Unlocking higher moments of parton distribution functions

Andrea Shindler

shindler@physik.rwth-aachen.de
shindler@lbl.gov

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Outline

Moments of parton distribution functions of any order from lattice QCD

Andrea Shindler*
Institute for Theoretical Particle Physics and Cosmology, TTK, RWTH Aachen University
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA and
Department of Physics, University of California, Berkeley, CA 94720, USA
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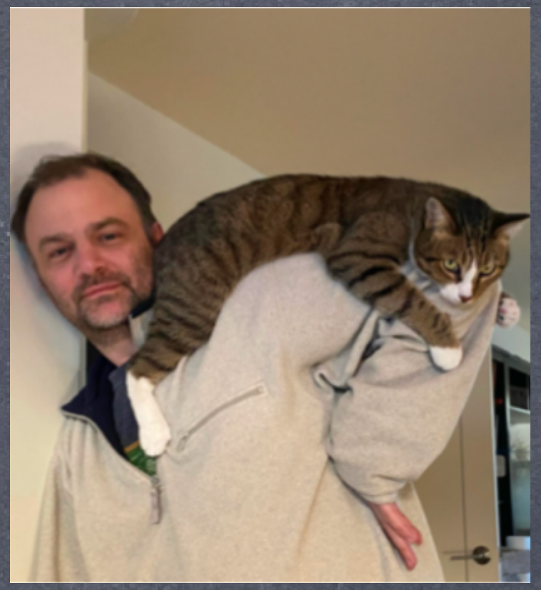
We describe a procedure to determine moments of parton distribution functions of any order in lattice QCD. The procedure is based on the gradient flow for fermion and gauge fields. The flowed matrix elements of twist-2 operators renormalize multiplicatively, and the matching with the physical matrix elements can be obtained using continuum symmetries and the irreducible representations of Euclidean 4-dimensional rotations. We calculate the matching coefficients at one-loop in perturbation theory for moments of any order in the flavor non-singlet case. We also give specific examples of operators that could be used in lattice QCD computations. It turns out that it is possible to choose operators with identical Lorentz indices and still have a multiplicative matching. One can thus use twist-2 operators exclusively with temporal indices, thus substantially improving the signal-to-noise ratio in the computation of the hadronic matrix elements.

2311.18704 [hep-lat]


Jangho Kim (FZJ)


Dimitra Pefkou (UCB)


Andre Walker-Loud (LBL)

OPEN LATtice initiative

PDF and Lattice QCD

- Connection between PDFs and hadronic matrix elements, which are calculable in lattice QCD, is established through the moments of the PDFs $\langle x^n \rangle$
- Lattice QCD calculations of the moments of the PDFs, provide, in principle, a mean for the complete reconstruction of the PDFs.
- This possibility has remained impractical due to the theoretical and numerical challenges associated with computing high moments.
- Continuum limit too difficult for $\langle x^n \rangle$ for $n>3$ -> Well known problem
  - For $n=2,3$ the need of non-vanishing external spatial momenta degrades the signal-to-noise ratio

Curci, Furmanski, Petronzio: 1980
Collins, Soper: 1982

Kronfeld, Photiadis: 1985
Martinelli, Sachrajda: 1987 - 1988
⋮

> reviews of Refs. [37, 38]). Direct calculations of distribution functions on a Euclidean lattice have not been feasible due to the time dependence of these quantities. A way around this limitation is the calculation on the lattice of moments of distribution functions (historically for PDFs and GPDs) and the physical PDFs can, in principle, be obtained from operator product expansion (OPE). Realistically, only the lowest moments of PDFs and GPDs can be computed (see e.g. [39–44]) due to large gauge noise in high moments, and also unavoidable power-divergent mixing with lower-dimensional operators. Combination of the two prevents a reliable and accurate calculation of moments beyond the second or third, and the reconstruction of the PDFs becomes unrealistic.

Cichy, Constantinou: 2019

Moments of the PDF: standard method

$$O_n^{rs}(x) = O_{\mu_1 \cdots \mu_n}^{rs}(x) = \overline{\psi}^r(x)\gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n\}}\psi^s(x)$$

- Calculate matrix elements using lattice QCD
  - Rotational group symmetry is broken into the hypercubic group H(4)
- Irreducible representations of O(4) generally become reducible representations of H(4) inducing unwanted mixings under renormalization
  - Irreps of H(4) allow mixing with lower dimensional operators and complicate mixings with operators of the same dimension

Beccarini et al.: 1995
Gockeler et al.: 1996

- Operators with different index combinations belong to different irreps of H(4)

O_3 $\quad \mu_1 = \mu_2 = \mu_3 \;\Rightarrow\; 1/a^2\delta_{\mu_i\mu_j}\cos(ap_{\mu_j})$ Kronfeld, Photiadis: 1985

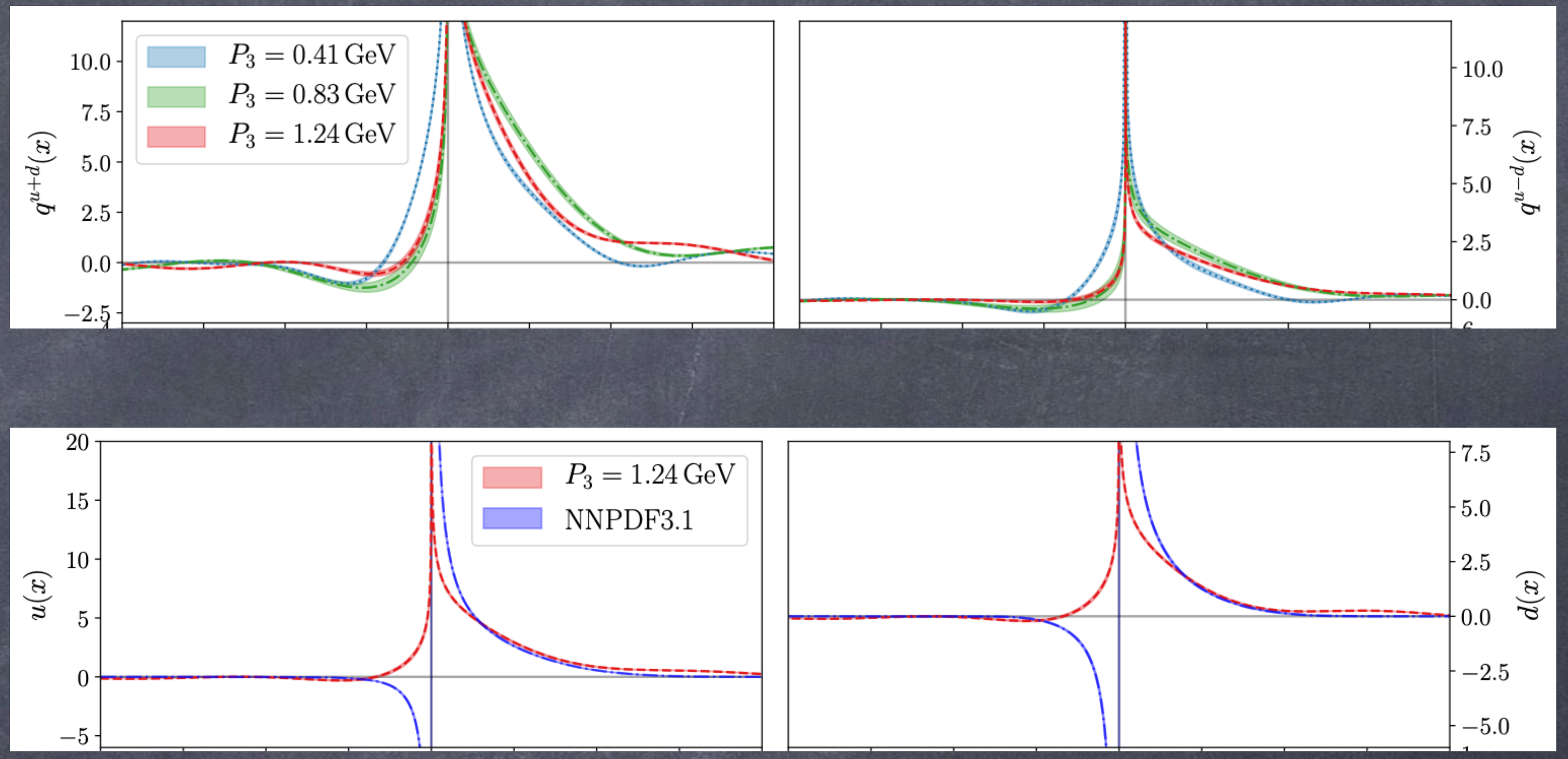
$\mu_1 \neq \mu_2 = \mu_3 \;\Rightarrow\; O_{\{411\}} - O_{\{433\}}$ Martinelli, Sachrajda: 1987

$\mu_1 \neq \mu_2 \neq \mu_3 \;\Rightarrow\; \langle h(p)|O_n|h(p)\rangle = 2\, p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1}\rangle_h(\mu)$

PDF and Lattice QCD

Approaches have been developed to determine the x-dependence of the PDFs

- Hadronic tensor — Liu, Dong: 1994
- Auxiliary scalar field — Aglietti et al.: 1998
- quasi-PDF (LaMET) — Ji: 2013
- pseudo-PDF — Radyushkin: 2017 - Karpie, Radyushkin, Orginos, Zafeiropoulos: 2017-2018
- Fictitious heavy quark — Detmold, Lin: 2005
- Auxiliary scalar quark — Braun, Müller: 2008
- Compton amplitude + OPE — Chambers et al.: 2017
- Good Lattice Cross Sections — Ma, Qiu: 2018
- hadron tensor method — Lian et al.: 2019
- PDF without Wilson line — Zhao: 2024
- These approaches allow in principle an indirect determination of the moments of PDF of nucleons and pions
- Recovery O(4) symmetry — Davoudi, Savage: 2012



Alexandrou et al. (ETMC): 2021

Egerer et al.: 2022

Gao et al.: 2020-2023

Method that addresses both the theoretical and numerical challenges faced in the past, which hindered the direct calculation of moments of any order from lattice QCD

Strategy

- Consider flowed twist-2 operators $O_n^{rs}(x,t) = \overline{\chi}^r(x,t)\gamma_{\{\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n\}}\chi^s(x,t)$

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- Renormalize flowed twist-2 operators —> renormalization is ALWAYS multiplicative.
  - Alternatively build ratios with the same fermion content

$$O_n^{rs}(t) = Z_n O_{n,B}^{rs}(t)\,, \quad Z_n = Z_\chi \qquad \left\langle \mathring{\overline{\chi}}_r(x,t)\overleftrightarrow{\not{D}}\mathring{\chi}_r(x,t)\right\rangle = -\frac{N_c}{(4\pi)^2 t^2}$$

