Higher moments of the pion parton distribution functions using gradient flow

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Introduction

- Precise knowledge of parton distribution functions (PDFs)

See Andrea Shindler's talk for more details

- Operator product expansion → Mellin moments of PDFs

  e.g., for unpolarized: $\langle h(p)|\mathcal{O}^{\mu_1\cdots\mu_n}|h(p)\rangle = 2\langle x^{n-1}\rangle p^{\{\mu_1}p^{\mu_2}\dots p^{\mu_n\}}$

$$\mathcal{O}^{\mu_1\cdots\mu_n} = i^{n-1}\bar{\psi}\gamma^{\{\mu_1}\overleftrightarrow{D}^{\mu_2}\dots\overleftrightarrow{D}^{\mu_n\}}\psi - \text{Traces}\,, \quad \langle x^{n-1}\rangle = \int_0^1 dx\, x^{n-1}\,(q(x) + (-1)^n\bar{q}(x))$$

- Euclidean lattice: Lorentz symmetry reduces to hypercubic symmetry
  20 irreducible representations → run into power divergent mixing with lower dimensional operators

- Investigate recent proposal [Shindler, 2311.18704] to circumvent this using gradient flow

Power divergent mixing

Göckeler et al, hep-lat/9602029

- Consider flavor non-singlet combinations that do not mix with gluon $\mathcal{O}_g^{\mu_1\cdots\mu_n} = F_{.\alpha}^{\{\mu_1}\overleftrightarrow{D}^{\mu_2}\ldots\overleftrightarrow{D}^{\mu_{n-1}}F^{\mu_n\}\alpha} - \text{Traces}$

- $n = 2:\ \tau_1^{(3)} \rightarrow \mathcal{O}_{44} - \text{Traces}, \ldots$
  $\tau_3^{(6)} \rightarrow \mathcal{O}_{\{14\}}, \ldots$ → requires boost in 1 direction

- $n = 3:\ \tau_1^{(4)} \rightarrow \mathcal{O}_{444} - \text{Traces}, \ldots$ → mixes with $\frac{1}{a}\bar{\psi}\gamma^\mu\psi$
  $\tau_2^{(4)} \rightarrow \mathcal{O}_{\{412\}}, \ldots$ → requires boost in 2 directions
  $\ldots$

- $n = 4:\ \tau_1^{(1)} \rightarrow \sum_\mu \mathcal{O}_{4444} - \text{Traces}, \ldots$ → mixes with $\frac{1}{\alpha^2}\bar{\psi}\psi, \ldots$
  $\tau_2^{(1)} \rightarrow \mathcal{O}_{\{1234\}}, \ldots$ → requires boost in 3 directions
  $\tau_1^{(3)} \rightarrow \mathcal{O}_{4444} - \text{Traces}, \ldots$ → mixes with $n = 2, \ldots$
  $\tau_3^{(6)} \rightarrow \mathcal{O}_{\{4111\}} - \text{Traces}, \ldots$ → mixes with $n = 2, \ldots$
  $\ldots$

Power divergent mixing

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  …

- $n > 4$: no irrep safe from power-divergent mixing

Flowed moments

Shindler, 2311.18704

matching coeffs computed perturbatively in small flow time expansion

$$\frac{\langle x^{n-1}\rangle(\mu)}{\langle x^{m-1}\rangle(\mu)} = \frac{c_m(t,\mu)}{c_n(t,\mu)} \frac{\langle h(p)|\mathcal{O}_{\mu_1\ldots\mu_n}(t)|h(p)\rangle}{\langle h(p)|\mathcal{O}_{\mu_1\ldots\mu_m}(t)|h(p)\rangle}$$

t: flow time

$$\mathcal{O}_{\mu_1\ldots\mu_n}(t) = \bar{\psi}(t)\gamma_{\mu_1}\overleftrightarrow{D}_{\{\mu_2}(t)\ldots\overleftrightarrow{D}_{\mu_n\}}(t)\psi(t) - \text{Traces}$$

Obtain $\langle h(p)|\mathcal{O}_{\mu_1\ldots\mu_n}(t)|h(p)\rangle$ on the lattice

Choose $\mu_1 = \mu_2 = \cdots = \mu_n = 4$

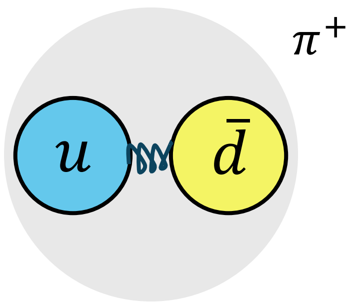
$$n = 2: \quad \bar{\psi}\gamma_4\overleftrightarrow{D}_4\psi - \frac{1}{3}\sum_{i=1}^{3}\bar{\psi}\gamma_i\overleftrightarrow{D}_i\psi$$

$$n = 3: \quad \bar{\psi}\gamma_4\overleftrightarrow{D}_4\overleftrightarrow{D}_4\psi - \frac{1}{12}\sum_{i=1}^{3}(\bar{\psi}\gamma_4\overleftrightarrow{D}_i\overleftrightarrow{D}_i\psi + \bar{\psi}\gamma_i\overleftrightarrow{D}_4\overleftrightarrow{D}_i\psi + \bar{\psi}\gamma_i\overleftrightarrow{D}_i\overleftrightarrow{D}_4\psi)$$

…

n	# of unique ops
2	4
3	10
4	40
5	136
6	544
7	2080

Ratios from lattice QCD



- Consider purely connected light-quark contribution u^+
  → if $m_u = m_d = m_s$, corresponds to non-singlet $\bar{u}u + \bar{d}d - 2\bar{s}s$

t: flow time
t_s: sink time
τ: operator insertion time

- 3-point function projected to zero momentum

$$\mathrm{C}^{\mathrm{3pt}}_{\mu_1\mu_2\ldots\mu_n}(t_s,\tau,t) = \sum_{\boldsymbol{x},\boldsymbol{y}} \langle \bar{u}(\boldsymbol{x},t_s)\gamma_5 d(\boldsymbol{x},t_s)\bar{u}(\boldsymbol{y},\tau,t)\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}(\boldsymbol{y},\tau,t)u\,(\boldsymbol{y},\tau,t)\bar{d}(0)\gamma_5 u(0)\rangle$$

$$\sim \boldsymbol{S}^{\mathrm{seq}}(t_s,\tau,t)\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}(\tau,t)\ldots\overleftrightarrow{D}_{\mu_n}(\tau,t)\boldsymbol{S}(\tau,t)$$

sequential propagator

$$\frac{\mathrm{C}^{\mathrm{3pt}}_{\mu_1\mu_2\ldots\mu_n}(t_s,\tau,t)}{\mathrm{C}^{\mathrm{3pt}}_{\mu_1\mu_2\ldots\mu_m}(t_s,\tau,t)} \propto \frac{\langle h(p)|\mathcal{O}_{\mu_1\ldots\mu_n}(t)|h(p)\rangle}{\langle h(p)|\mathcal{O}_{\mu_1\ldots\mu_m}(t)|h(p)\rangle}$$

for $\tau \gg 0, t_s - \tau \gg 0, \ldots$

Compute $\boldsymbol{S}^{\text{seq}}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\boldsymbol{S}$

$$\overleftrightarrow{D}_\mu = (\overrightarrow{D}_\mu - \overleftarrow{D}_\mu)/2$$

$$S^{\text{seq}}(x)\overrightarrow{D}_\mu(x)S(x) = \frac{1}{2}(S^{\text{seq}}(x)U_\mu(x)S(x+\hat{\mu}) - S^{\text{seq}}(x)U_\mu^\dagger(x-\hat{\mu})S(x-\hat{\mu}))$$

$$-S^{\text{seq}}(x)\overleftarrow{D}_\mu(x)S(x) = \frac{1}{2}(S^{\text{seq}}(x-\hat{\mu})U_\mu(x-\hat{\mu})S(x) - S^{\text{seq}}(x+\hat{\mu})U_\mu^\dagger(x)S(x))$$

naively compute 4^{n-1} terms

Compute $\boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\boldsymbol{S}$

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$x \to x - \hat{\mu}$ $\quad$ $x \to x + \hat{\mu}$

$$-S^{\mathrm{seq}}(x)\overleftarrow{D}_\mu(x)S(x) = \frac{1}{2}\left(S^{\mathrm{seq}}(x-\hat{\mu})U_\mu(x-\hat{\mu})S(x) - S^{\mathrm{seq}}(x+\hat{\mu})U_\mu^\dagger(x)S(x)\right)$$

naively compute 4^{n-1} terms → compute only 2^{n-1} terms (those in $\boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overrightarrow{D}_{\mu_2}\ldots\overrightarrow{D}_{\mu_n}\boldsymbol{S}$)

