Lattice extraction of the TMD soft function and CS kernel with the auxiliary field representation of the Wilson line

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with

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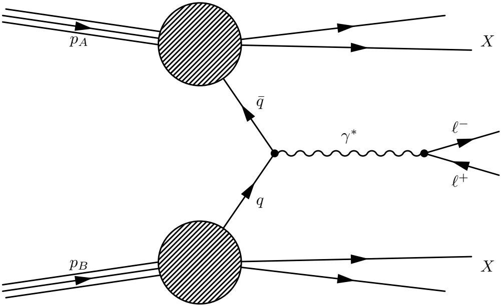
Outline

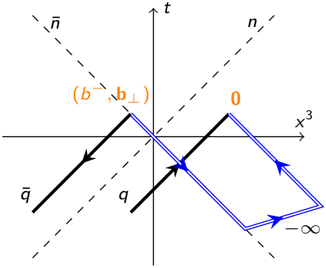
Drell-Yan process

- $\Lambda_{\rm QCD} \lesssim |\vec{q}_\perp| \ll Q$
- Collins-Soper (CS) scale ζ_a, ζ_b, where $\zeta_a \zeta_b = Q^4$
- Rapidity divergences

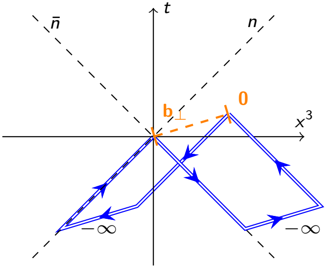


$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}\left(Q^2,\mu\right) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} f_i\left(x_a, \vec{b}_\perp, \mu, \zeta_a\right) f_j\left(x_b, \vec{b}_\perp, \mu, \zeta_b\right)$$
$$\times\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\rm QCD}^2}{Q^2}\right)\right]$$

Beam and soft function



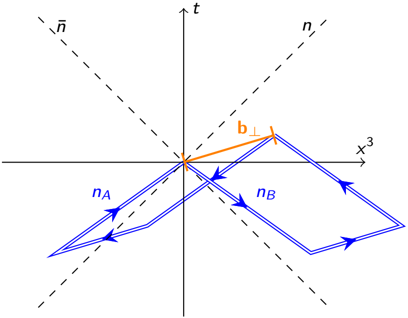
Staple gauge link for beam function



Wilson loop for soft function

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}\left(Q,\mu\right) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} B_i\left(x_a, \vec{b}_\perp, \mu, \frac{\zeta_a}{\nu^2}\right) B_j\left(x_b, \vec{b}_\perp, \mu, \frac{\zeta_b}{\nu^2}\right)$$
$$\times S_i\left(b_\perp, \mu, \nu\right)\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

Off lightcone regulator



- Spacelike Wilson lines:

$$n_A \equiv n - e^{-y_A}\,\bar{n},$$
$$n_B \equiv \bar{n} - e^{+y_B}\,n$$

- Timelike direction also possible

$$f_i(x_b, \vec{b}_\perp, \mu, \zeta_b) = \lim_{\epsilon \to 0} Z^i_{\mathrm{UV}}(\mu, \epsilon, \zeta_b) \lim_{\substack{y_A \to +\infty \\ y_B \to -\infty}} B_i\left(x_b, \vec{b}_\perp, \epsilon, y_B, xP^+\right)$$
$$\times \sqrt{\frac{S_i(b_\perp, \epsilon, y_A - y_n)}{S_i(b_\perp, \epsilon, y_A - y_B)\, S_i(b_\perp, \epsilon, y_n - y_B)}}$$

Collins-Soper kernel

- Collins-Soper (CS) kernel governs the rapidity evolution of the TMDPDF

$$K\left(b_{\perp}, \mu\right)=\frac{\mathrm{d} f_q\left(x, b_{\perp}, \mu, \zeta\right)}{\mathrm{d} \log \zeta}$$

- Can be obtained from the soft function

$$S_q\left(b_{\perp}, y_A, y_B, \mu\right) \overset{\substack{y_A \rightarrow+\infty \\ y_B \rightarrow-\infty}}{=} S_I\left(b_{\perp}, \mu\right) e^{2 K\left(b_{\perp}, \mu\right)\left(y_A-y_B\right)}$$

