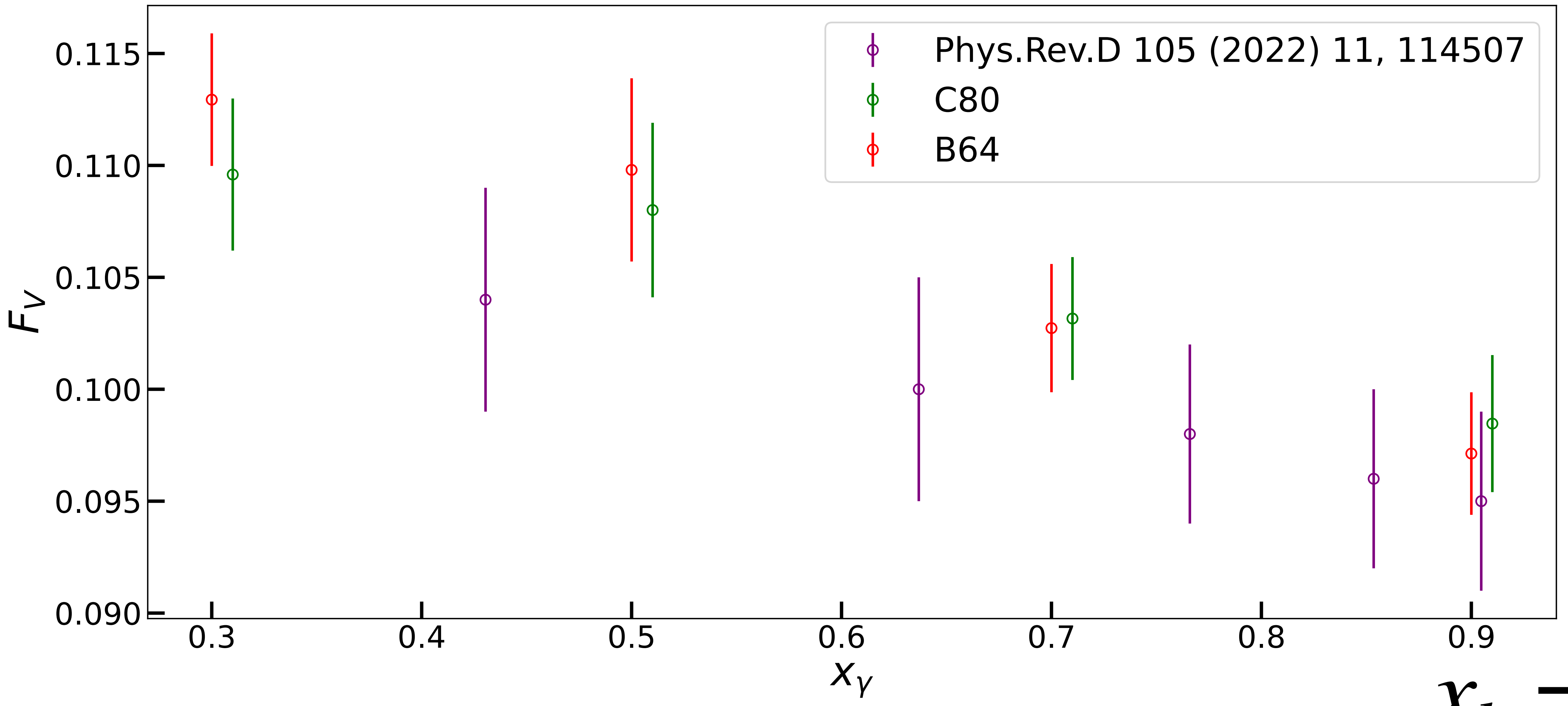
Kernel reconstruction

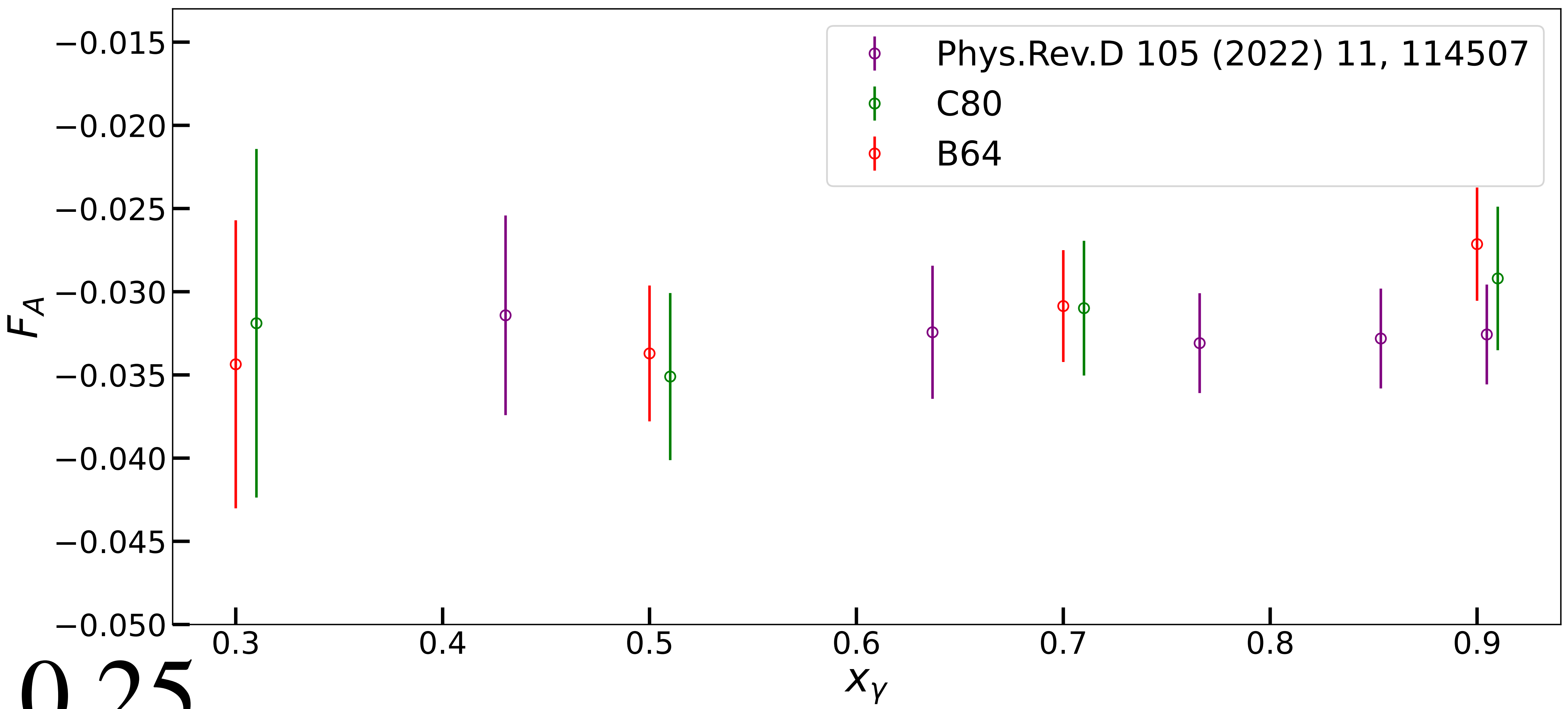
$$A[\mathbf{g}] = \int_{E^{th}}^{\infty} dE' \left| \sum_{t=t_w+1}^{T/2} g_t(E,\varepsilon) e^{-E't} - K(E'-E,\varepsilon) \right|^2$$

Statistical error

$$B[\mathbf{g}] \propto \sum_{t_1,t_2=t_w+1}^{T/2} g_{t_1}(E,\epsilon) g_{t_2}(E,\epsilon) Cov(t_1,t_2)$$

Comparison at a fixed virtuality





$x_k = 0.25$