Compute $\boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\boldsymbol{S}$: pipeline

$$F_\mu S(x) = U_\mu(x)S(x+\hat{\mu})$$
$$B_\mu S(x) = -U_\mu^\dagger(x-\hat{\mu})S(x-\hat{\mu})$$

$$S(x) \xrightarrow{\vec{D}_{\mu_n}} \begin{cases} F_{\mu_n}S(x) \xrightarrow{\vec{D}_{\mu_{n-1}}} \begin{cases} F_{\mu_{n-1}}F_{\mu_n}S(x) \xrightarrow{\vec{D}_{\mu_{n-2}}} \cdots \\ B_{\mu_{n-1}}F_{\mu_n}S(x) \xrightarrow{\vec{D}_{\mu_{n-2}}} \cdots \end{cases} \\ B_{\mu_n}S(x) \xrightarrow{\vec{D}_{\mu_{n-1}}} \begin{cases} F_{\mu_{n-1}}B_{\mu_n}S(x) \xrightarrow{\vec{D}_{\mu_{n-2}}} \cdots \\ B_{\mu_{n-1}}B_{\mu_n}S(x) \xrightarrow{\vec{D}_{\mu_{n-2}}} \cdots \end{cases} \end{cases} \Bigg\} \; \tilde{S}^i_{\mu_2\ldots\mu_n}(x),\ i = 1,\ldots,2^n$$

$$C^i_{\mu_1\ldots\mu_n}(\tau) = \sum_{\boldsymbol{x}} \mathrm{Tr}[S^{\mathrm{seq}}(\boldsymbol{x},\tau)\gamma_{\mu_1}\tilde{S}^i_{\mu_2\ldots\mu_n}(\boldsymbol{x},\tau)]$$

t and t_s dependence implicit
(t_s fixed by seq source)

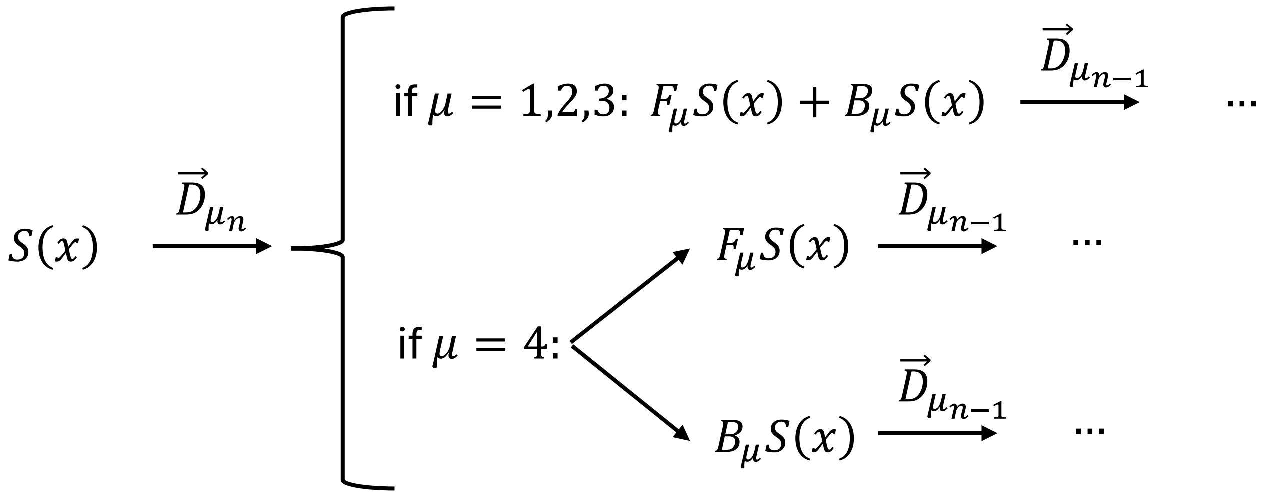
$$C^i_{\mu_1\ldots\mu_n}(\tau) \to \{C_{\mu_1\ldots\mu_n}(\tau;1{,}1,\ldots,1), C_{\mu_1\ldots\mu_n}(\tau;1{,}1,\ldots,0), \ldots, C_{\mu_1\ldots\mu_n}(\tau;0{,}0,\ldots,0)\}$$

$$\to \boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\boldsymbol{S} = \sum_{\ell_i\epsilon\{0,1\}}\frac{1}{2^{n+n_t-1}}\sum_{k=0}^{n_t}\binom{n_t}{k} C_{\mu_1\ldots\mu_n}(t-\Delta t(\mu_2,\ldots,\mu_n;\ell_2,\ell_3,\ldots,\ell_n)+k;\ \ell_2,\ell_3,\ldots,\ell_n)$$

$$\Delta t(\mu_2,\ldots,\mu_n;\ell_2,\ell_3,\ldots,\ell_n) = \sum_{i \text{ with } \mu_i=4} \ell_i$$

Compute $\boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\boldsymbol{S}$: pipeline

$\boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overrightarrow{D}_{\mu_2}\ldots\overrightarrow{D}_i\boldsymbol{S} = \boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overrightarrow{D}_{\mu_2}\ldots\overleftarrow{D}_i\boldsymbol{S}$, for $i = \{1,2,3\}$
→ further reduces total #



Compute $\boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\boldsymbol{S}$: pipeline

n	# of unique ops
2	4
3	10
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$\boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\vec{D}_{\mu_2}\ldots\vec{D}_i\boldsymbol{S} = \boldsymbol{S}^{\mathrm{seq}}\gamma_{\mu_1}\vec{D}_{\mu_2}\ldots\overleftarrow{D}_i\boldsymbol{S}$, for $i = \{1,2,3\}$

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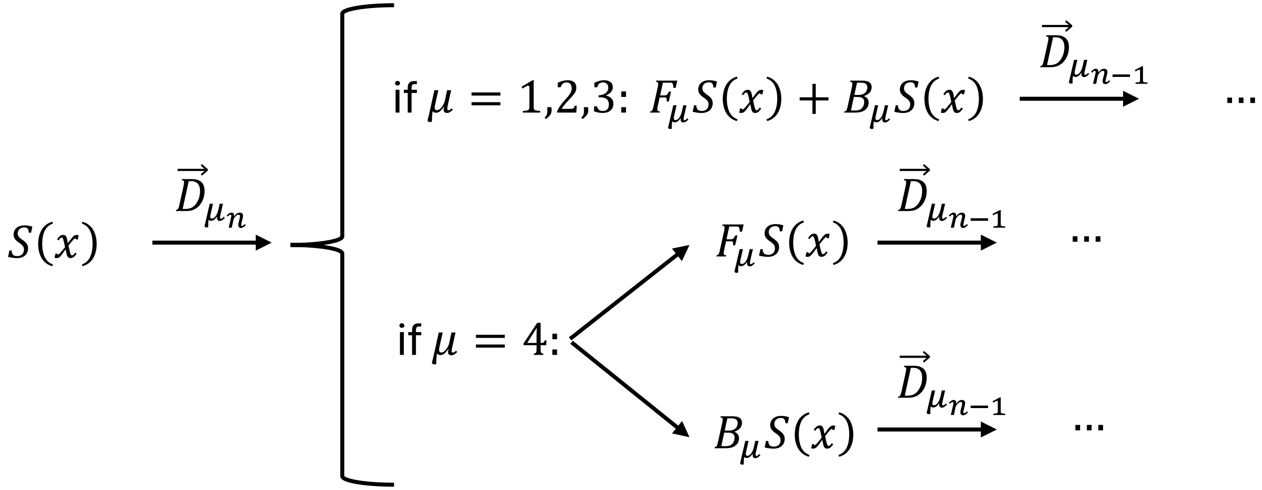


→ If ordered in a tree-search manner, the shifted propagators $\tilde{S}^{i}_{\mu_2\ldots\mu_n}$ can be reused instead of computed for each unique op starting from S

e.g. for $n = \{2,3,4\}$:

μ_1: γ-index

$\mu_1 1 \rightarrow \mu_1 11 \rightarrow \mu_1 111 \rightarrow \mu_1 112 \rightarrow \mu_1 113 \rightarrow \mu_1 114$

$\rightarrow \mu_1 12 \rightarrow \mu_1 121 \rightarrow \mu_1 122 \rightarrow \mu_1 123 \rightarrow \mu_1 124$

$\rightarrow \mu_1 13 \rightarrow \mu_1 131 \rightarrow \mu_1 132 \rightarrow \mu_1 133 \rightarrow \mu_1 134$

$\rightarrow \mu_1 14 \rightarrow \mu_1 141 \rightarrow \mu_1 142 \rightarrow \mu_1 143 \rightarrow \mu_1 144$