- By direct computation:

$$K\left(b_{\perp}, \mu\right)=\lim _{\substack{y_A \rightarrow+\infty \\ y_B \rightarrow-\infty}} \frac{1}{2} \frac{\partial}{\partial y_n} \log \left(\frac{S_q\left(b_{\perp}, y_n, y_B, \mu\right)}{S_q\left(b_{\perp}, y_A, y_n, \mu\right)}\right)$$

Lattice extraction of TMDPDFs

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = C_q\left(x\tilde{P}^z, \mu\right)\exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_\perp)\log\frac{\tilde{\zeta}}{\zeta}\right] f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)$$
$$\times\left\{1 + \mathcal{O}\left(\frac{1}{(x\tilde{P}^z b_\perp)^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(x\tilde{P}^z)^2}\right)\right\}$$

[Ebert, *et. al.*, 2019], [Ebert, *et. al*, 2022]

- C_q is a perturbatively calculable matching kernel
- $\tilde{\zeta} = x^2 m_h^2 e^{2(y_{\tilde{P}} + y_B - y_n)}$

quasi-TMDPDF

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = \tilde{f}_q^{\mathrm{naive}}\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right)\sqrt{\frac{\tilde{S}_q^{\mathrm{naive}}(b_\perp, \mu)}{S_q(b_\perp, \mu, 2y_n, 2y_B)}}$$

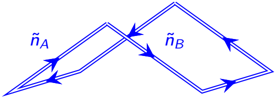
- $\tilde{f}_q^{\mathrm{naive}}$ and $\tilde{S}_q^{\mathrm{naive}}$ are lattice calculable objects
- S_q is the Collins soft function

Outline

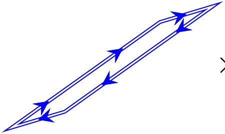
Our approach

$$\tilde{n}_A = \left(in_A^0, 0, 0, n_A^3\right), \quad \tilde{n}_B = \left(in_B^0, 0, 0, n_B^3\right)$$

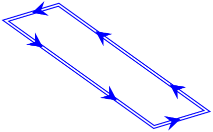
$S_{\text{ratio}}\left(\vec{b}_\perp, \tilde{n}_A, \tilde{n}_B, a, \tau\right) =$



divided by $\sqrt{\quad}$ of:



$\times$



- Ratio gives correct dependence on $b_\perp$
- Removes linear divergences associated with finite length Wilson lines
- Ensures power counting in $b_\perp^4/\tau^4$
- Approaches lattice time τ independent result for large τ

Connection to Minkowski space

- At large lattice time, τ:

$$S_{\mathrm{ratio}}\left(b_\perp, \tilde{n}_A, \tilde{n}_B, a, \tau\right) = S_{\mathrm{lat}}\left(b_\perp, r_a, r_b, a\right) + \mathcal{O}\left(\frac{b_\perp^4}{\tau^4}\right)$$

- Direct mapping to rapidity variables in Collins' scheme:

$$r_a = \frac{n_A^3}{n_A^0} = \frac{1+e^{-2y_A}}{1-e^{-2y_A}}, \quad r_b = \frac{n_B^3}{n_B^0} = \frac{1+e^{2y_B}}{1-e^{2y_B}}$$

- Construct matching between lattice and continuum renormalization schemes

$$S\left(b_\perp, y_A, y_B, \mu\right) = C(r_a, r_b, \mu, a) \times S_{\mathrm{lat}}\left(b_\perp, r_a, r_b, a\right)$$

- Obtain CS kernel from:

$$S\left(b_\perp, y_A, y_B, \mu\right) = S_I\left(b_\perp, \mu\right) e^{2K(b_\perp,\mu)(y_A-y_B)}$$

Outline

Auxiliary field definition of the Wilson line

Write Wilson line in terms of one dimensional 'fermions' that live along the path:

$$
\begin{aligned}
&P \exp\left\{-ig\int_{s_i}^{s_f} \mathrm{d}s n^\mu A_\mu(y(s))\right\} \\
&= Z_\psi^{-1}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\psi\bar{\psi}\exp\left\{ig\int_{s_i}^{s_f}\mathrm{d}s\bar{\psi}i\partial_s\psi - \bar{\psi}n\cdot A\psi\right\}
\end{aligned}
$$

