Systematic effects in the lattice calculation of inclusive semileptonic decays

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In collaboration with
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Today's agenda

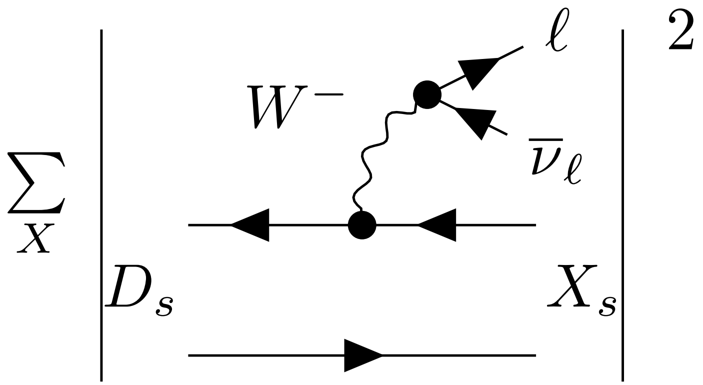
- Quick review on lattice formulation of inclusive decays

- Systematic errors in the analysis
  1. Finite-volume effects
  2. Finite polynomial approximation

- Summary & Outlook

Introduction

On-going analysis of inclusive semileptonic decay rate $D_s \to X_s \ell \nu_\ell$

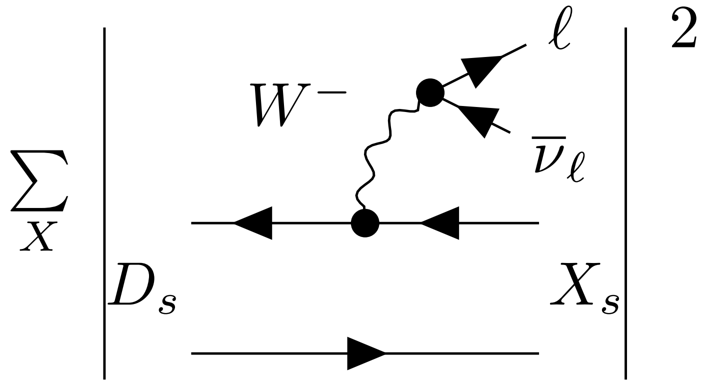


$$\frac{d\Gamma}{dq^2 dq_0^2 dE_l} = \frac{G_F^2 |V_{cs}|^2}{8\pi^2} L_{\mu\nu} W^{\mu\nu}$$

$L_{\mu\nu}$: Leptonic tensor (analytically known)

$W^{\mu\nu}$: Hadronic tensor (nonperturbative QCD)

On-going analysis of inclusive semileptonic decay rate $D_s \to X_s \ell \nu_\ell$



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Idea [P. Gambino & S. Hashimoto, 2005.13730]

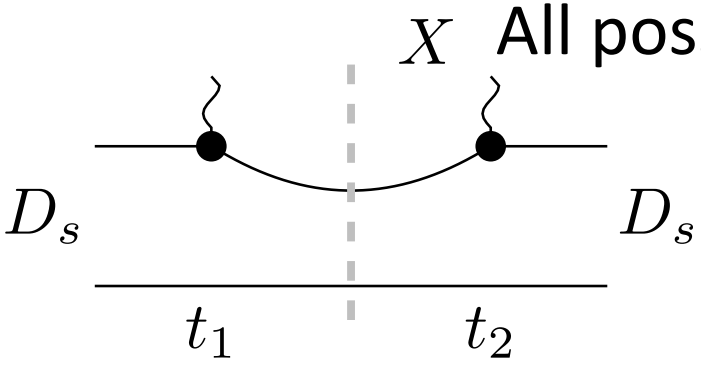
Smeared spectral density

$\rho_s(\omega)$

Smearing ≙ phase space integral

Approximation using **4Pt function** correlation function

All possible states



$$\Gamma \sim \int_0^{q^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, [\,]_{\text{Lattice}}$$

Inclusive Decays - Continuum

Total decay rate [2211.16830, 2305.14092]

$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2} \sum_{l=0}^{2} \bar{X}^{(l)}(\boldsymbol{q}^2)$$

$\bar{X}^{(l)}(\boldsymbol{q}^2)$ integral over energy of hadronic final states

Kernel function

$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega) K^{(l)}_{\mu\nu}(\boldsymbol{q},\omega)$$