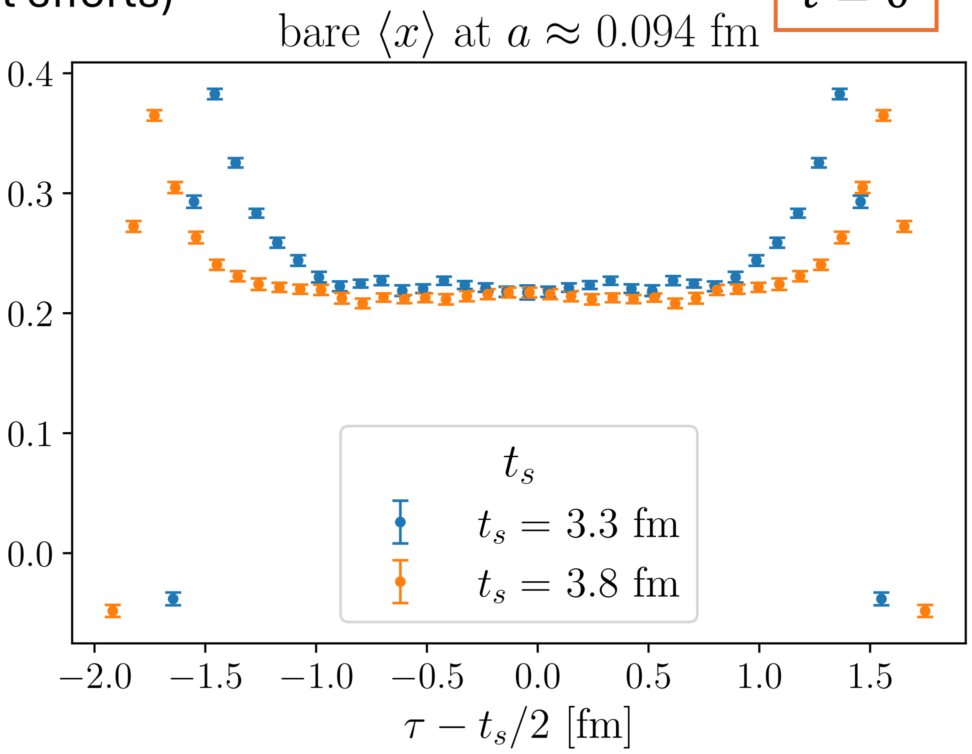
$\mu_1 2 \rightarrow \mu_1 21 \rightarrow \mu_1 121 \rightarrow \mu_1 122 \rightarrow \mu_1 123 \rightarrow \mu_1 124$

$\rightarrow \cdots$

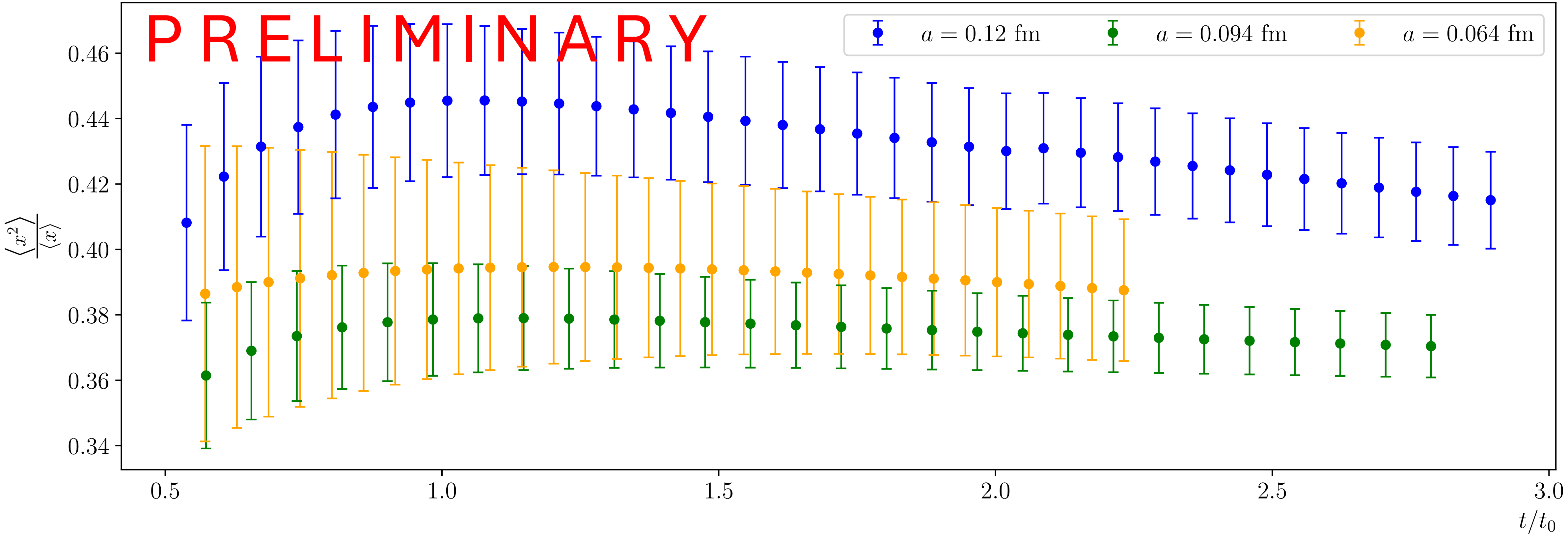
Calculation details

- Three SWF ensembles generated by OpenLat [https://openlat1.gitlab.io]
  (see Giovanni Pederiva's poster for more information on the OpenLat efforts)
- $m_\pi \approx 410$ MeV, $m_u = m_d = m_s$
  $a \approx 0.12$ fm, 0.094 fm, 0.064 fm
- One stochastic point-source per configuration
  $a \approx 0.12$ fm: $N_{cfg} = 119$, $t_s/a \in \{25,30,35,40\}$
  $a \approx 0.094$ fm: $N_{cfg} = 210$, $t_s/a \in \{35,40\}$
  $a \approx 0.064$ fm: $N_{cfg} = 22$, $t_s/a \in \{30,40\}$
- Flow integration step 0.01
  Measurements equally spaced in t up to $\sqrt{8t} \approx 0.6$ fm, $t/t_0 \approx 2.5$