[Gervais, Nevau 1980], [Aref'eva 1980]

Auxiliary field propagator:

$$
in\cdot DH_n(y) = \delta(y) \xrightarrow{\text{Euclidean space}} -i\tilde{n}\cdot D_E H_{\tilde{n}}(y) = \delta(y), \qquad \tilde{n} = (in_0, \vec{n})
$$

Meaningful solution only obtained with a UV cutoff [Aglietti, *et. al.* 1992], [Agglietti, 1994]

Auxiliary field propagator in Euclidean space

$$K(\tau) = \left(1 - \frac{H_0|_\tau}{2n}\right)^n U_4^\dagger(\tau - 1)\left(1 - \frac{H_0|_{\tau-1}}{2n}\right)^n$$

$$H_0\psi(x) = -iv \cdot \Delta^\pm\psi(x) = -\sum_\mu \frac{iv_\mu}{2}\left[U_\mu(x)\,\psi(x+\hat{\mu}) - U_{-\mu}(x)\,\psi(x-\hat{\mu})\right]$$

$$G(\mathbf{x}, \tau, \mathbf{x}', \tau') = K(\tau)G(\mathbf{x}, \tau - 1, \mathbf{x}', \tau')$$

For $n \to \infty$ and $a \to 0$, this will produced the same continuum equation as before.

We get correction terms of $\mathcal{O}\left(a^2, \frac{a^2}{n}\right)$
[Horgan, *et. al.*, 2009]

Outline

Details of lattice calculation

- Using $N_f = 2 + 1$ flavor PACS-CS configurations
- non-perturbatively $\mathcal{O}(a)$-improved Wilson quark action and Iwasaki gauge action
- $32^3 \times 64$ lattice with $a = 0.0907(13)$ fm
- 400 configurations
- thyp2 smearing
- Up to 32 sources per configuration
- Using GPT/GRID

[PACS-CS '09, '11]

Numerator and denominator factors

- At large τ, we expect

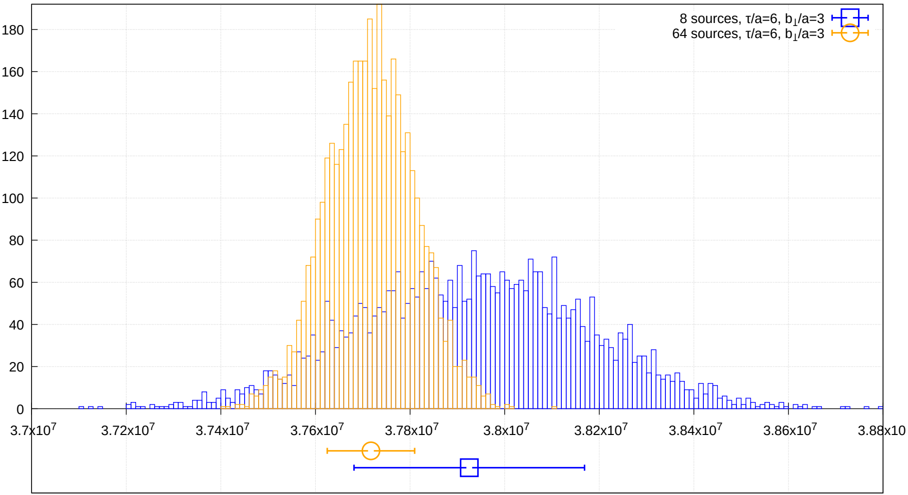
$$S_{\text{num}} \overset{\tau\to\infty}{\sim} e^{2\tau(r_a+r_b)/a} \big/ \tau^4$$