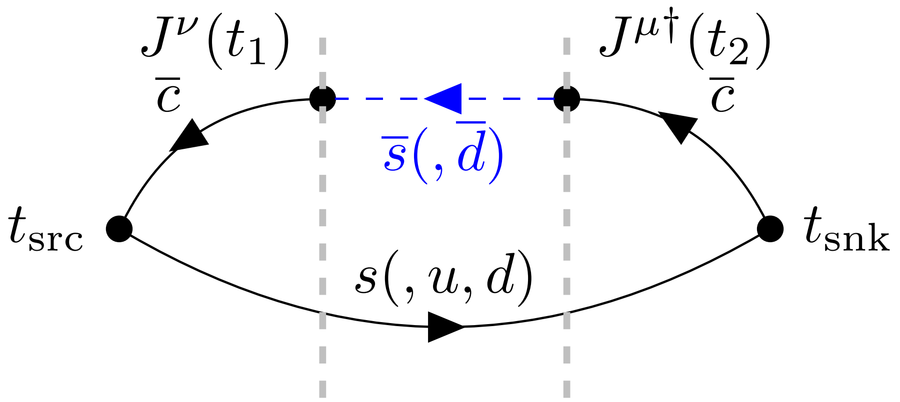
$$k^{(l)}_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max} - \omega)$$

Analytically known

l-th power of ω and $\boldsymbol{q}^2$

Step function

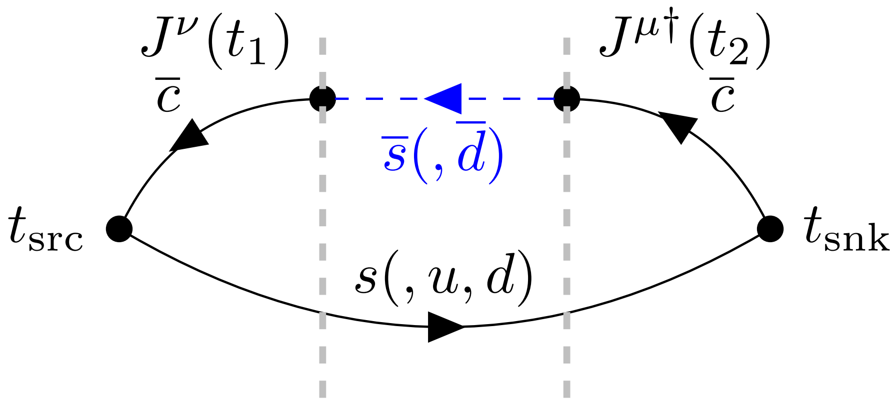
Inclusive decays – Lattice



- t_{src}, t_2, t_{snk} fixed
- $t_{src} \leq t_1 \leq t_2$
- $t = t_2 - t_1$

$$C_{\mu\nu}(\boldsymbol{q}, t) = \int_0^\infty d\omega\, W_{\mu\nu}(\boldsymbol{q}, \omega)\, e^{-\omega t}$$

Inclusive decays – Lattice



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Continuum expression

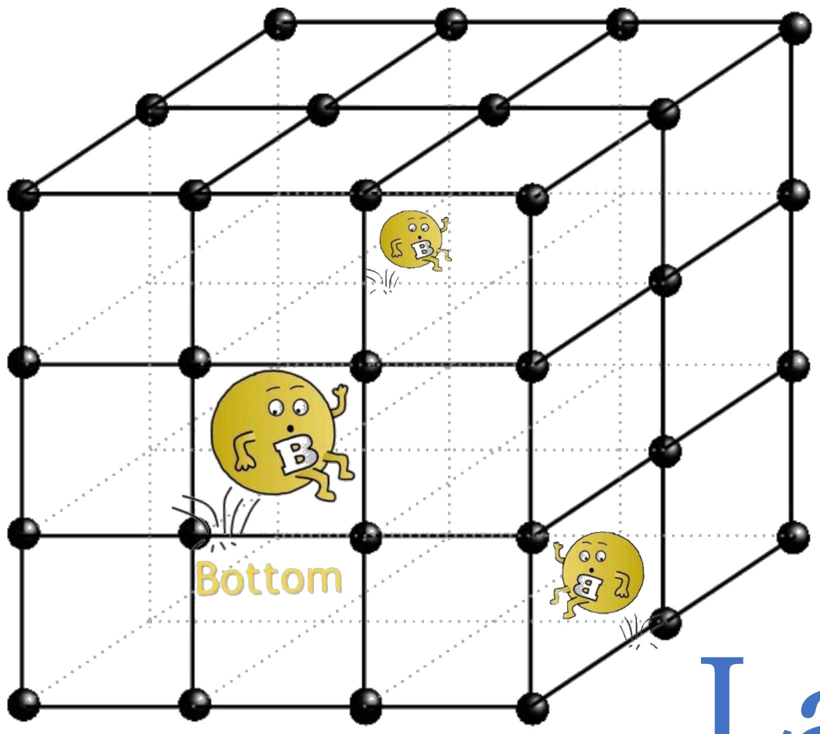
$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) K_{\mu\nu}^{(l)}(\boldsymbol{q}, \omega)$$