Makino, Suzuki: 2014

NNLO

Harlander, Kluth, Lange: 2018

Artz et al.: 2019

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Artz et al.: 2019

- Construct fields based on irreps of O(4) -> symmetrized and traceless

$$\widehat{\mathring{O}}_n^{rs}(x,t) = \bar{\mathring{\chi}}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\mathring{\chi}^s(x,t) - \text{terms with } \delta_{\mu_i\mu_j}$$

Continuum limit is finite for any n

$$\left\langle h(p)|\widehat{\mathring{O}}_n(t)|h(p)\right\rangle = 2p_{\mu_1}\cdots p_{\mu_n}\left\langle x^{n-1}\right\rangle_h(t)$$

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- Perform a short flow time expansion. Matching constrained by continuum symmetries for traceless operators

$$\widehat{\mathring{O}}_n^{rs}(t) = c_n(t,\mu)\widehat{O}_n^{rs,\mathrm{MS}}(\mu) + \mathcal{O}(t)$$

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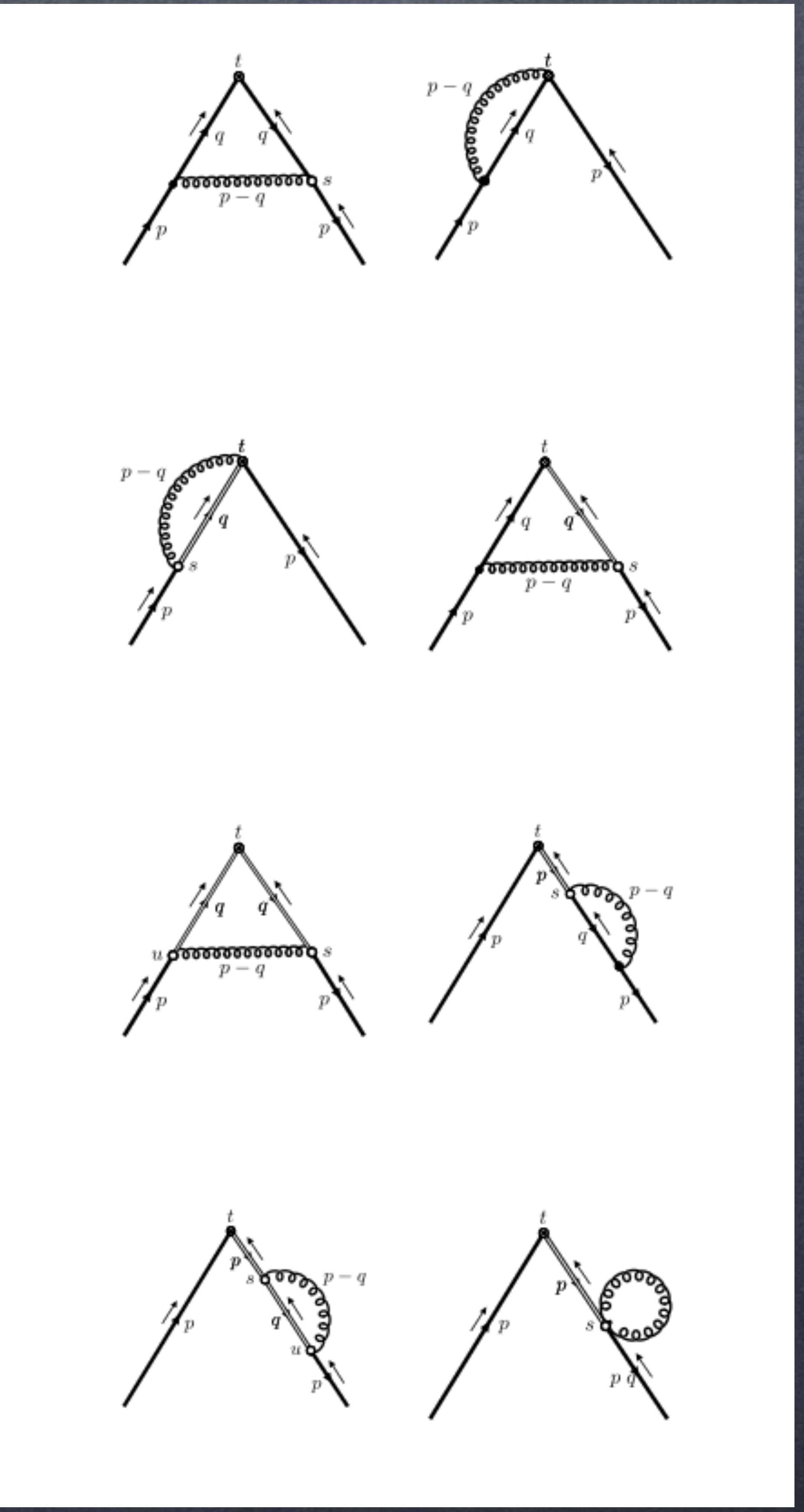
$$\hat{\mathring{O}}_n^{rs}(t) = c_n(t,\mu)\hat{O}_n^{rs,\mathrm{MS}}(\mu) + \mathcal{O}(t)$$

- Calculate matching coefficients in PT

Matching coefficients

Matching equations $$\left\langle \psi^r \mathring{\hat{O}}_n^{rs}(t)\overline{\psi}^s \right\rangle = c_n(t,\mu)\left\langle \psi^r \hat{O}_n^{rs,\mathrm{MS}}(t=0,\mu)\overline{\psi}^s \right\rangle$$

A.S.: 2023



- Expand integrands of loop integrals in all scales excluding t
  - Analytic structure altered —> distortion of IR structure
  - in matching equation the IR modification drops out in the difference
  - Expanding loop integrals in the RHS vanish in DR —> UV and IR are identical
  - The LHS is UV-finite, beside the renormalization of the bare parameters and flowed fermion fields
  - The IR singularities on the LHS exactly match the UV MS counterterms

$$c_n(t,\mu) = 1 + \frac{\overline{g}^2(\mu)}{(4\pi)^2} c_n^{(1)}(t,\mu) + O(\overline{g}^4)$$

$$c_n^{(1)}(t,\mu) = C_F\left[\gamma_n \log\left(8\pi\mu^2 t\right) + B_n\right]$$

$$\gamma_n = 1 + 4\sum_{j=2}^{n}\frac{1}{j} - \frac{2}{n(n+1)}$$

$$B_n = \frac{4}{n(n+1)} + 4\frac{n-1}{n}\log 2 + \frac{2-4n^2}{n(n+1)}\gamma_E - \frac{2}{n(n+1)}\psi(n+2) + \\ + \frac{4}{n}\psi(n+1) - 4\psi(2) - 4\sum_{j=2}^{n}\frac{1}{j(j-1)}\frac{1}{2^j}\phi(1/2,1,j) - \log(432)$$

$$\phi(z,s,a) = \sum_{k=0}^{\infty}\frac{z^k}{(k+a)^s}$$

Gross, Wilczek: 1974

$n=2$ Makino, Suzuki: 2014

O(a) improvement

$$\widehat{\mathring{O}}_n^{rs}(x,t) = \bar{\mathring{\chi}}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\mathring{\chi}^s(x,t) - \text{terms with } \delta_{\mu_i\mu_j}$$

$$\left\langle h(p)|\widehat{\mathring{O}}_n(t)|h(p)\right\rangle = 2p_{\mu_1}\cdots p_{\mu_n}\left\langle x^{n-1}\right\rangle_h(t)$$

- Hadronic matrix elements of flowed operators beside the O(a) from the lattice theory are affected by O(am) and short-distance O(a)
- The O(am) are independent on n (depend only on the fermion content)
- With ratios discretization effects are $O(a^2)$ —> clover fermions are back in the game

$$\frac{\left\langle x^{n-1}\right\rangle_h(t)}{\left\langle x^{m-1}\right\rangle_h(t)} \quad n \neq m \quad n \geq 3,\ m \geq 2$$

O(a) improvement

$$\hat{\mathring{O}}_n^{rs}(x,t) = \bar{\mathring{\chi}}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\mathring{\chi}^s(x,t) - \text{terms with } \delta_{\mu_i\mu_j}$$

$$\left\langle h(p)|\hat{\mathring{O}}_n(t)|h(p)\right\rangle = 2p_{\mu_1}\cdots p_{\mu_n}\left\langle x^{n-1}\right\rangle_h(t)$$

- Hadronic matrix elements of flowed operators beside the O(a) from the lattice theory are affected by O(am) and short-distance O(a)
- The O(am) are independent on n (depend only on the fermion content)
- With ratios discretization effects are O(a²) —> clover fermions are back in the game

$$\frac{\left\langle x^{n-1}\right\rangle_h(t)}{\left\langle x^{m-1}\right\rangle_h(t)} \quad n \neq m \quad n \geq 3,\ m \geq 2$$

→ Finite continuum limit and O(a) improved

Strategy

$$\left\langle h(p)|\widehat{\mathring{O}}_n(t)|h(p)\right\rangle = 2\, p_{\mu_1}\cdots p_{\mu_n} \left\langle x^{n-1}\right\rangle_h (t)$$

Continuum limit is finite for any n

$$\left\langle x^{n-1}\right\rangle_h^{\mathrm{MS}}(\mu) = c_n(t,\mu)^{-1}\left\langle x^{n-1}\right\rangle_h (t) + \mathcal{O}(t)$$