$$S_A \overset{\tau\to\infty}{\sim} e^{4(\tau r_a - iz)/a} \big/ \tau^4$$

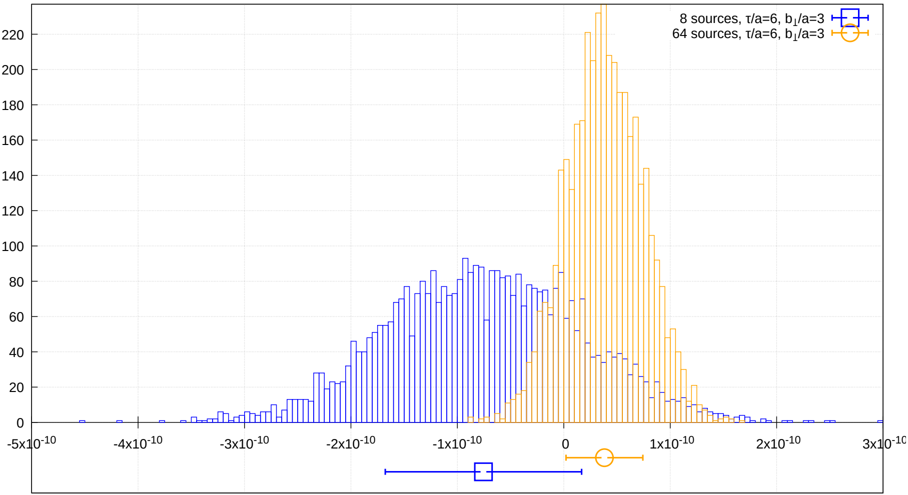
$$S_B \overset{\tau\to\infty}{\sim} e^{4(\tau r_b + iz)/a} \big/ \tau^4$$

- We expect to see real and imaginary contributions to the denominator factors
- Combined denominator factor $\sqrt{S_A S_B}$, should be purely real