Approximate Kernel in polynomials of $e^{-\omega}$

$$K(\omega, \boldsymbol{q}) \simeq k_0 + k_1 e^{-\omega} + \ldots + k_N e^{-N\omega}$$

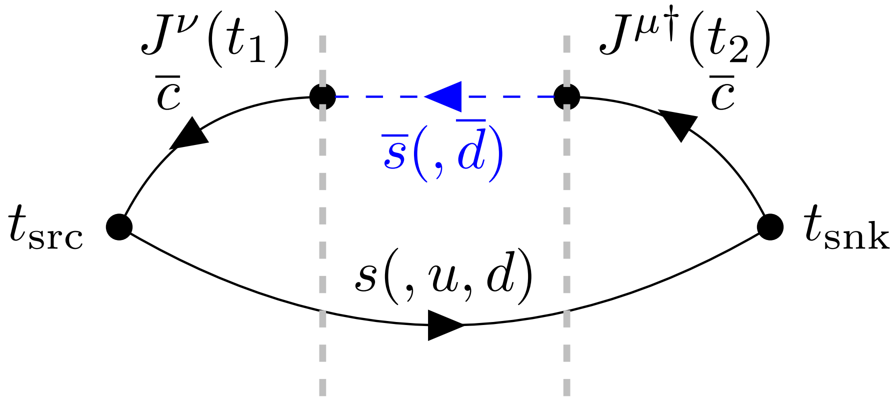
C(0) C(N)

$$\bar{X}^{(l)}(\boldsymbol{q}^2) \sim k_0 \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) + \cdots + k_N \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) e^{-N\omega}$$



Lattice Setup

Simulations conducted on Fugaku using Grid [P. Boyle et al., https://github.com/paboyle/Grid] and Hadrons [A. Portelli et al., https://github.com/aportelli/Hadrons] software packages



Lattice setup:

- Lattice size: $48^3 \times 96$
- Lattice Spacing: $a = 0.055$ fm
- $M_\pi \simeq 300$ MeV

Simulation:

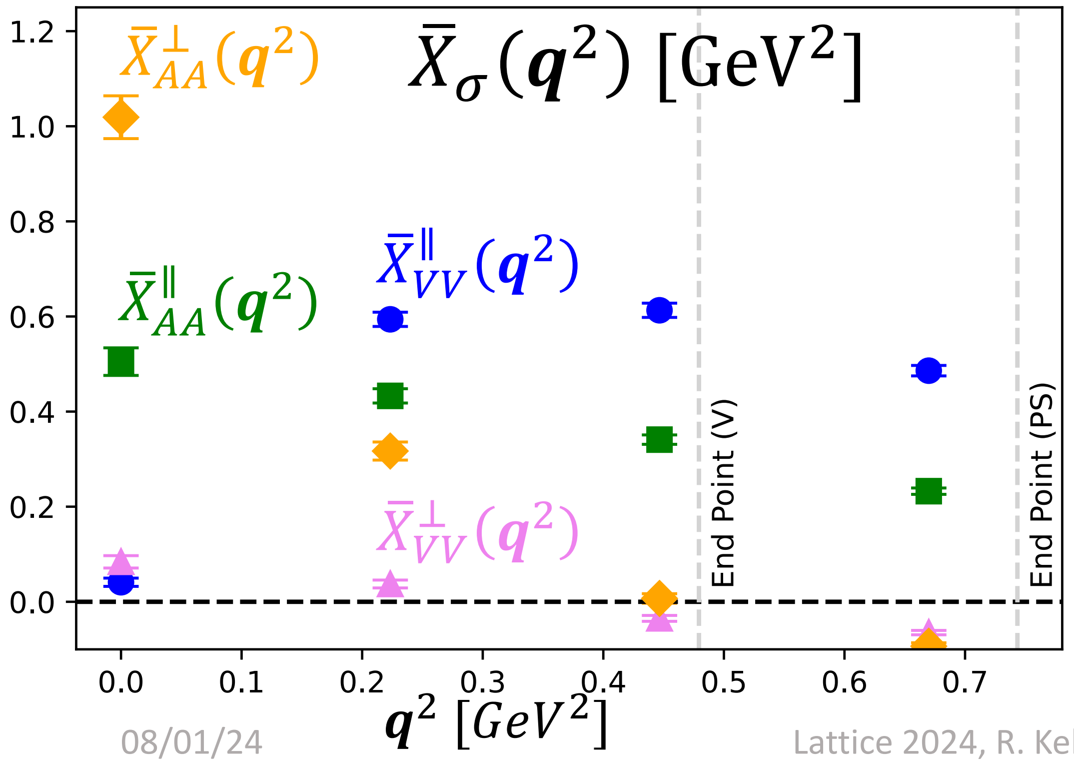
- 2+1 Möbius domain-wall fermions
- s, c quarks simulated at near-physical values
- Cover whole kinematical region $\boldsymbol{q} = (0,0,0) \rightarrow (1,1,1)$