$t = 0$

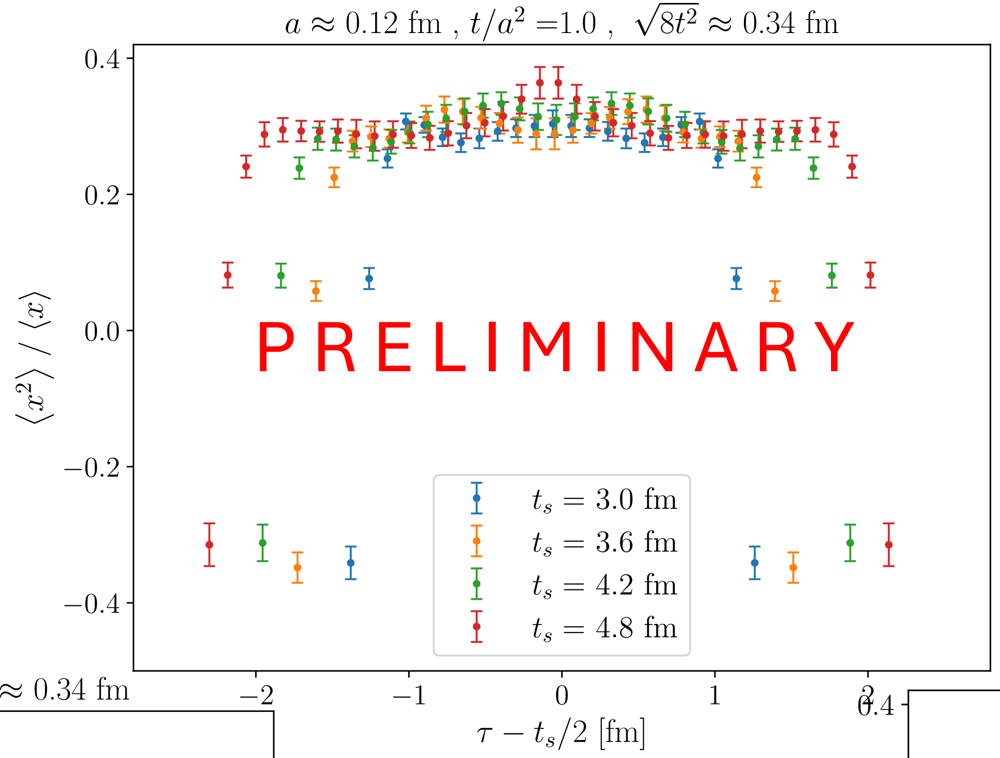


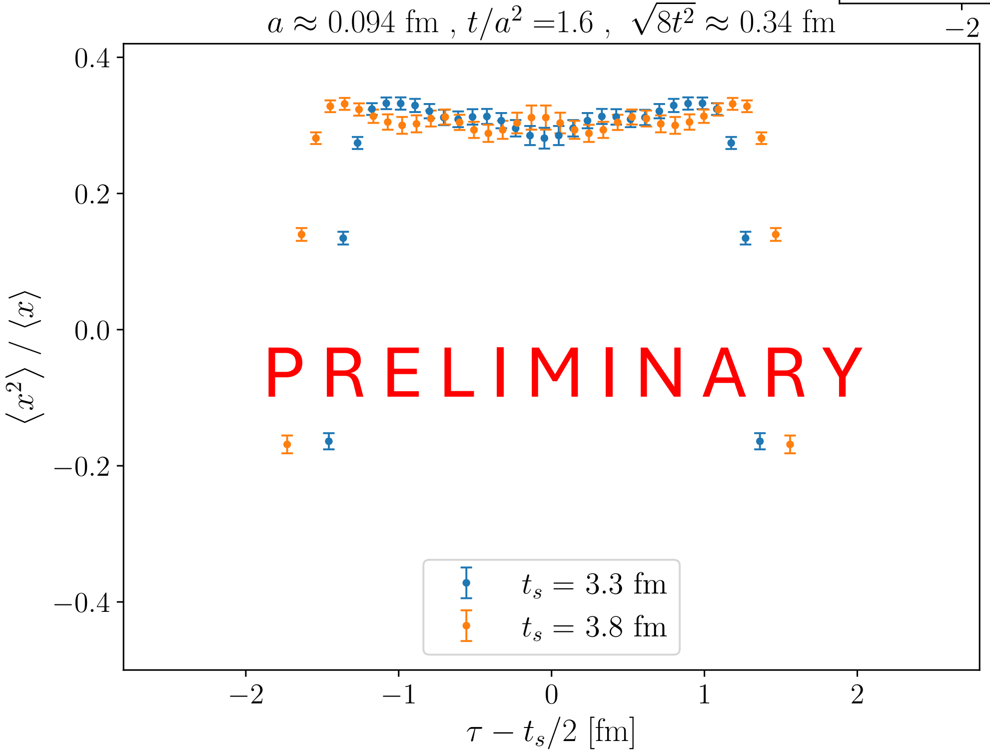
$$\frac{\langle x^2\rangle_{\overline{\mathrm{MS}}}}{\langle x\rangle_{\overline{\mathrm{MS}}}}(\mu = 2\ \mathrm{GeV})\ \text{at}\ t_f = 40a, \tau = 20a$$

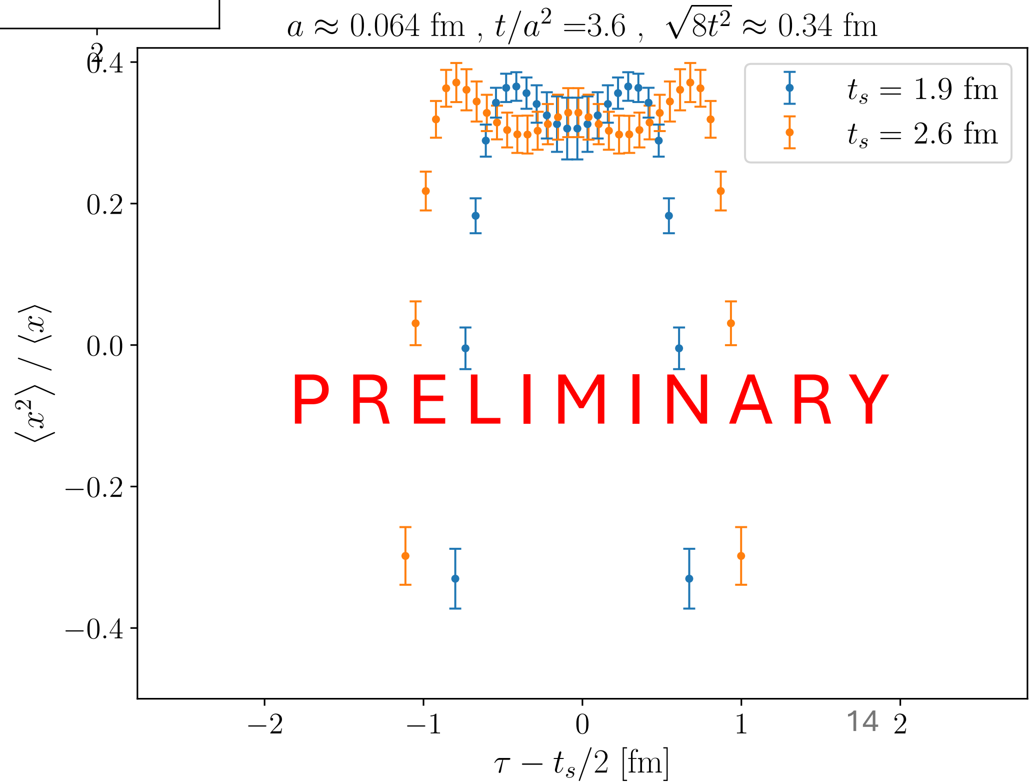


$\langle x^2 \rangle_{\overline{\text{MS}}} / \langle x \rangle_{\overline{\text{MS}}} (\mu = 2 \text{ GeV})$