Matching is multiplicative for any n

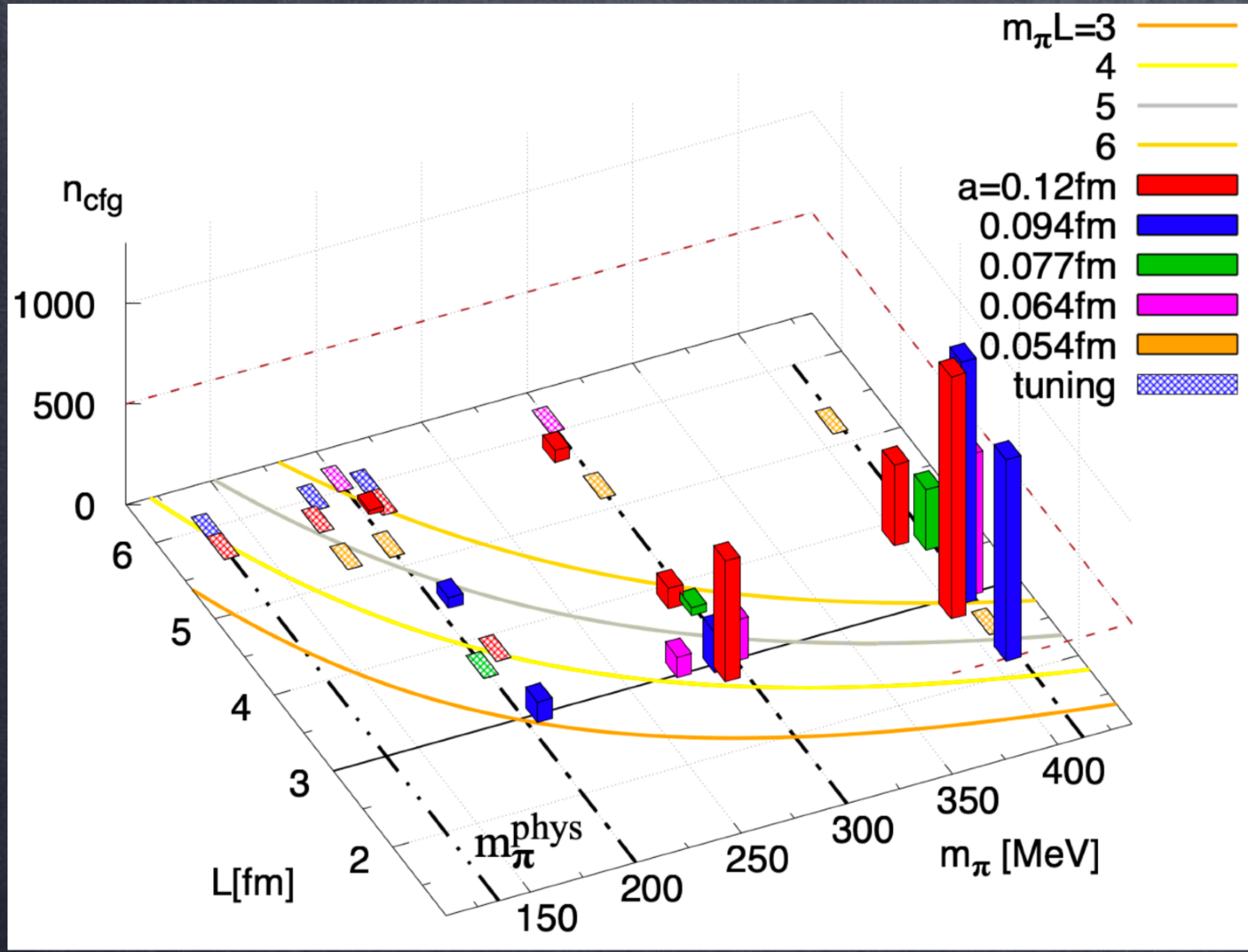
n=4 $\widehat{O}_{4444} = O_{4444} - \frac{3}{4}O_{\{\alpha\alpha 44\}} + \frac{1}{16}O_{\{\alpha\alpha\beta\beta\}}$

Vanishing spatial momenta for any n

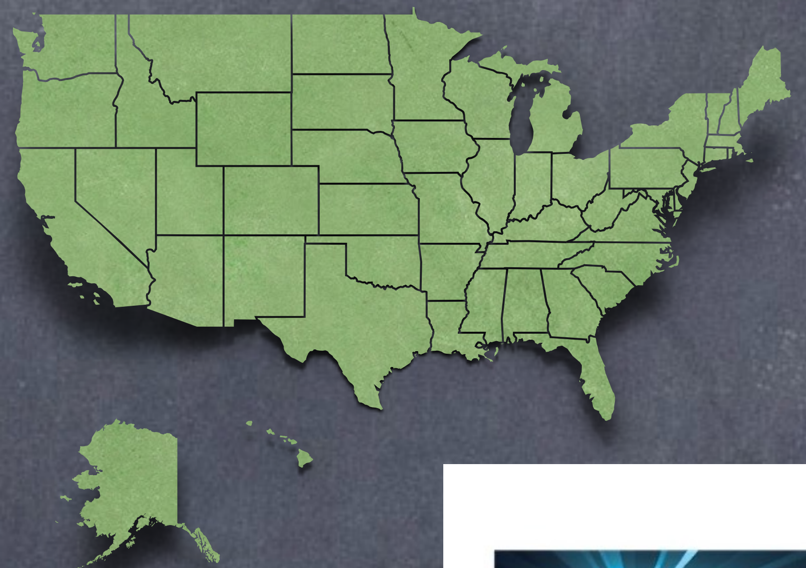
$$\left\langle x^{n-1}\right\rangle_h^{\mathrm{MS}}(\mu) = \left\langle x^{m-1}\right\rangle_h^{\mathrm{MS}}(\mu)\frac{c_m(t,\mu)}{c_n(t,\mu)}\frac{\left\langle x^{n-1}\right\rangle_h (t)}{\left\langle x^{m-1}\right\rangle_h (t)}\,, \quad m\neq n \quad n\geq 3\; m\geq 2$$

OpenLAT

- OpenLat: open science initiative. Gauges with SWF open to the whole community



https://openlat1.gitlab.io



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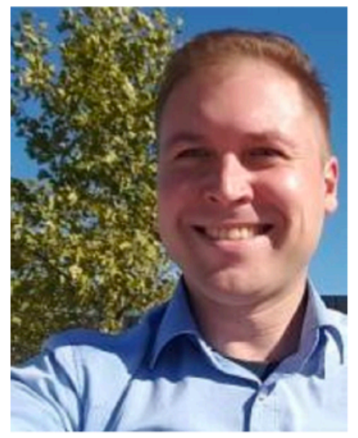
EuroHPC

Irene Joliot-Curie

Jangho Kim (FZJ)

Dimitra Pefkou (UCB)

Francesca Cuteri



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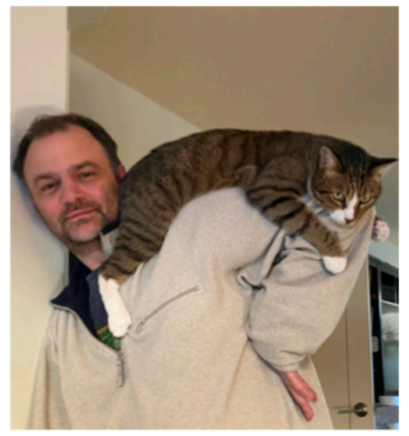
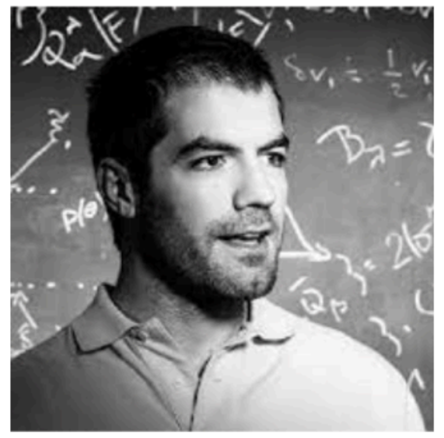
Andrea Shindler

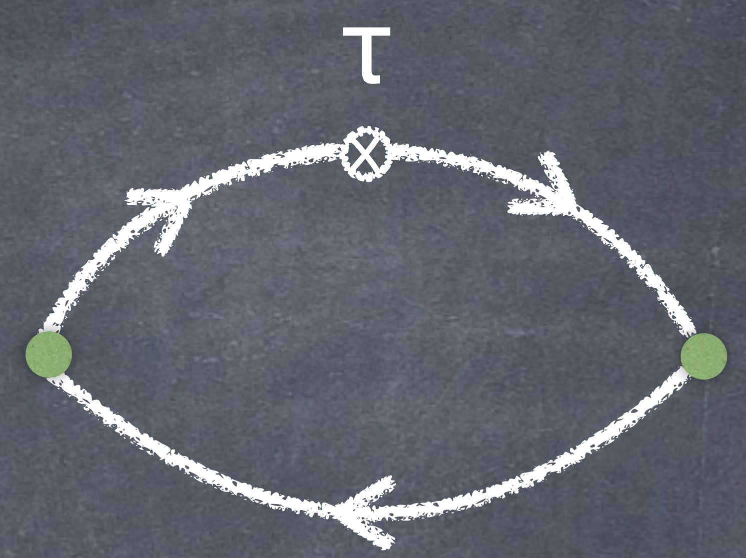


André Walker-Loud



Savvas Zafeiropoulos

Flowed moments $\langle x^2\rangle/\langle x\rangle$ for "pions"



Lattice parameters $N_f = 3$ $m_{PS} \simeq 410$ MeV

$a \simeq 0.12$ fm $L \simeq 2.9$ fm $t_s/a = 40$ $\tau = t_s/2$

$a \simeq 0.094$ fm $L \simeq 3$ fm $t_s/a = 40$ $\tau = t_s/2$

$a \simeq 0.064$ fm $L \simeq 3.1$ fm $t_s/a = 40$ $\tau = t_s/2$

Statistics (sources x gauges)

$a \simeq 0.12$ fm $1 \times 119 = 119$ $\sim 3.5 - 7\%$

$a \simeq 0.094$ fm $1 \times 210 = 210$ $\sim 2.6 - 6\%$

$a \simeq 0.064$ fm $1 \times 22 = 22$ $\sim 5.6 - 12\%$

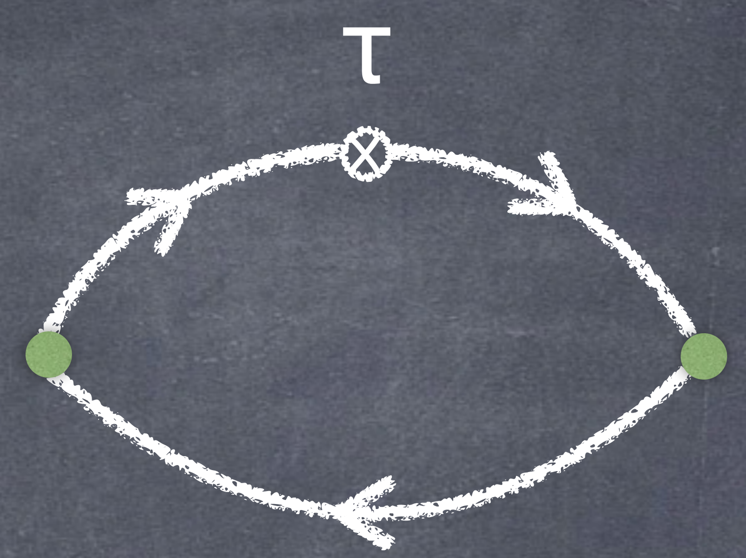
Lattice parameters (ETMC)