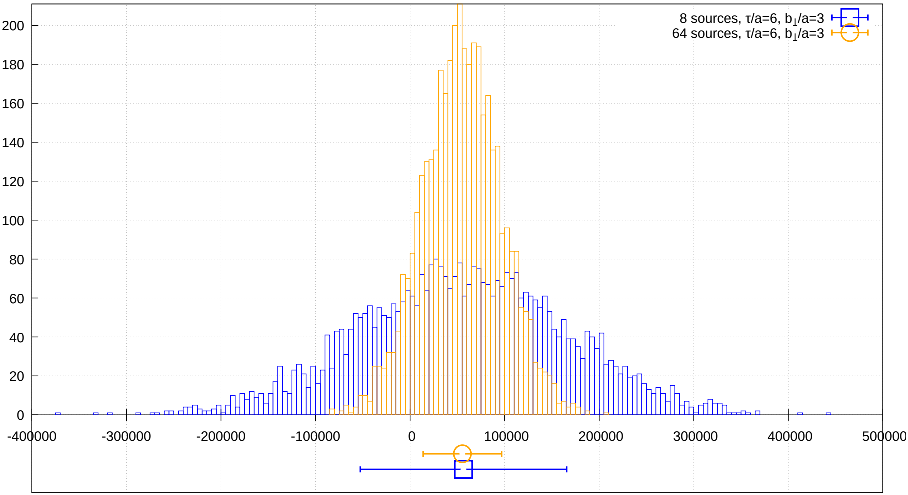
Numerator, real part



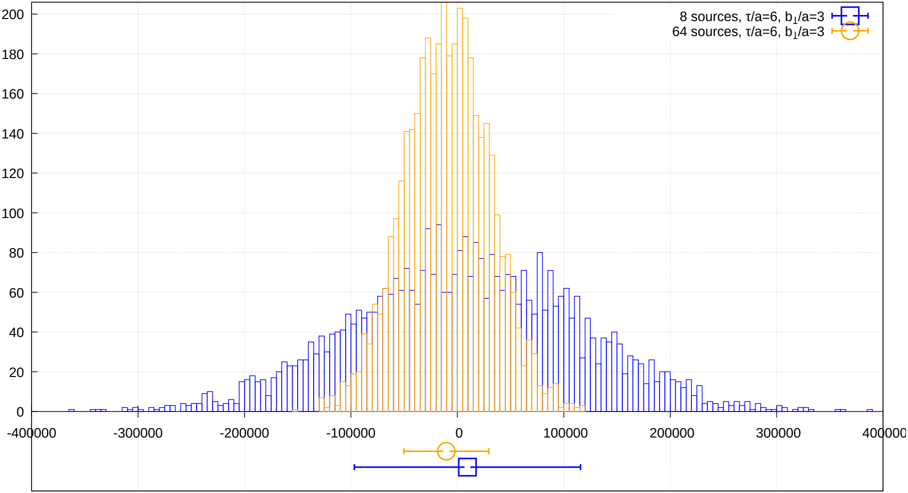
Numerator, imaginary part



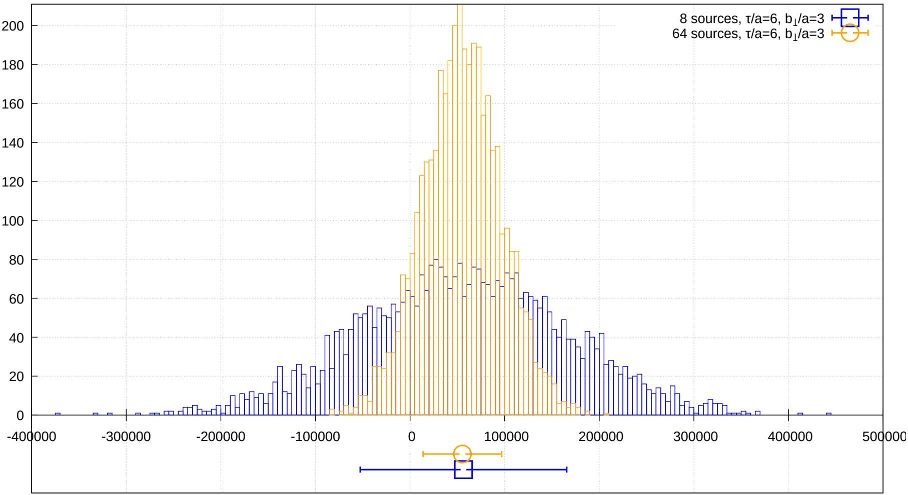
Denominator A, real part



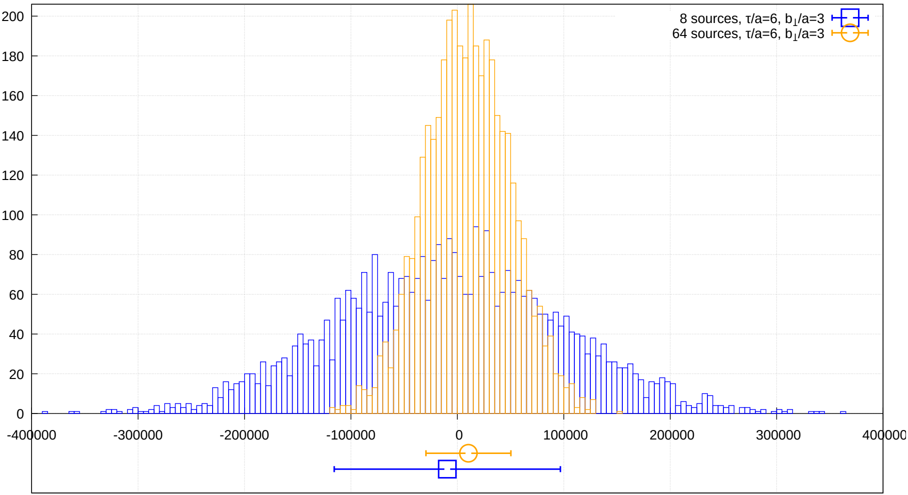
Denominator A, imaginary part



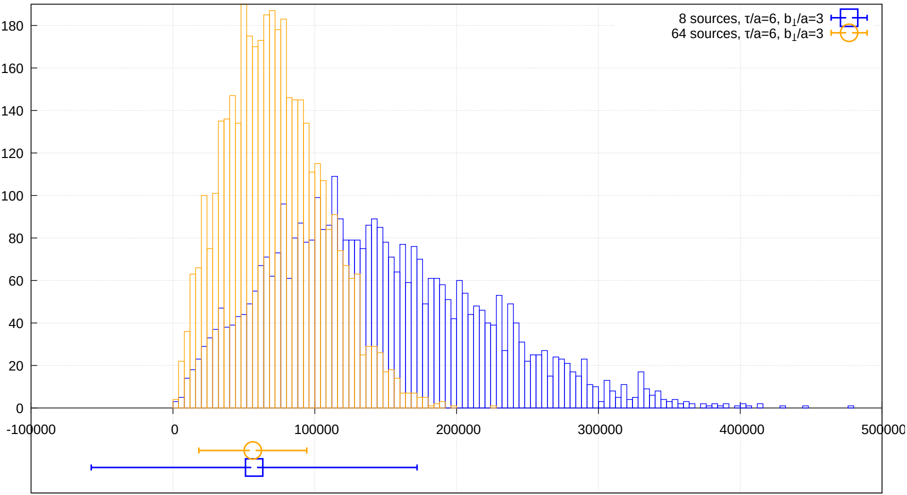
Denominator B, real part



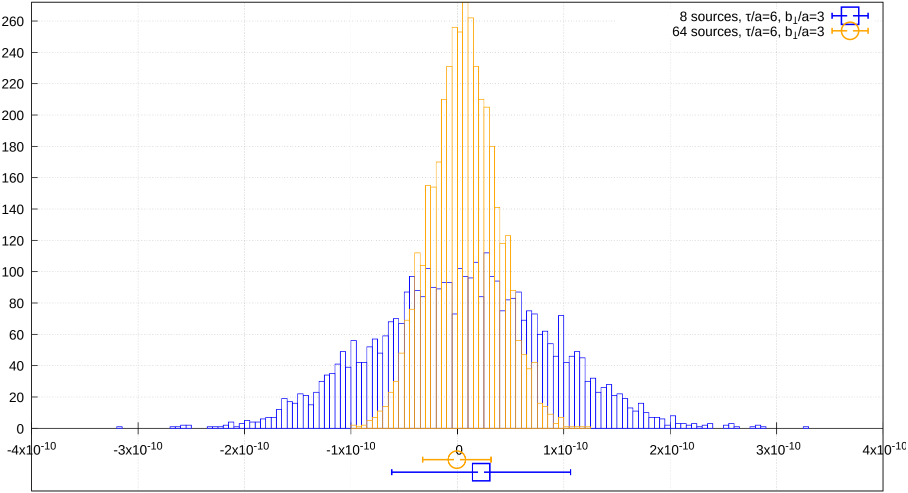
Denominator B, imaginary part



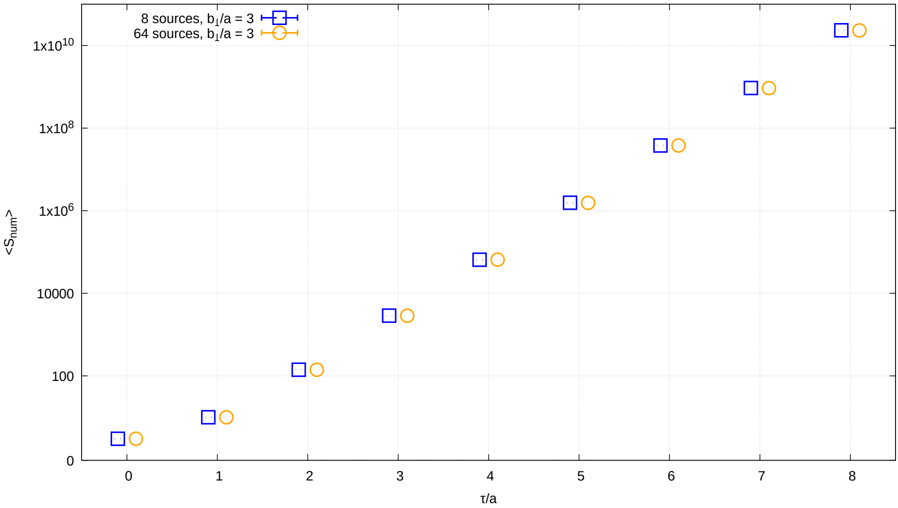
Full denominator, real part