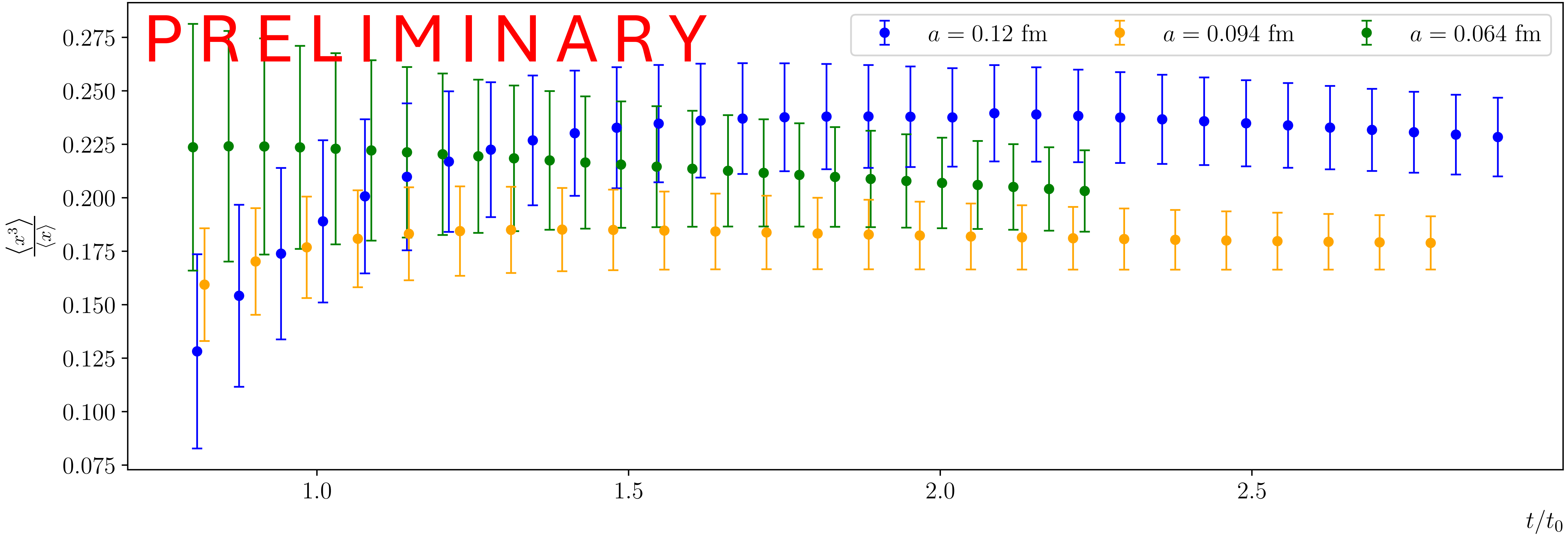
at fixed $t \approx 0.34$ fm





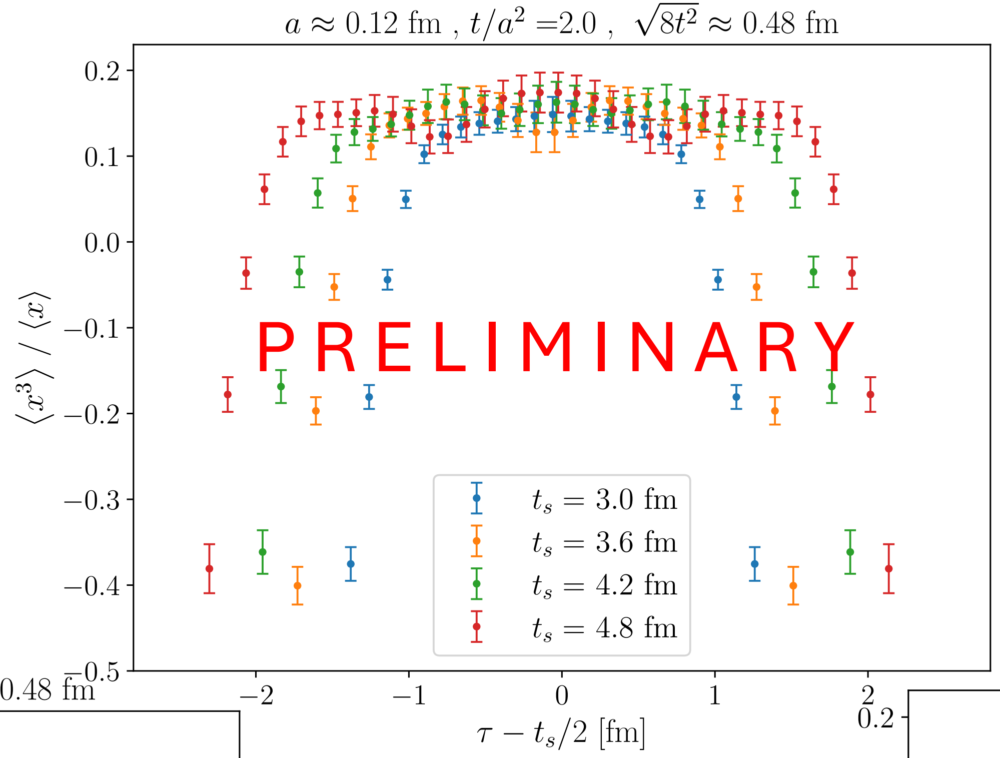


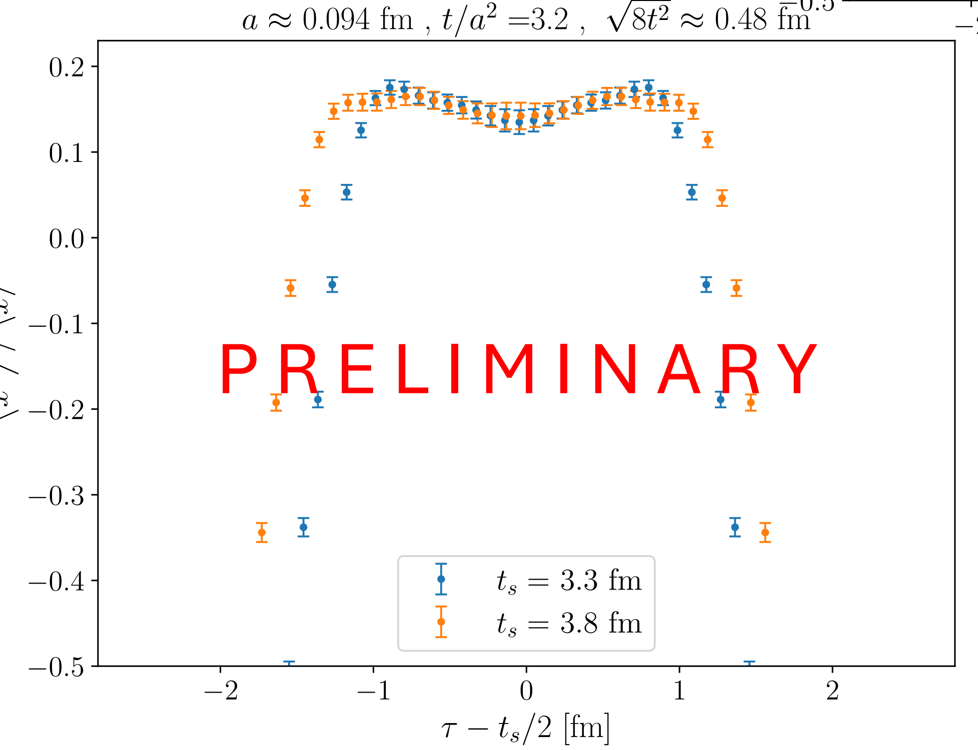
$$\frac{\langle x^3\rangle_{\overline{\mathrm{MS}}}}{\langle x\rangle_{\overline{\mathrm{MS}}}}(\mu = 2\text{ GeV})\text{ at } t_f = 40a, \tau = 20a$$

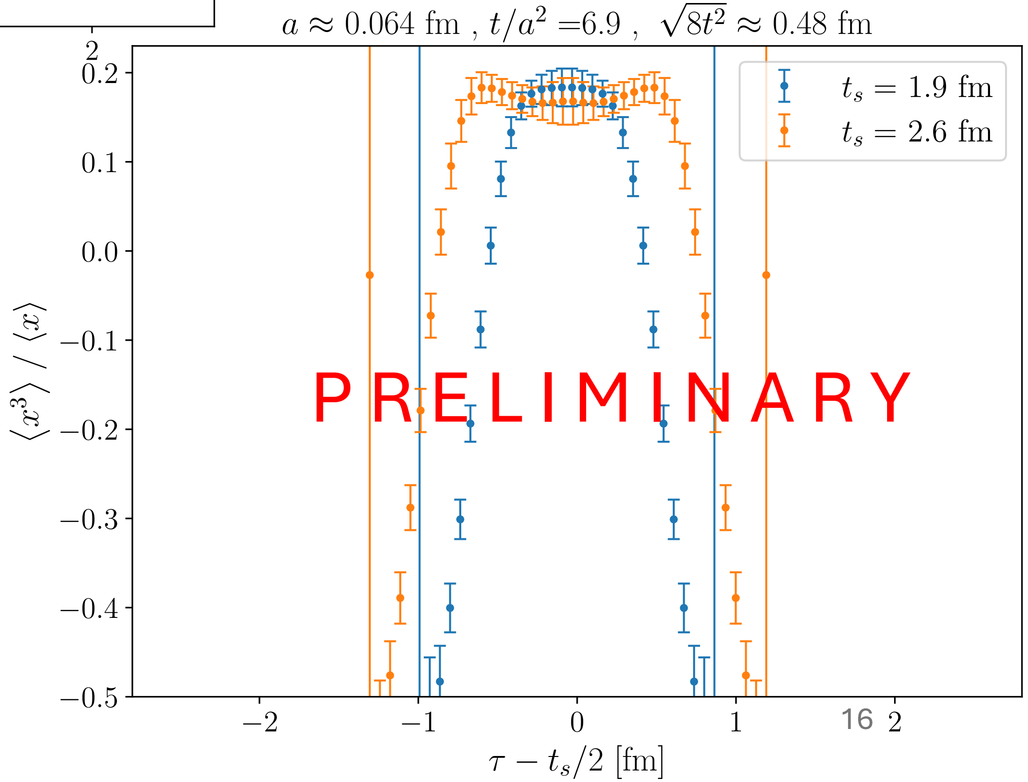


$\langle x^3 \rangle_{\overline{\mathrm{MS}}} / \langle x \rangle_{\overline{\mathrm{MS}}} (\mu = 2\ \mathrm{GeV})$

at fixed $t \approx 0.48$ fm







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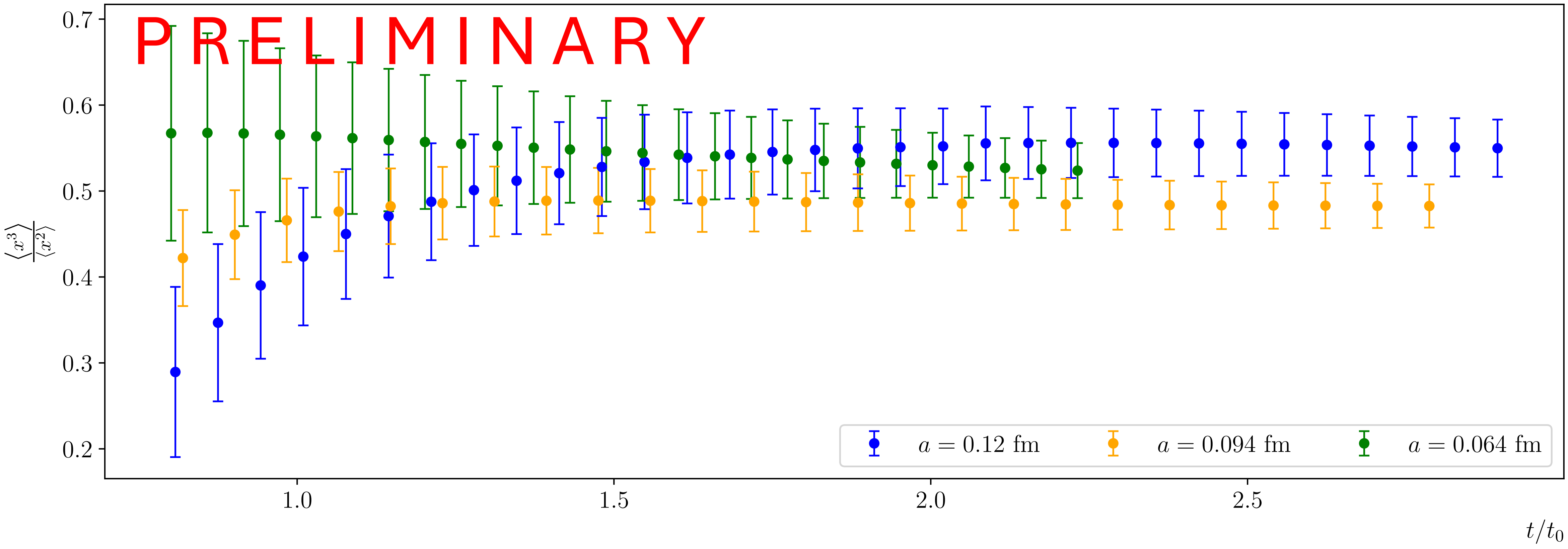