$a \simeq 0.093$ fm $L \simeq 3$ fm $m_\pi \simeq 260$ MeV

Statistics (sources x gauges)

$32 \times 122 = 3904$ $\sim 27\%$

Flowed moments <x^2>/<x> for "pions"



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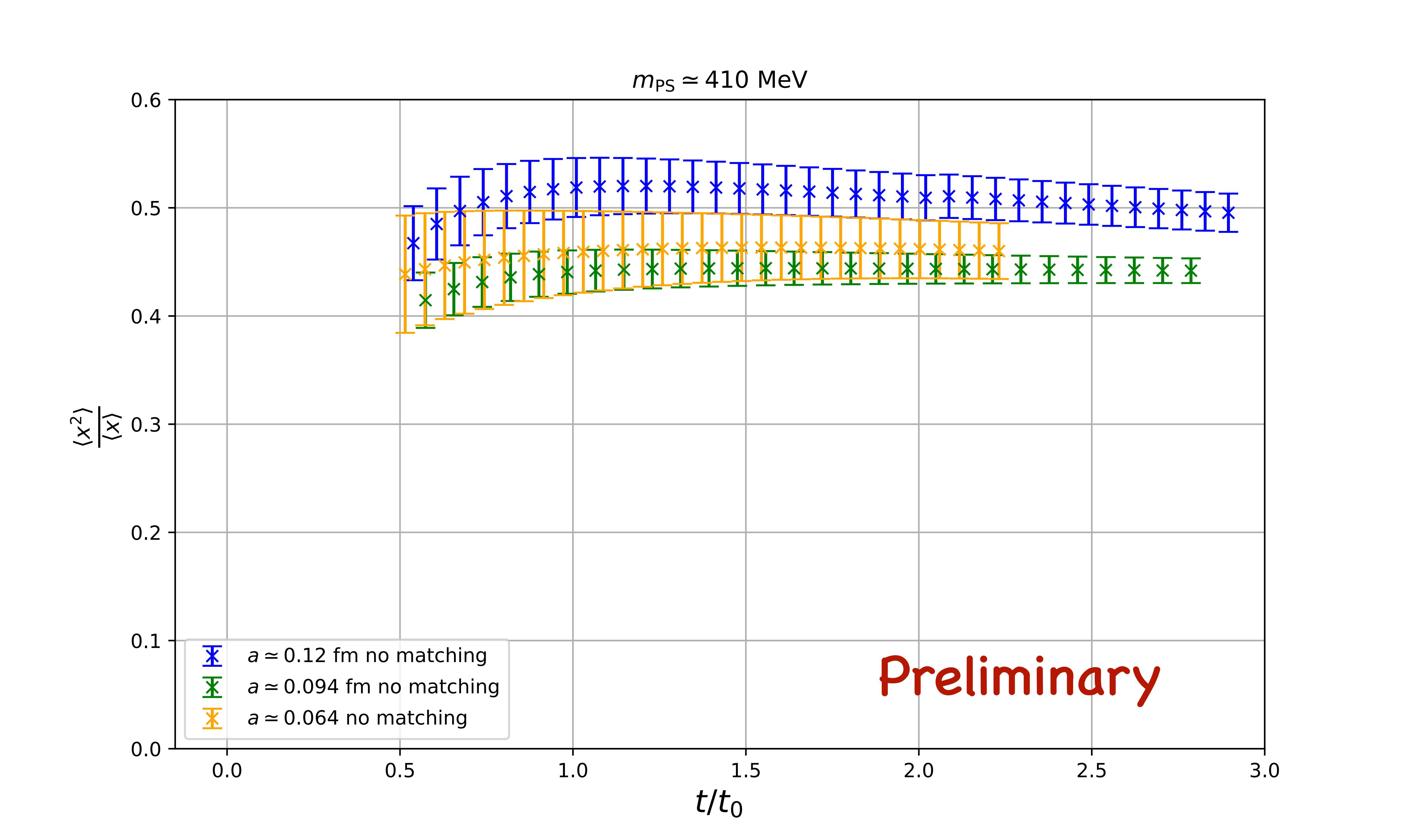
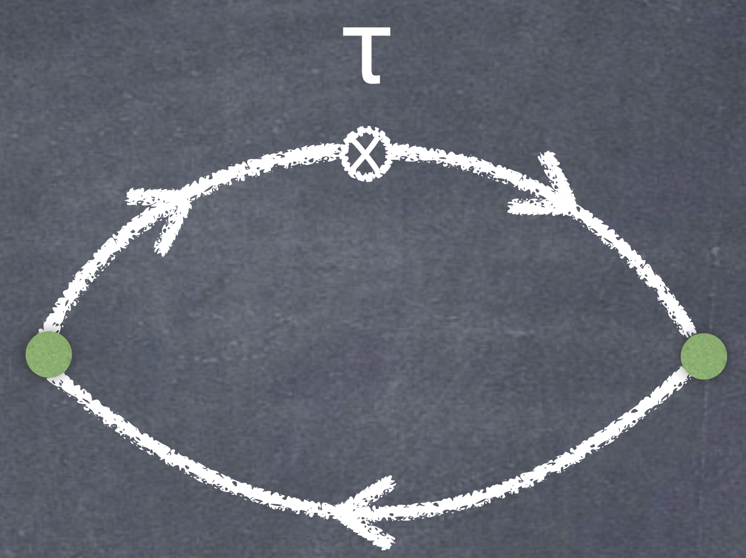


Flowed moments <x²>/<x> for "pions"



$$\frac{\langle x^2 \rangle_{\overline{MS}}(\mu)}{\langle x \rangle_{\overline{MS}}(\mu)}$$

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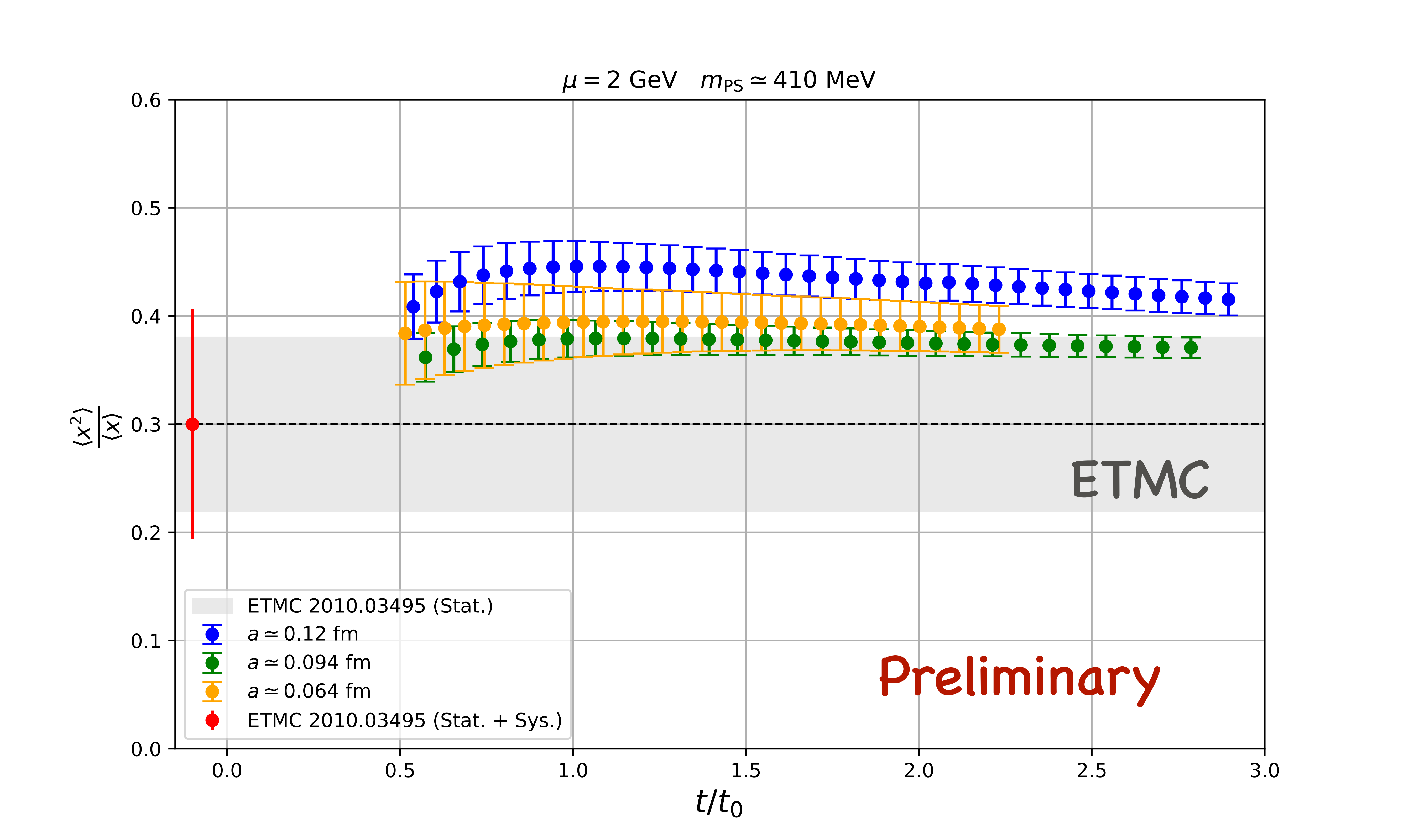

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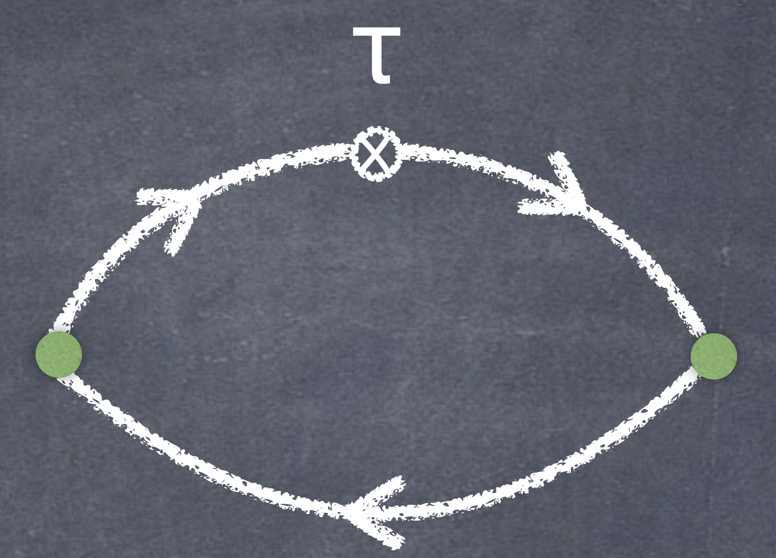