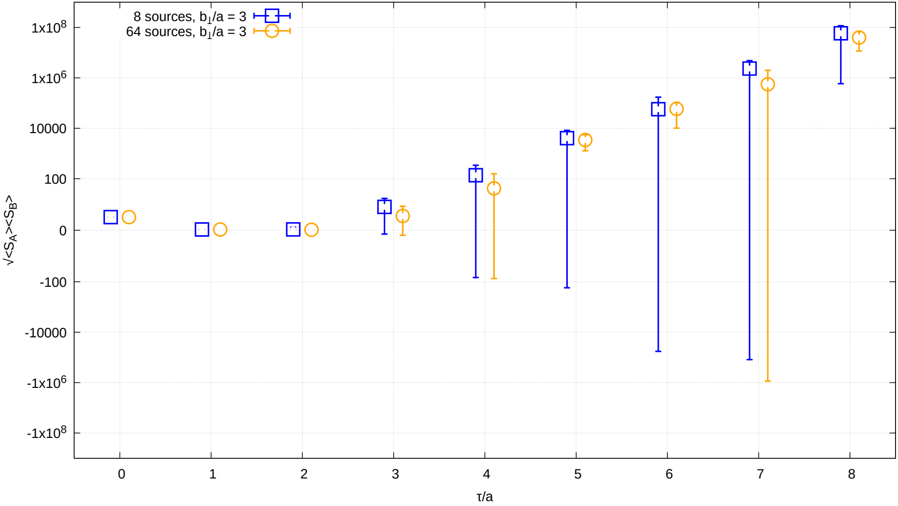
Full denominator, imaginary part



Numerator, τ dependence



Full denominator, τ dependence



Conclusion and outlook

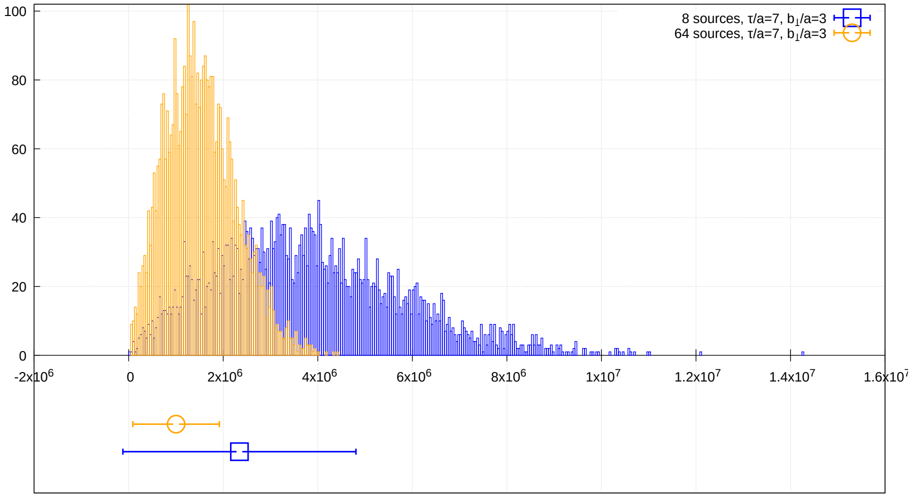
- Important to have multiple methods for computing the soft function
- Euclidean space calculation of soft function has a direct mapping to Minkowski space result
- Preliminary numerical investigation shows a decrease in error with increased statistics
- Expect to see a signal with ~ 1000 sources

Thank you!

Group members

Anthony Francis (NYCU), Issaku Kanamori (R-CCS, RIKEN), C.-J. David Lin (NYCU), WM (NYCU), Yong Zhao (Argonne)

Full denominator, real part



Coordinate space

Perform integration in coordinate space:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2} = \int_0^\infty \mathrm{d}u \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-uk^2} e^{-(b+s\tilde{n}_A-t\tilde{n}_B)^2/4u}$$
$$= \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}} \frac{1}{((b+s\tilde{n}_A - t\tilde{n}_B)^2/4)^{d/2-1}}$$