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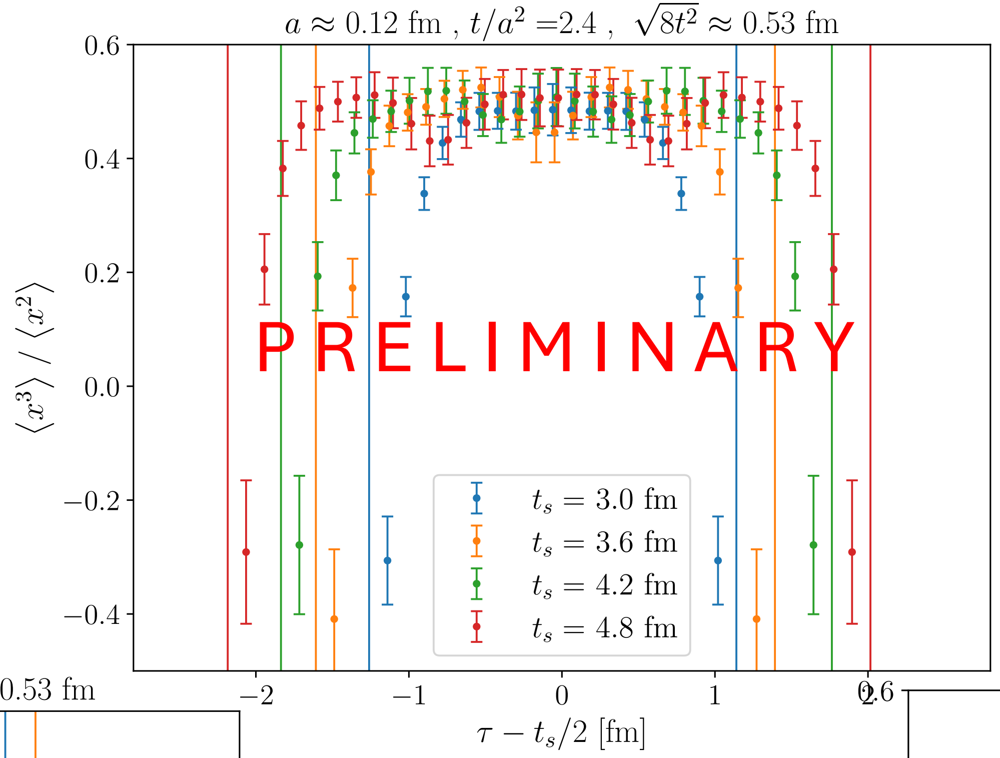
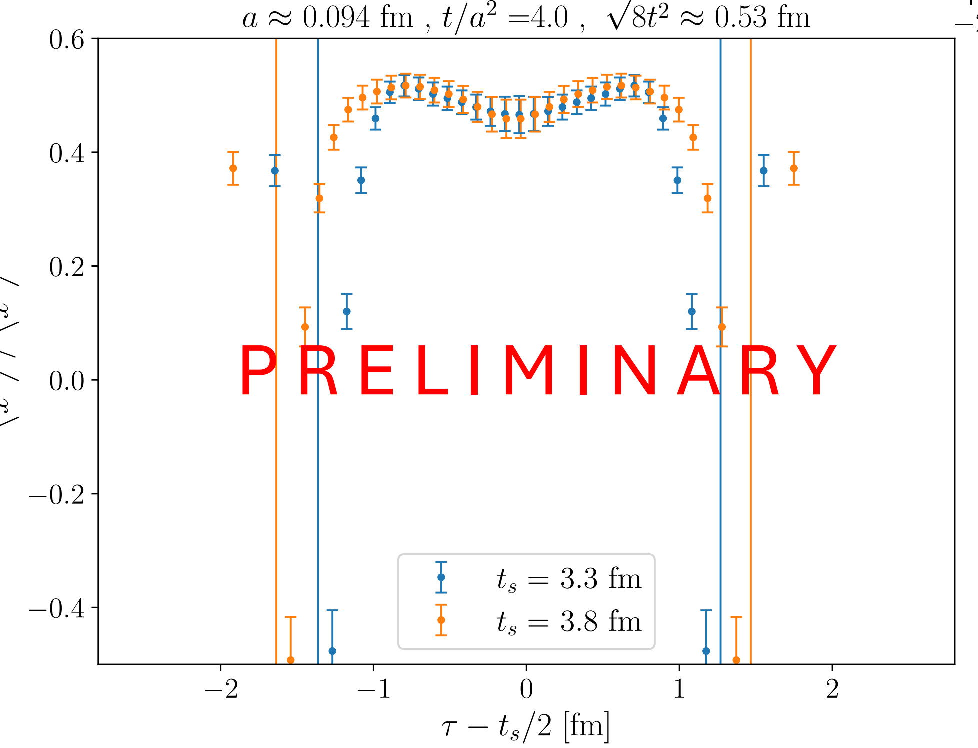
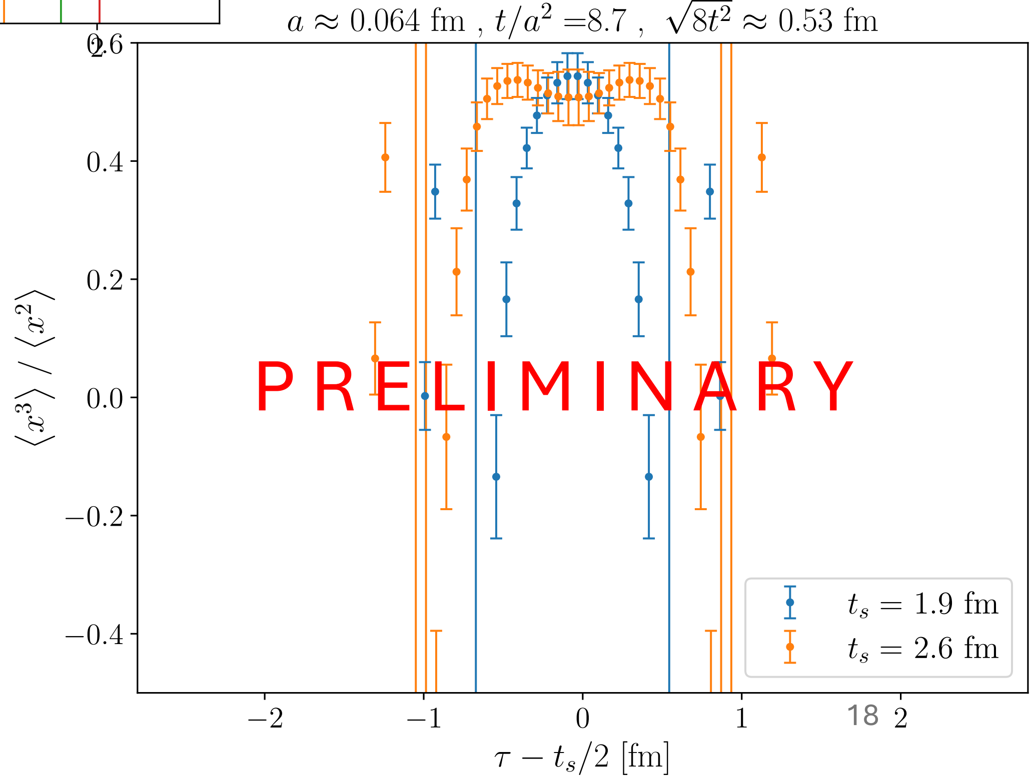