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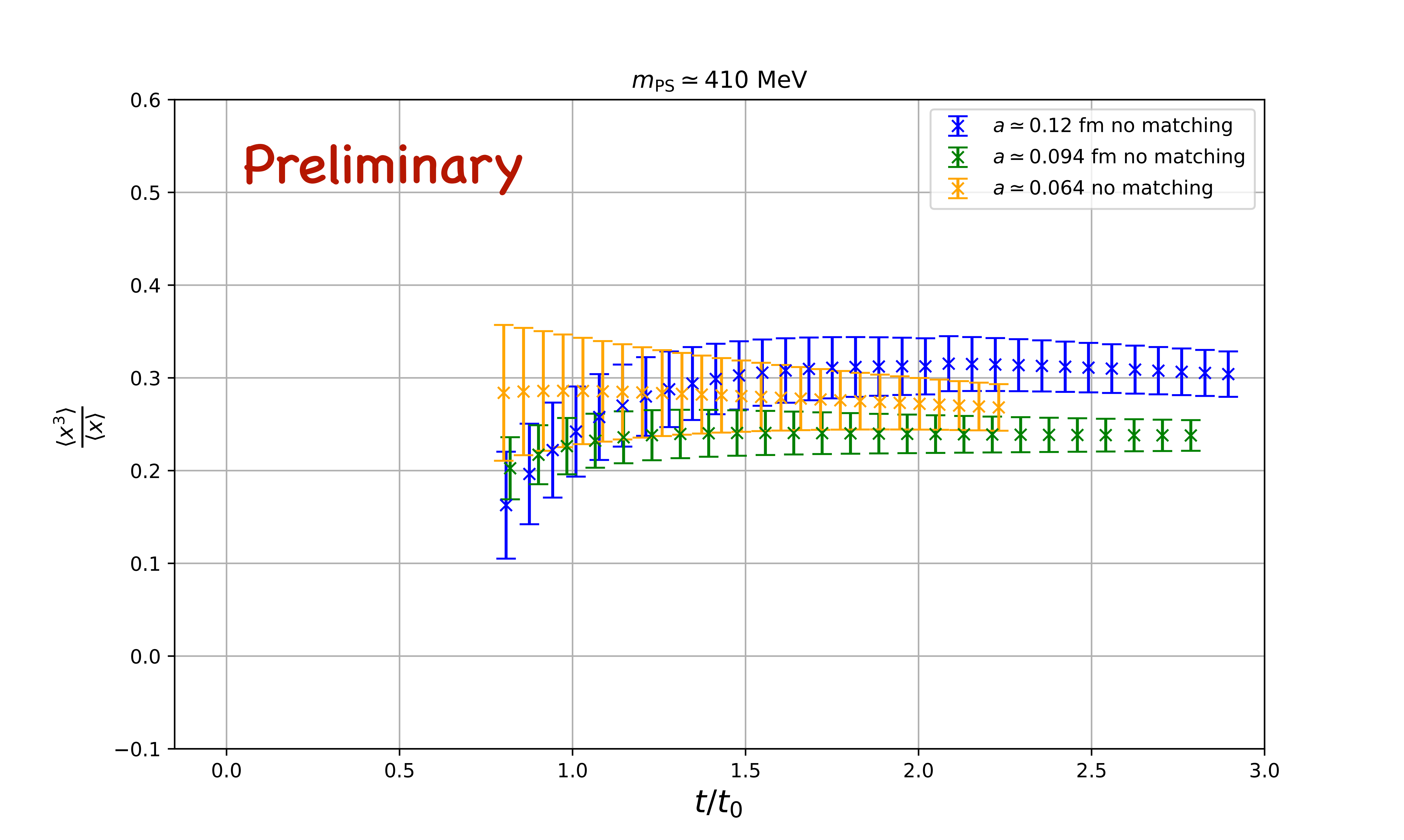
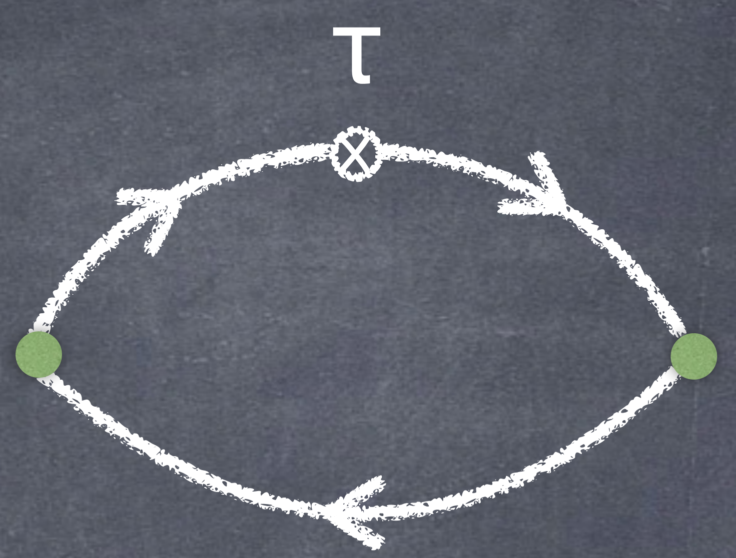
Statistics (sources x gauges)

$72 \times 122 = 8784$ $\sim 75\%$



Flowed moments <x³>/<x>



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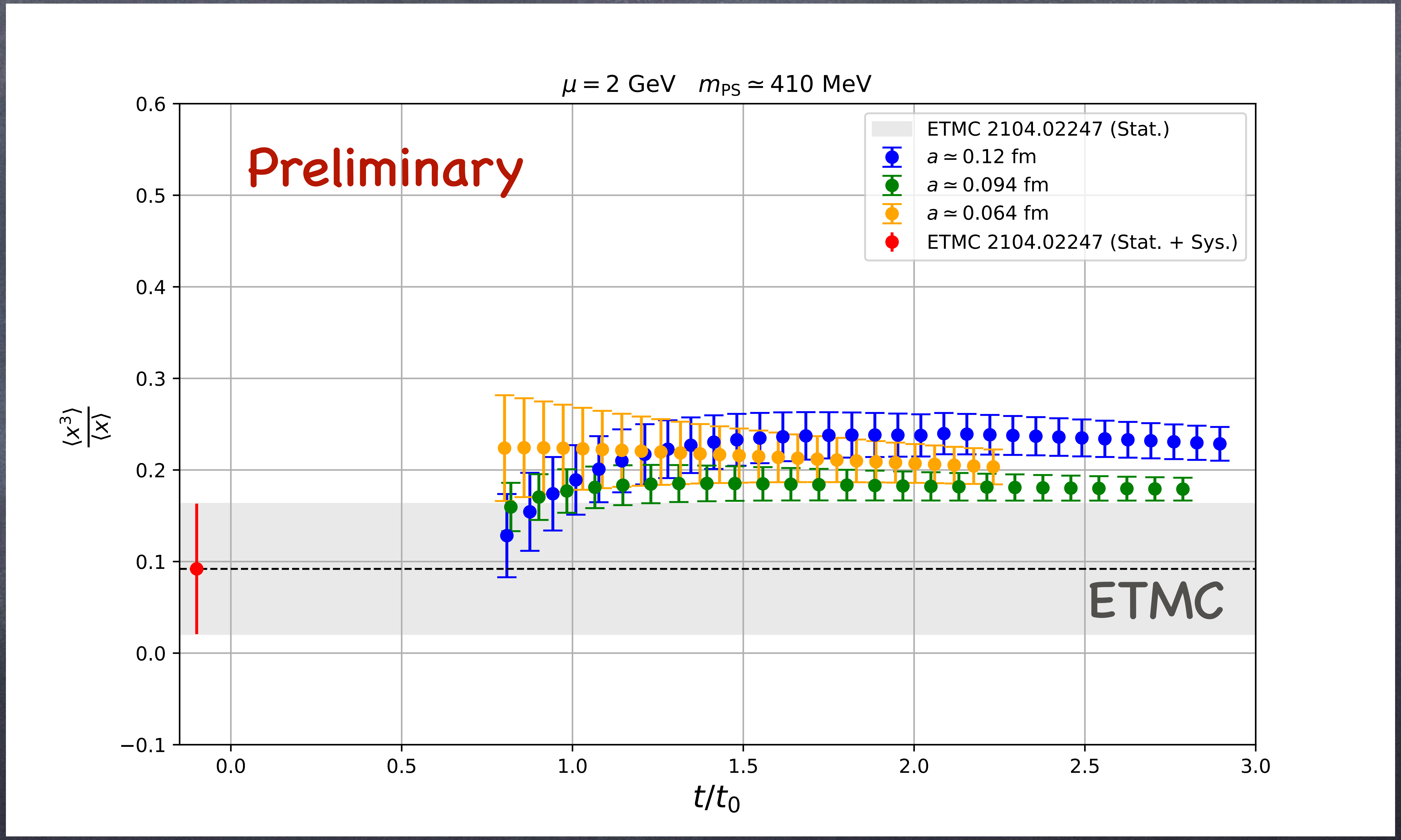
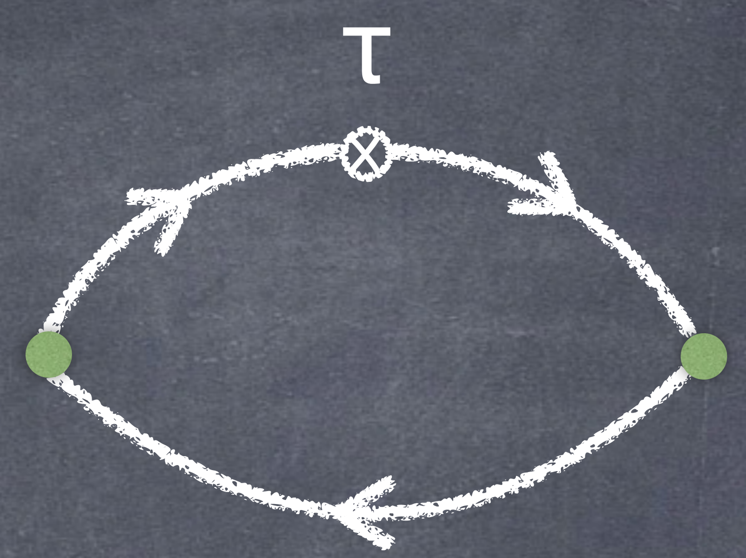