'u' integral only valid for

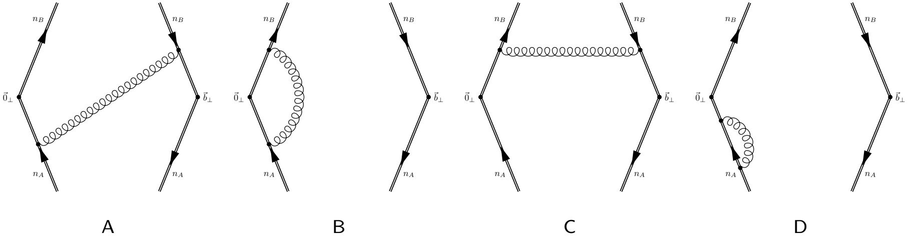
$$(s\tilde{n}_A - t\tilde{n}_B)^2 = s^2((n_A^3)^2 - (n_A^0)^2) + t^2((n_B^3)^2 - (n_B^0)^2) + st(n_A^3 n_B^3 + n_A^0 n_B^0) > 0$$

Euclidean space integral only finite when:

$$|n_A^3| > |n_A^0|, \quad |n_B^3| > |n_B^0|, \quad n_A^3 n_B^3 + n_A^0 n_B^0 > 0$$

$$\rightarrow \quad |r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$$

Soft function in Euclidean space at one loop



Calculation in coordinate space at one loop:

$$S^{(1)}\left(b_\perp, \epsilon, r_a, r_b\right)$$
$$= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 + \log\left|\frac{(r_a-1)(r_b-1)}{(r_a+1)(r_b+1)}\right| \frac{r_a r_b + 1}{r_a + r_b}\right\}$$

$$|r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$$

Finite L Wilson lines

For $L \to \infty$ and $r_a, r_b \to 1$:

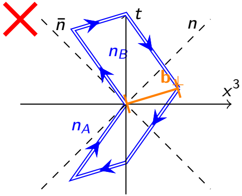
$$S\left(b_\perp, a, r_a, r_b, L\right) = 1 + \frac{\alpha_s C_F}{2\pi}\left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)}\log\left(\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)}\right)\right)\log\left(\frac{b_\perp^2}{a^2}\right)$$
$$+ \frac{\alpha_s C_F}{2\pi}\Bigg\{ -4\log\left(\frac{b_\perp^2}{a^2}\right) + 2\frac{\pi b_\perp}{a} + 2\frac{\pi\left(|n_A| + |n_B|\right)L}{b_\perp}$$
$$-2\frac{\pi\left(|n_A| + |n_B|\right)L}{a}$$
$$+ 2\frac{b_\perp^2}{L^2}\left(C_1 - \frac{1}{3}\right)\Bigg\} + \mathcal{O}\left(\frac{b_\perp^4}{L^4}, \alpha_s^2\right)$$

$$C_1 = 1 - \frac{1}{2}\frac{1}{b_0^2(r_b^2 - 1)} - \frac{1}{2}\frac{1}{a_0^2(r_a^2 - 1)} \implies \frac{b_\perp^2}{L^2} \ll r_{a,b} - 1, \qquad r_{a,b} = \frac{n_{A,B}^3}{n_{A,B}^0}$$

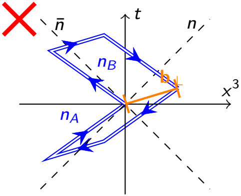
- Ratio removes problem terms

$$S_{\mathrm{ratio}}\left(\vec{b}_\perp, r_a, r_b, a, L\right) = \frac{\tilde{S}\left(\vec{b}_\perp, r_a, r_b, a, L\right)}{\sqrt{\tilde{S}\left(\vec{b}_\perp, r_a, -r_a, a, L\right)\tilde{S}\left(\vec{b}_\perp, -r_b, r_b, a, L\right)}}$$

What can we reconstruct in Minkowski space?



$|r_a| < 1, \quad |r_b| < 1, \quad n_A^0 n_B^0 (r_a r_b + 1) < 0$



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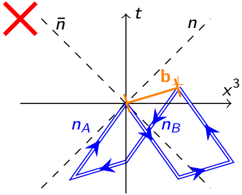

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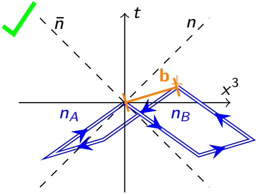

$|r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$

but it's okay: $S_{\mathrm{DY}} = S_{\mathrm{SIDIS}}$