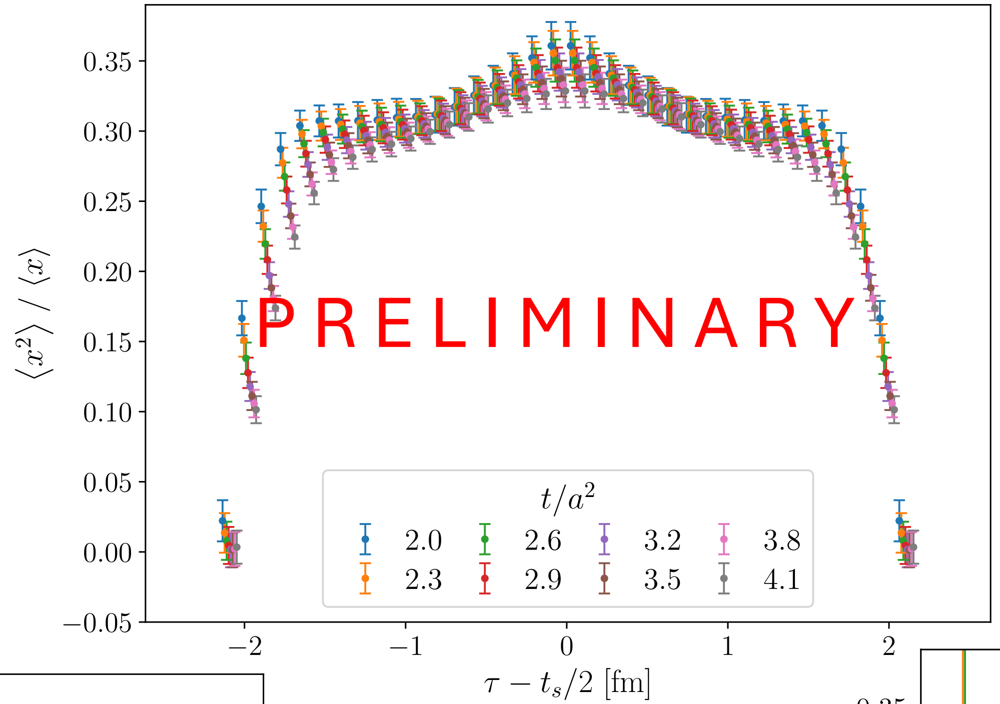


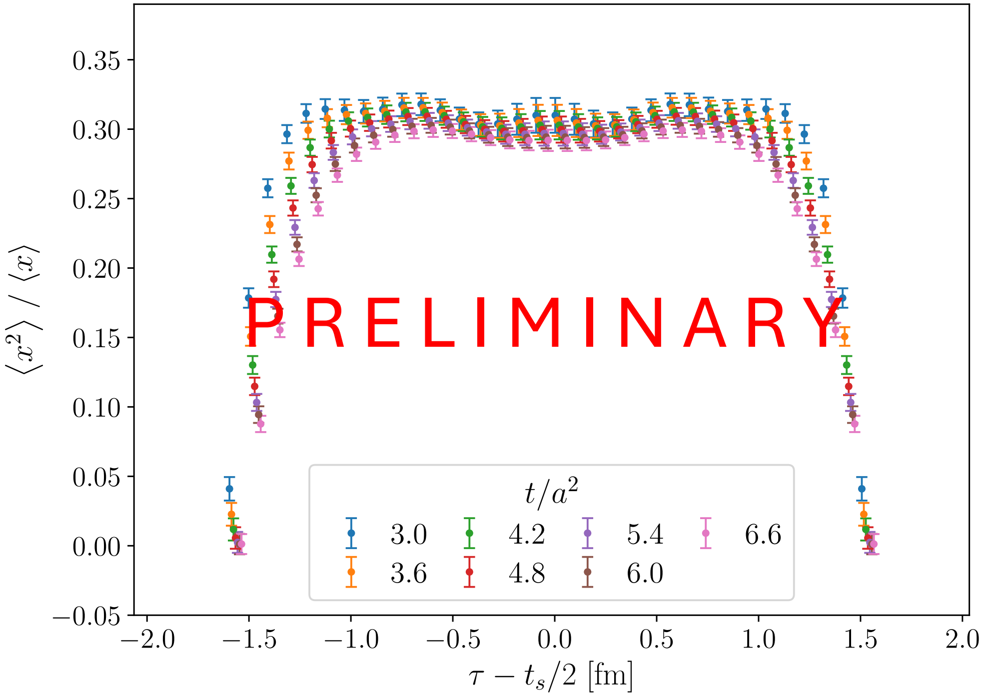


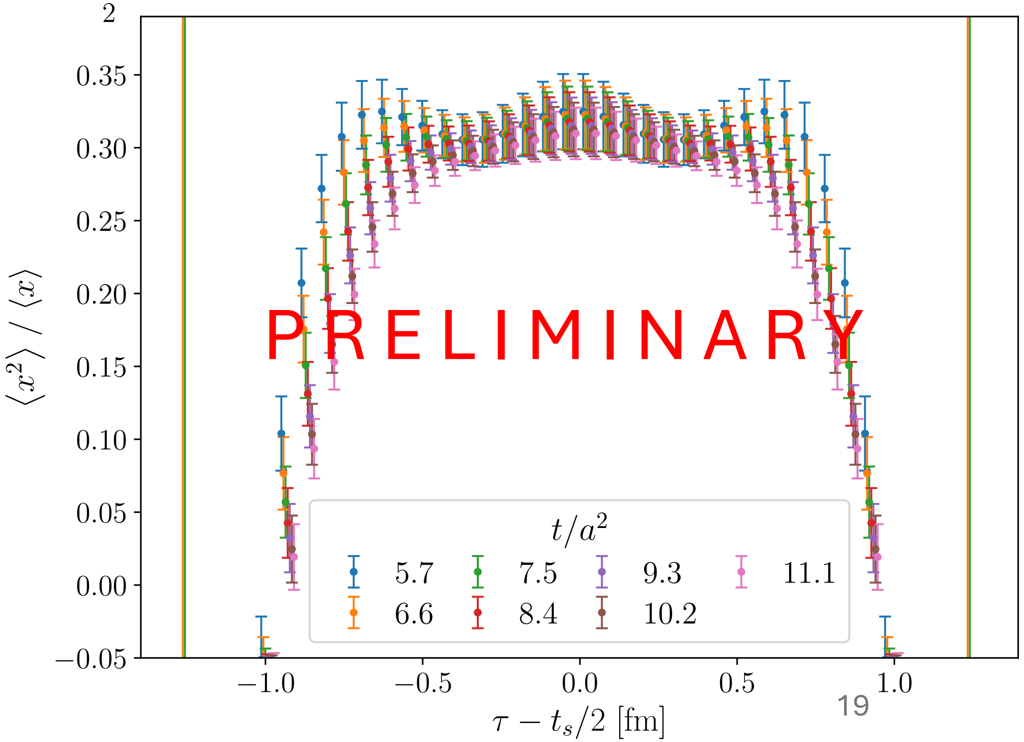
flow-dependent plateaus

$\langle x^2\rangle_{\overline{\rm MS}}/\langle x\rangle_{\overline{\rm MS}}(\mu = 2\ \text{GeV})$

at fixed $t_s \approx 40a$



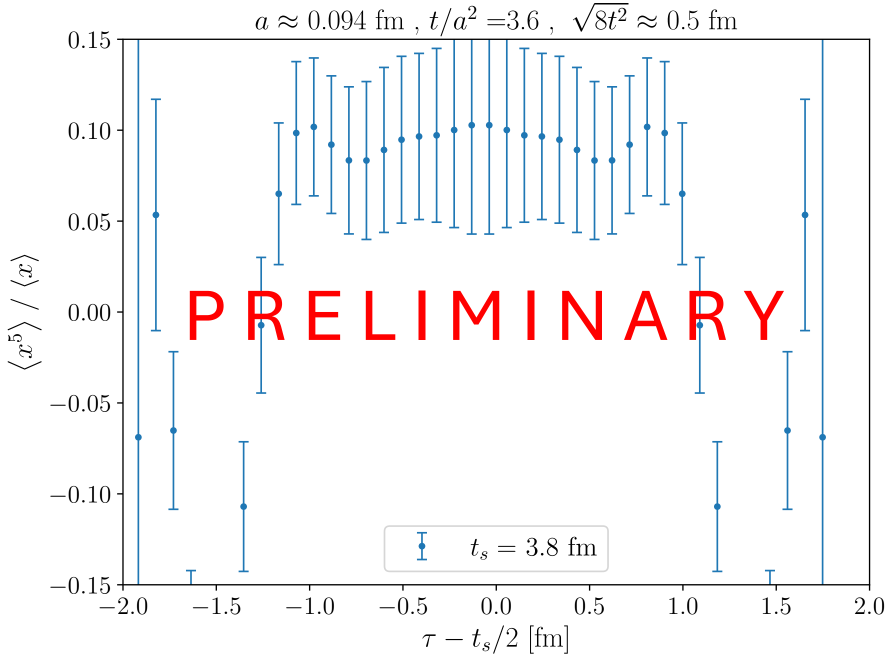




higher moments?

$\langle x^5 \rangle_{\overline{\mathrm{MS}}} / \langle x \rangle_{\overline{\mathrm{MS}}} (\mu = 2\ \mathrm{GeV})$

at $a \approx 0.094$ fm, vary t



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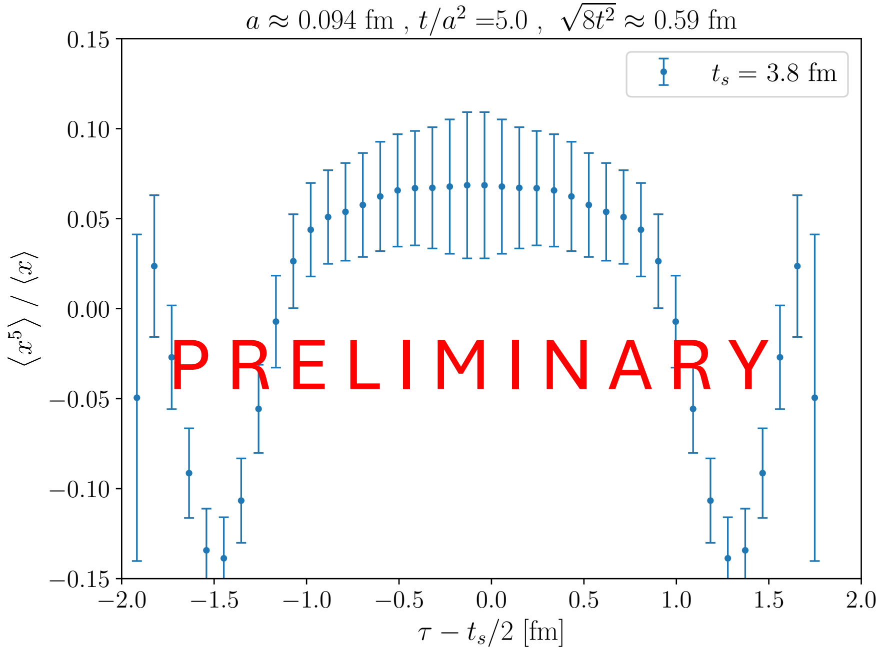