Flowed moments <x³>/<x²>



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$a \simeq 0.064$ fm $L \simeq 3.1$ fm $t_s/a = 40$ $\tau = t_s/2$

Statistics (sources x gauges)

$a \simeq 0.12$ fm $1 \times 119 = 119$ $\sim 6 - 13\%$

$a \simeq 0.094$ fm $1 \times 210 = 210$ $\sim 5 - 13\%$

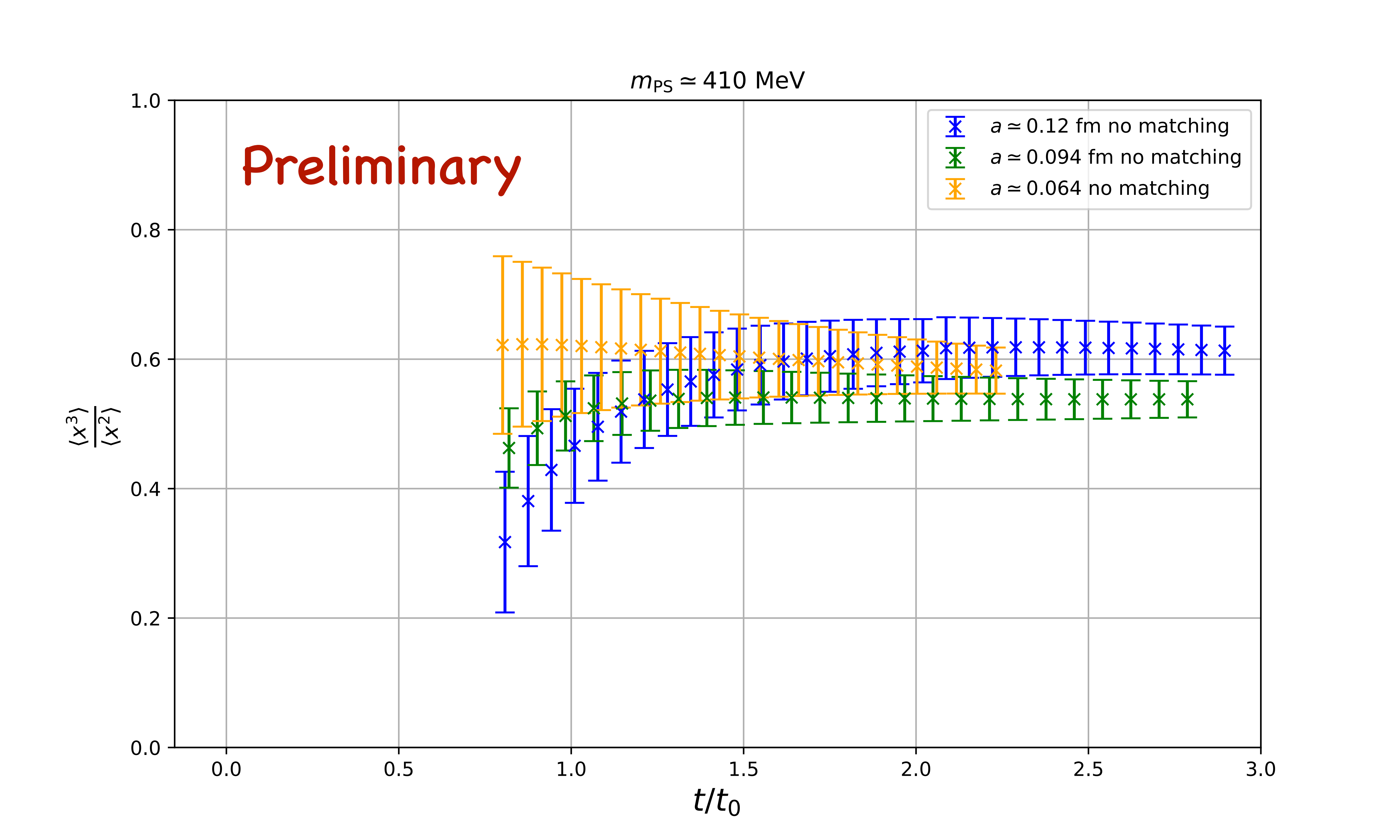
$a \simeq 0.064$ fm $1 \times 22 = 22$ $\sim 6 - 22\%$

Lattice parameters (ETMC)

$a \simeq 0.093$ fm $L \simeq 3$ fm $m_\pi \simeq 260$ MeV

Statistics (sources x gauges)

$72 \times 122 = 8784$ $\sim 77\%$



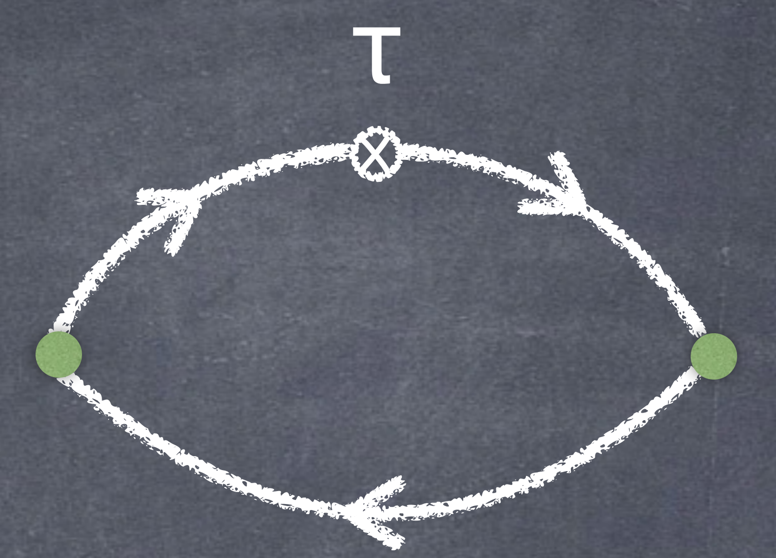
Flowed moments <x³>/<x²>

Lattice parameters $N_f = 3 \quad m_{PS} \simeq 410$ MeV

$a \simeq 0.12$ fm $\quad L \simeq 2.9$ fm $\quad t_s/a = 40 \quad \tau = t_s/2$

$a \simeq 0.094$ fm $\quad L \simeq 3$ fm $\quad t_s/a = 40 \quad \tau = t_s/2$

$a \simeq 0.064$ fm $\quad L \simeq 3.1$ fm $\quad t_s/a = 40 \quad \tau = t_s/2$



$$\frac{\langle x^3 \rangle_{\overline{MS}}(\mu)}{\langle x^2 \rangle_{\overline{MS}}(\mu)}$$

Statistics (sources x gauges)

$a \simeq 0.12$ fm $\quad 1 \times 119 = 119 \quad \sim 6-13\%$

$a \simeq 0.094$ fm $\quad 1 \times 210 = 210 \quad \sim 5-13\%$

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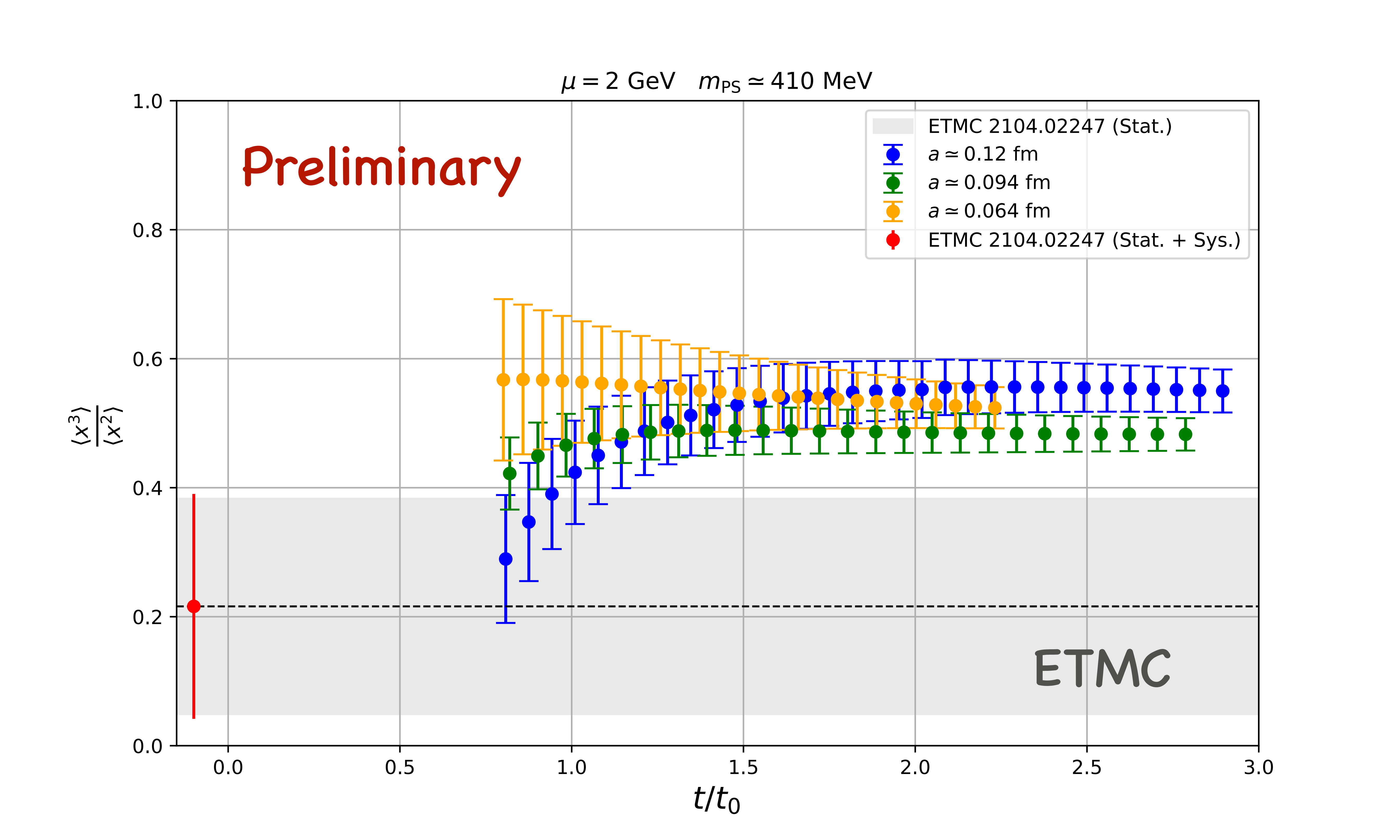


Lattice parameters (ETMC)

$a \simeq 0.093$ fm $\quad L \simeq 3$ fm $\quad m_\pi \simeq 260$ MeV

Statistics (sources x gauges)

$72 \times 122 = 8784 \quad \sim 77\%$

Outlook

- Continuum limit at isosymmetric point for pion
- Calculate up to $\langle x^6 \rangle$
- Reconstruct PDF
- Nucleon matrix elements -> excited state contamination
- Singlet unpolarized PDF -> gluon PDF
- Extension to 2-loops of perturbative matching
- Many other potential applications for hadron structure calculations

Summary

- New method to calculate moments of PDF from lattice QCD
- Method is general and can be used with any lattice action
- We make use of an intermediate regulator (GF) that simplifies the continuum limit
- After recovering O(4) symmetry the matching is done using continuum PT
- Matrix elements can be all calculated with vanishing external momenta
- Ratios of matrix elements improve further continuum limit and S/N
- Very promising. Stay tuned and do not leave this room

Backup Slides

Gradient flow

Narayanan, Neuberger:2006

Lüscher 2010

Lüscher, Weisz 2011

$x_\mu = (\mathbf{x}, x_4) \quad t \to \text{flow-time} \quad [t] = -2$

$A_\mu(x) = A_\mu^a(x) T^a \;\to\; \text{gluon fields}$

$$\partial_t B_\mu(x,t) = D_\nu G_{\nu\mu}(x,t)$$
$$B_\mu(x,t)|_{t=0} = A_\mu(x)$$

$D_\nu = \partial_\nu + [B_\nu(x,t), \cdot]$

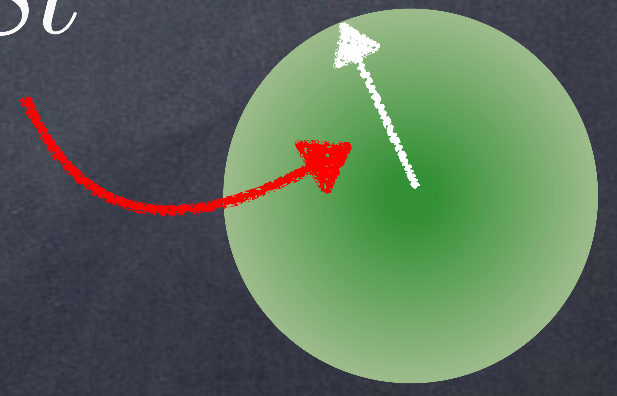
$G_{\mu\nu}(x,t) = \partial_\mu B_\nu(x,t) - \partial_\nu B_\mu(x,t) + [B_\mu, B_\nu]$

$$\partial_t B_\mu = \partial_\nu \partial_\nu B_\mu$$

$$B_\mu(x,t) = \int d^4y \; K(x-y;t) A_\mu(y)$$

$$K(x;t) = \int \frac{d^4p}{(2\pi)^4} \mathrm{e}^{ipx} \mathrm{e}^{-tp^2} = \frac{\mathrm{e}^{-x^2/4t}}{(4\pi t)^2}$$

- Gaussian damping at large momenta
- Smoothing at short distance over a range $\sqrt{8t}$



$B_\mu(x,t) \qquad t > 0 \qquad$ finite

Continuum limit is finite

Lüscher, Weisz: 2011

Gradient flow

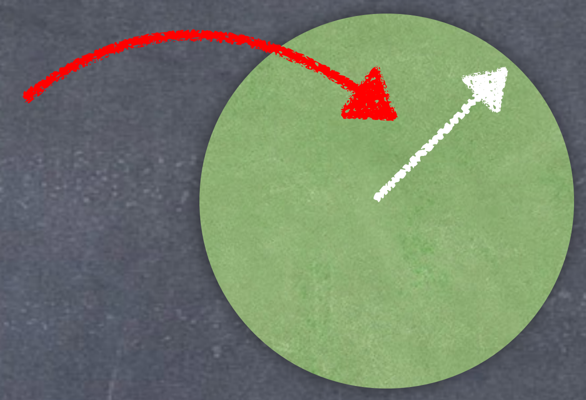
Lüscher: 2013

$x_\mu = (\mathbf{x}, x_4) \quad t \to \text{flow-time} \quad [t] = -2$

$$\chi(x,t) = \int d^4y K(x-y,t)\psi(y) \quad K(x,t) = \frac{1}{4\pi t^2} \mathrm{e}^{-\frac{x^2}{4t}}$$

$$\partial_t \chi(x,t) = \Delta \chi(x,t)$$
$$\partial_t \bar{\chi}(x,t) = \bar{\chi}(x,t)\overleftarrow{\Delta}$$
$$\chi(x,t=0) = \psi(x)$$
$$\bar{\chi}(x,t=0) = \bar{\psi}(x)$$

- Smoothing over a range $\sqrt{8t}$



- Gaussian damping at large momenta

$$\chi_R(x,t) = Z_\chi^{1/2}\chi(x,t)$$

$$\mathcal{O}(x,t) = \overline{\chi}(x,t)\Gamma(x,t)\chi(x,t) \qquad \mathcal{O}_R = Z_\chi \mathcal{O}$$

$$\Delta = D_{\mu,t}D_{\mu,t} \qquad D_{\mu,t} = \partial_\mu + B_{t,\mu}$$

$$\Sigma_t = \langle \overline{\chi}(x,t)\chi(x,t)\rangle \qquad \Sigma_{t,R} = Z_\chi \Sigma_t$$

No additive divergences

Continuum limit finite after normalizing fermion fields

Flowed twist-2 operators

$$O_n^{rs}(x,t) = \overline{\chi}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\chi^s(x,t)$$

$$O_n^{rs}(t) = Z_n O_{n,B}^{rs}(t)\,, \quad Z_n = Z_\chi$$

$$\left\langle \mathring{\overline{\chi}}_r(x,t)\overleftrightarrow{\not{D}}\mathring{\chi}_r(x,t)\right\rangle = -\frac{N_c}{(4\pi)^2 t^2}$$

Makino, Suzuki: 2014

$$\chi^{r,\mathrm{MS}}(x,t) = (8\pi t)^{\epsilon/2}\,\zeta_\chi^{1/2}\mathring{\chi}^r(x,t)$$

$$\overline{\chi}^{r,\mathrm{MS}}(x,t) = (8\pi t)^{\epsilon/2}\,\zeta_\chi^{1/2}\mathring{\overline{\chi}}_r(x,t)$$

$$\zeta_\chi = 1 - \frac{\overline{g}^2}{(4\pi)^2}C_F\left(3\log(8\pi\mu^2 t) - \log(432)\right)$$

$$D = 4 - 2\epsilon$$

$$C_F = \frac{N_c^2 - 1}{2N_c}$$

$$\log\mu^2 = \log\overline{\mu}^2 + \gamma_E - \log 4\pi$$

NNLO

Harlander, Kluth, Lange: 2018

Artz et al.: 2019

O(4) irreducible representations

GL(4) irrep $$T_{\{\mu_1\cdots\mu_n\}} = \frac{1}{n!} \sum_{\substack{\sigma\in\text{all}\\ \text{permutations}}} T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}$$

In O(4) an additional operation is allowed that commutes with orthogonal trafo: contraction of 2 indices

$$T^{(12)}_{\mu_1\cdots\mu_n} = T_{\alpha\alpha\mu_3\cdots\mu_n} = \delta_{\mu_1\mu_2} T_{\mu_1\cdots\mu_n}$$ rank n-2 tensor

Subspace of traceless tensors is invariant under O(4), i.e. the traceless rank n tensors are transformed among themselves under O(4)

Always possible to decompose $$T_{\mu_1\cdots\mu_n} = \widehat{T}_{\mu_1\cdots\mu_n} + \underbrace{\delta_{\mu_1\mu_2} T^{(12)}_{\mu_1\cdots\mu_n} + \cdots}_{n(n-1)/2 \text{ terms}}$$ Invariant under O(4)

E.g. $$\widehat{T}_{\mu_1\mu_2} = T_{\mu_1\mu_2} - \frac{1}{4}\delta_{\mu_1\mu_2} T_{\alpha\alpha} \qquad \widehat{T}_{\mu_1\mu_2\mu_3} = T_{\mu_1\mu_2\mu_3} - \frac{1}{6}\left[\delta_{\mu_1\mu_2} T_{\alpha\alpha\mu_3} + \delta_{\mu_1\mu_3} T_{\alpha\mu_2\alpha} + \delta_{\mu_2\mu_3} T_{\mu_1\alpha\alpha}\right]$$

Traceless tensors invariant under vector index permutations —> starting point to construct all the irreducible representations of O(4) (Young symmetrizers)

Traceless and symmetrized rank-n tensors are an irreducible representation of O(4)

Strategy - Short flow-time expansion

Lüscher: 2013

$$[\mathcal{O}_i(t)]_{\mathrm{R}} = \sum_i c_{ij}(t,\mu)\,[\mathcal{O}_i(t=0,\mu)]_{\mathrm{R}} + O(t)$$

LQCD

PT - LQCD

$$c_{ij}(t,\mu) = \delta_{ij} + \frac{\alpha_s(\mu)}{4\pi} c_{ij}^{(1)}(t,\mu) + O(\alpha_s^2)$$

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LQCD PT - LQCD

$$c_{ij}(t,\mu) = \delta_{ij} + \frac{\alpha_s(\mu)}{4\pi} c_{ij}^{(1)}(t,\mu) + O(\alpha_s^2)$$

- Calculation of matrix elements with flowed fields
  - Multiplicative renormalization (no power divergences and no mixing)
- Calculation of Wilson coefficients
  - Insert OPE in off-shell amputated 1PI Green's functions
- Power divergences subtracted non-perturbatively (LQCD)
- Determination of the physical renormalized matrix element at zero flow-time

A.S., Luu, de Vries: 2014-2015
Dragos, Luu, A.S. de Vries: 2018-2019
Rizik, Monahan, A.S.: 2018-2020
A.S.: 2020
Kim, Luu, Rizik, A.S.: 2020
Mereghetti, Monahan, Rizik, A.S., Stoffer: 2021
Monahan, Rizik, A.S., Stoffer: 2023
A.S.: 2023

O(a) improvement

$$O_n^{rs}(x) = O^{rs}_{\mu_1\cdots\mu_n}(x) = \overline{\psi}^r(x)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\psi^s(x)$$