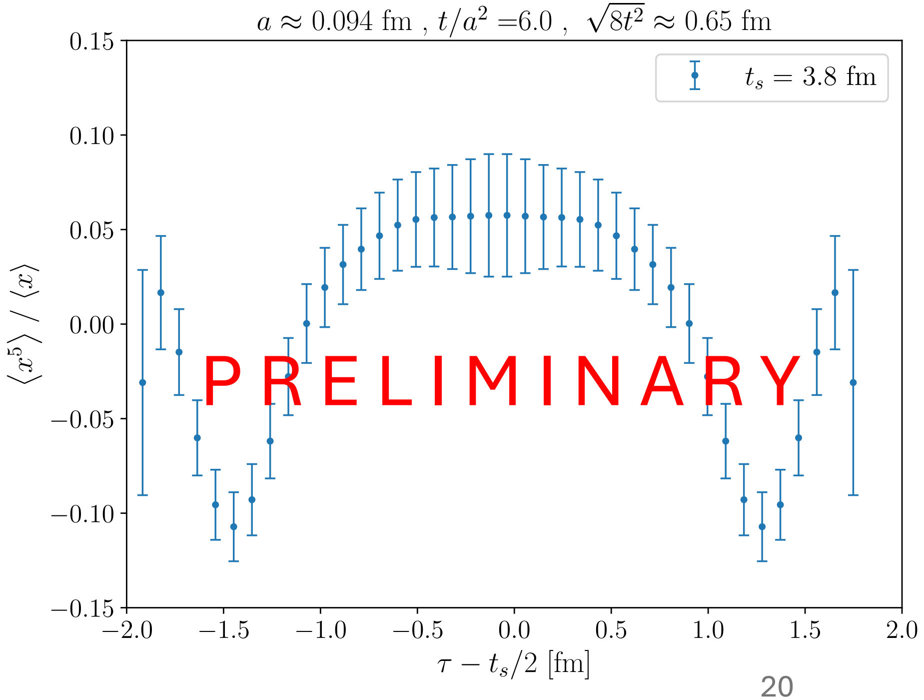

higher moments?

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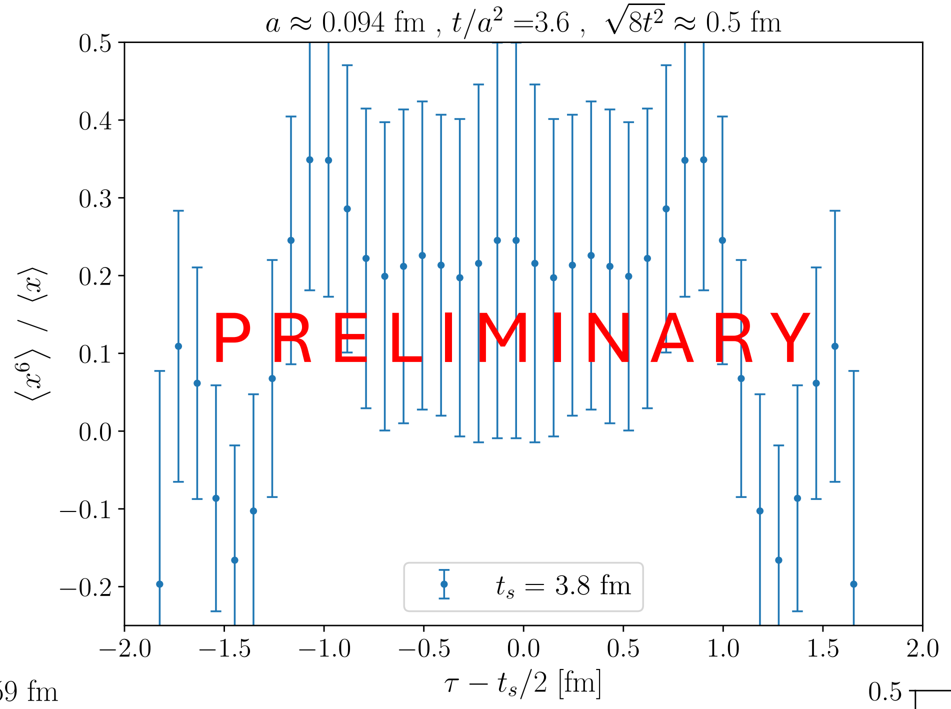
at $a \approx 0.094$ fm, vary t



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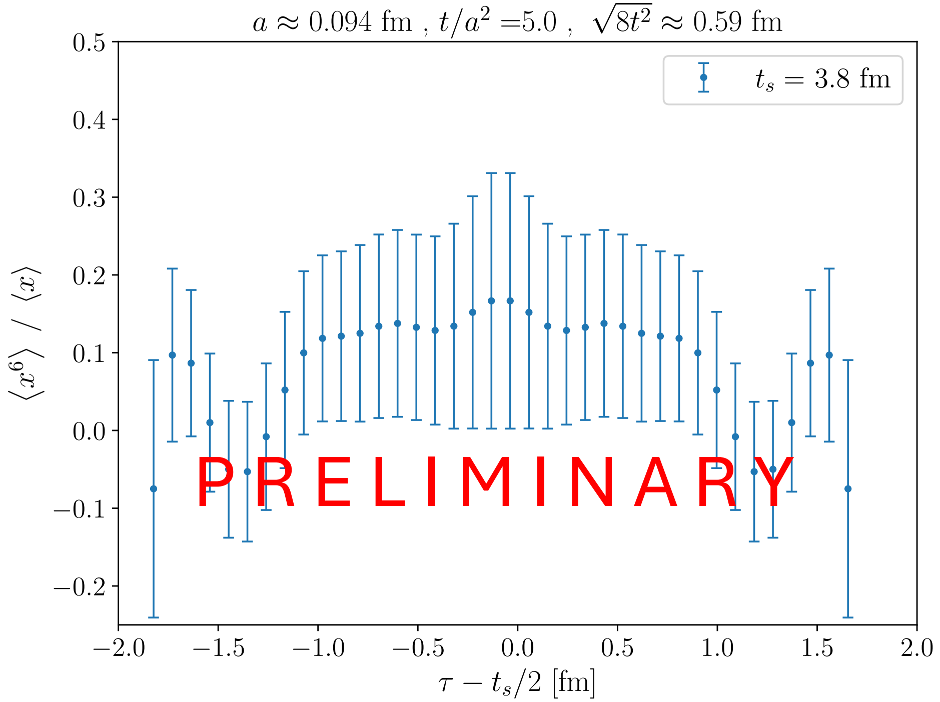

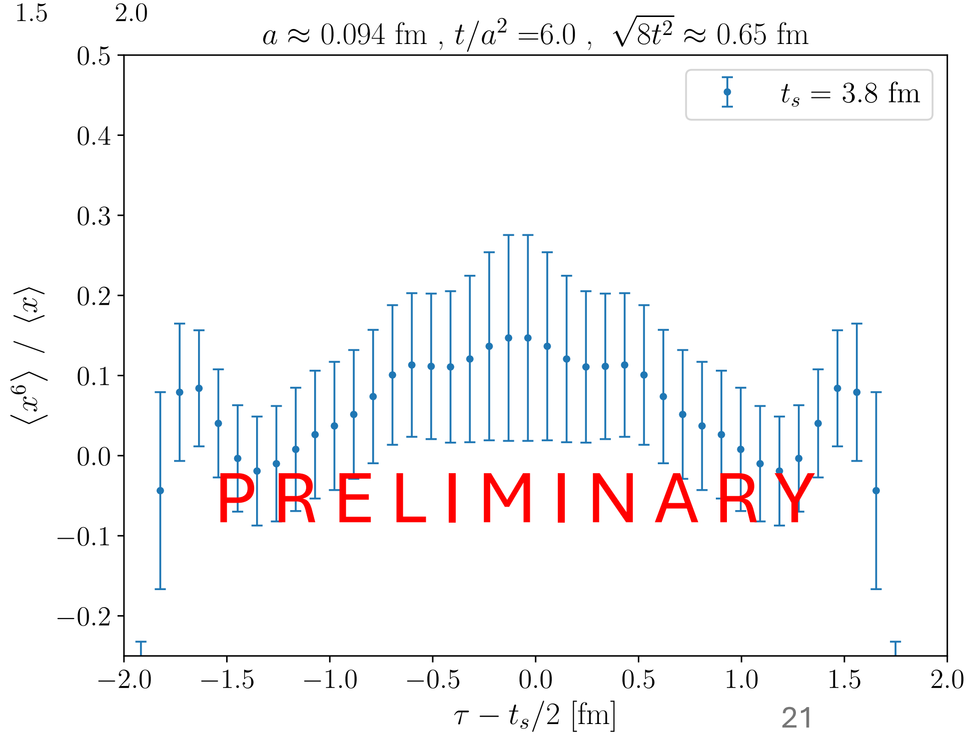

Summary and conclusion

- Preliminary investigation of proposal to using the gradient flow for obtaining precise, higher moments of PDFs

- Promising results for the pion flavor non-singlet moments using three SWF ensembles generated by OpenLat at $m_\pi \approx 410\ \mathrm{MeV}$ and three different lattice spacings

- This work: increase of statistics to resolve up to $n = 6$ or 7, careful investigation of systematics

- Future: nucleon
  flavor singlet
  off-forward: generalized form factors– moments of GPDs

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Thank you!