- Beside the O(a) from the lattice theory twist-2 fields are affected by specific O(a) that depend on n
- Improvement coefficients are known only for n=2 and only in PT
- Only GW fermions or Wtm at maximal twist removes these O(a)

$$\hat{\mathring{O}}_n^{rs}(x,t) = \mathring{\overline{\chi}}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\mathring{\chi}^s(x,t) - \text{terms with } \delta_{\mu_i\mu_j}$$

$$\left\langle h(p)|\hat{\mathring{O}}_n(t)|h(p)\right\rangle = 2p_{\mu_1}\cdots p_{\mu_n}\left\langle x^{n-1}\right\rangle_h(t)$$

- Hadronic matrix elements of flowed operators beside the O(a) from the lattice theory are affected by O(am) and short-distance O(a)
- The O(am) are independent on n (depend only on the fermion content)
- With ratios discretization effects are O(a²) —> clover fermions are back in the game

$$\frac{\left\langle x^{n-1}\right\rangle_h(t)}{\left\langle x^{m-1}\right\rangle_h(t)} \quad n \neq m \quad n \geq 3,\ m \geq 2$$

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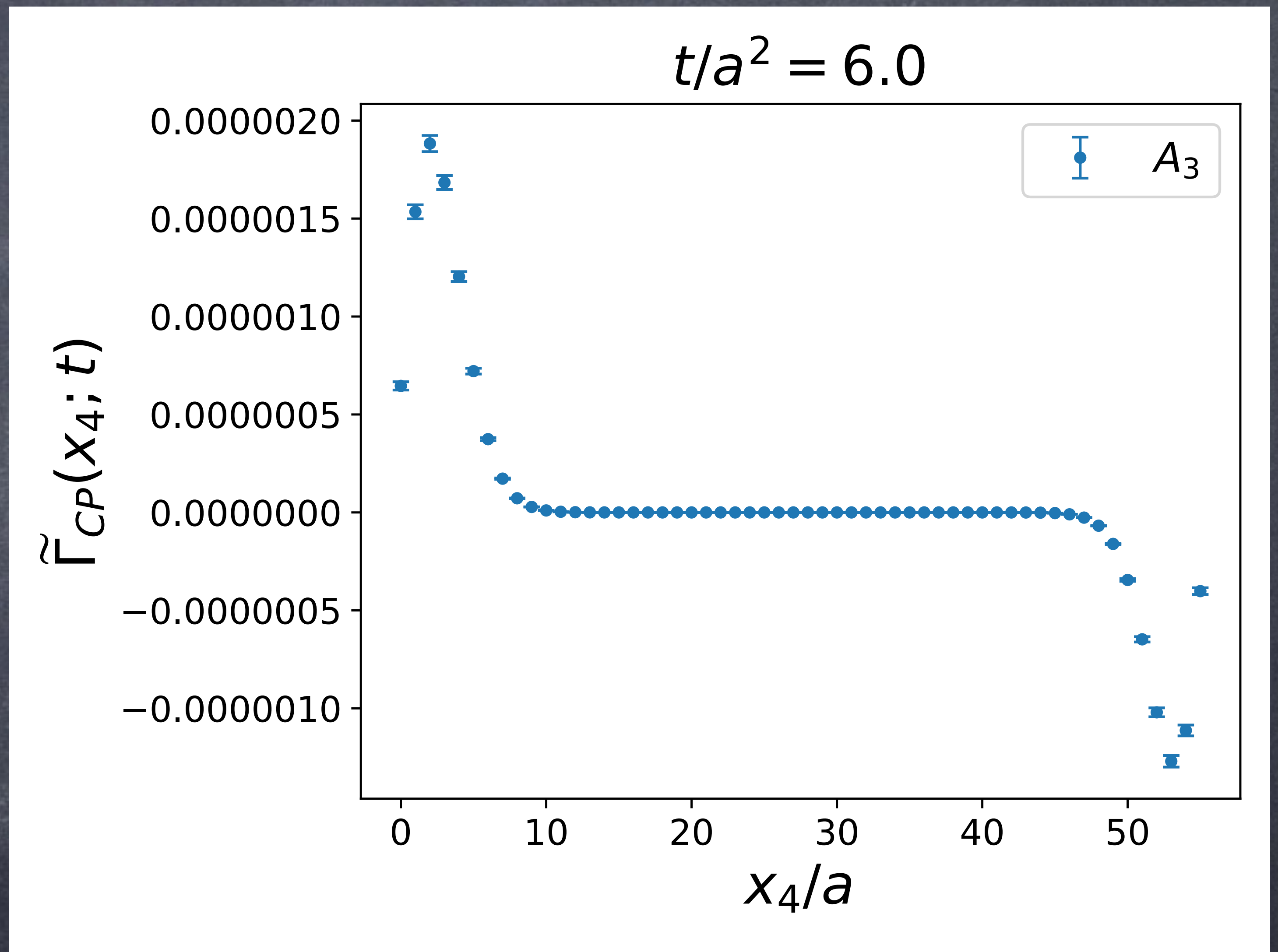
$$\Gamma_{CP}(x_4;t) = a^3\sum_{\mathbf{x}}\left\langle O_{\mathrm{C}}^{ij}(x_4,\mathbf{x};t)P^{ji}(0,\mathbf{0};0)\right\rangle$$

$$\widetilde{\Gamma}_{CP}(x_4;t) = a^3\sum_{\mathbf{x}}\left\langle O_{\mathrm{C}}^{ij}(x_4,\mathbf{x};t)\widetilde{P}^{ji}(0,\mathbf{0};0)\right\rangle$$

$$\widetilde{P}^{ij}(x) = \overline{\lambda}_i(x)\gamma_\mu\gamma_5\psi_j(x) + \overline{\psi}_i(x)\gamma_\mu\gamma_5\lambda_j(x)$$



Kim, Luu, Rizik, A.S.: 2021

O(a) improvement

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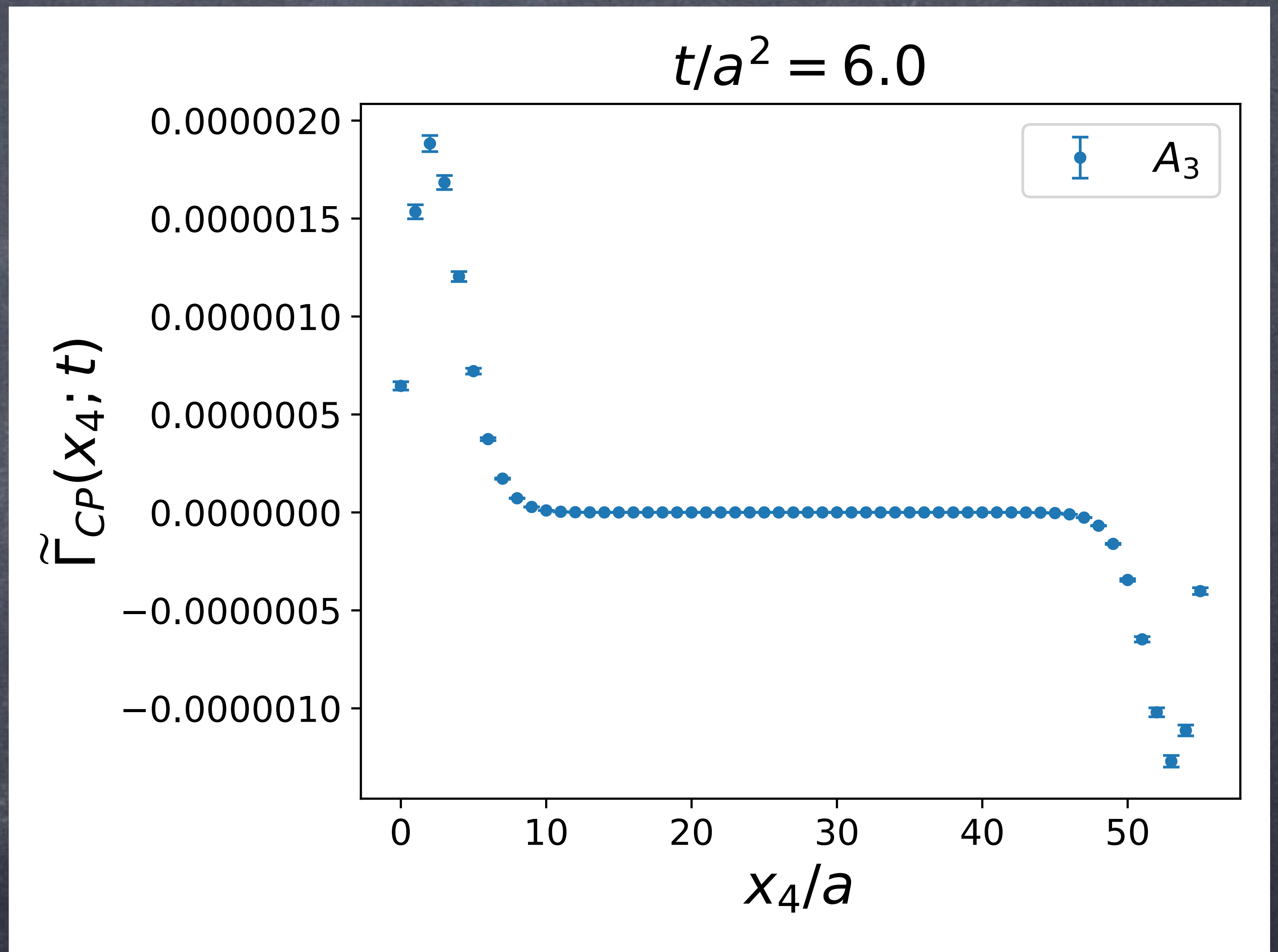

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→ **Finite continuum limit and O(a) improved**

Potential systematics

$$\langle x^{n-1}\rangle_h^{\rm MS}(\mu) = \langle x^{m-1}\rangle_h^{\rm MS}(\mu)\frac{c_m(t,\mu)}{c_n(t,\mu)}\frac{\langle x^{n-1}\rangle_h(t)}{\langle x^{m-1}\rangle_h(t)}\,, \quad m\neq n \quad n\geq 3 \; m\geq 2$$

Finite volume effects — at finite a the extension of the local operators is (n-1)a — $n\sim 10-12$

Discretization errors — $\sqrt{8t}\gtrsim na$

Perturbative matching — $\mu = 2\ \mathrm{GeV}$

$$c_n^{\rm NLL}(t,\mu,\overline{g}(\mu)) = c_n(t,q,\overline{g}(q)){\rm exp}\left\{-\int_{\overline{g}(\mu)}^{\overline{g}(q)} dx\,\frac{\gamma_n(x)}{\beta(x)}\right\}$$