Numerical Results

The differential rate $\bar{X} \sim \frac{d\Gamma}{d\boldsymbol{q}^2}$

$$\bar{X}_\sigma(\boldsymbol{q}^2) = \sum_{l=0}^{2} \left\langle D_s(\boldsymbol{0}) \middle| \tilde{J}_\mu^\dagger(-\boldsymbol{q}) K_\sigma^{(l)}(\hat{H}, \boldsymbol{q}^2) \tilde{J}_\nu(\boldsymbol{q}) \middle| D_s(\boldsymbol{0}) \right\rangle$$

$N = 10, \sigma = 0.1$



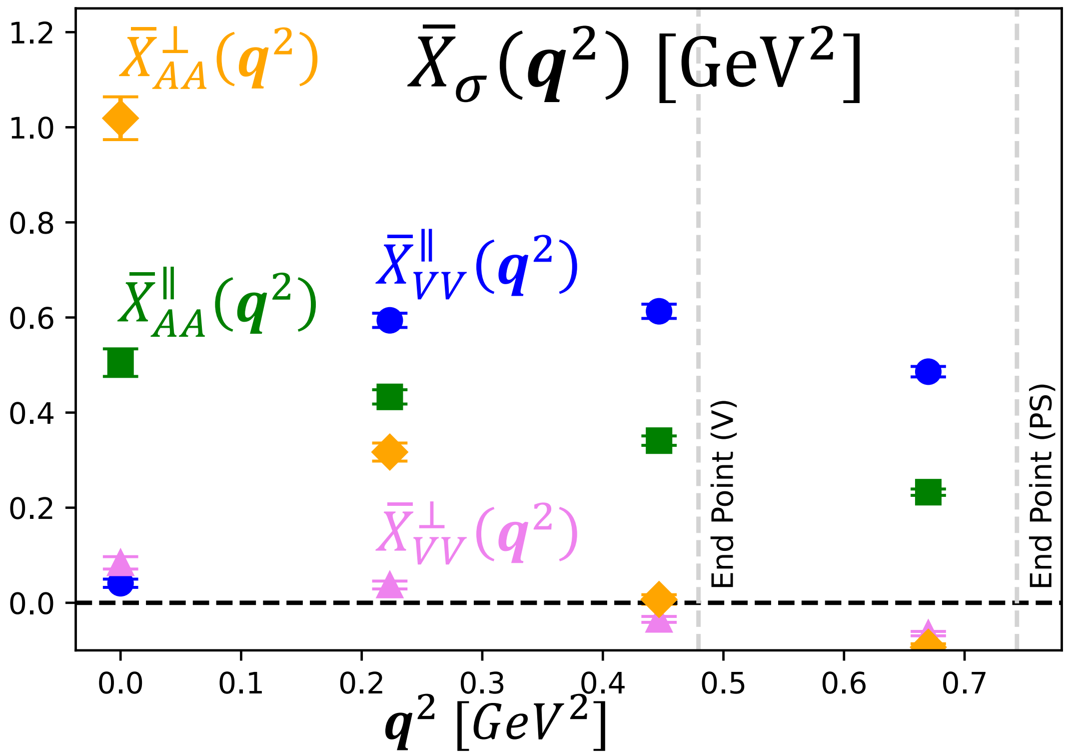
Decomposed $\bar{X}_\sigma(\boldsymbol{q}^2)$:

- Vector (VV) & Axial-vector (AA)
- ∥ and ⊥ polarization with respect to $\boldsymbol{q}$

Decomposition allows for comparison with ground state limit

Systematic errors - Finite volume

$N = 10, \sigma = 0.1$



Infinite volume limit? [2312.16442]

- In finite volume spectral density is a sum of delta peaks

Computing $\bar{X}_{\sigma}(\boldsymbol{q}^2)$ requires ordered limits

$$\lim_{\sigma \to 0} \lim_{V \to \infty} \bar{X}_{\sigma}(\boldsymbol{q}^2)$$

Necessary data not available

➡ Estimate finite-volume effects using a model (non-interacting two-body states)

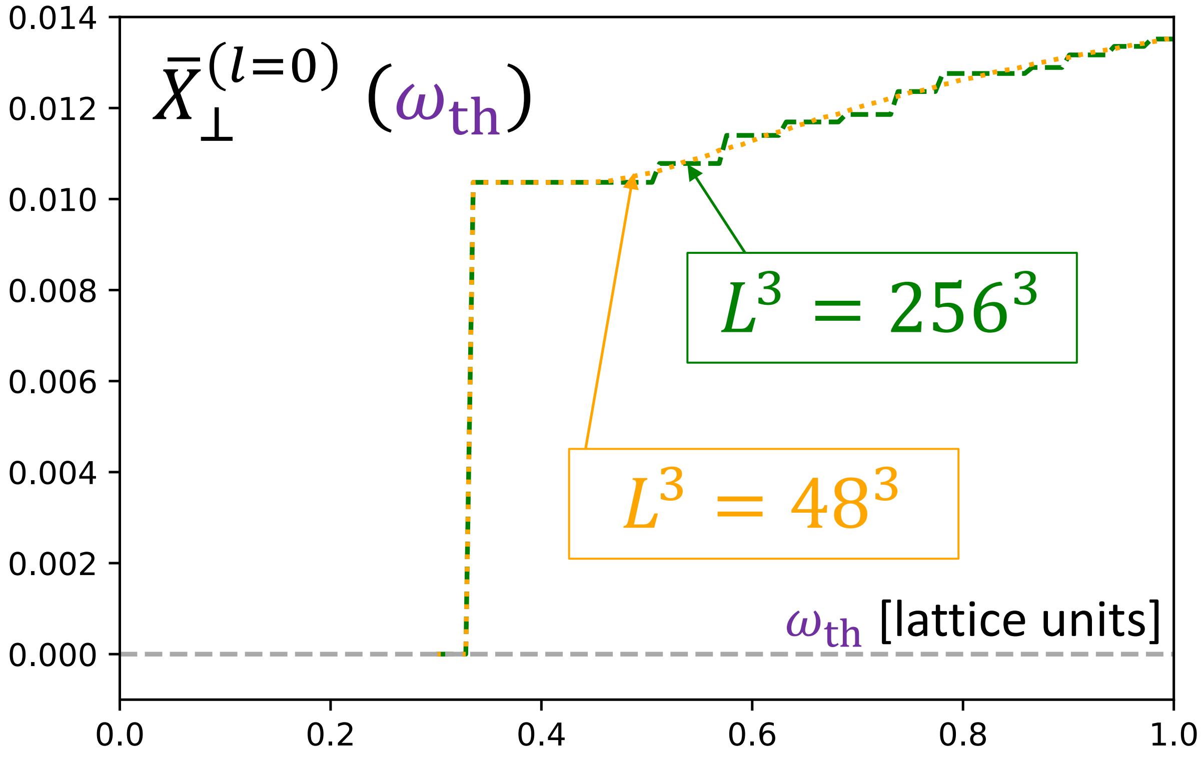
Finite volume – Model analysis

$\bar{X}^{\perp}_{AA}(\boldsymbol{q}^2)$ for $\boldsymbol{q} = (0,0,1)$

Model-based results

$$\bar{X}^{(l)}(\boldsymbol{q}^2) \sim \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega) k_{\sigma}^{(l)}(\boldsymbol{q},\omega)\theta(\omega_{\text{th}} - \omega)$$

Test by (artificially) varying the upper limit of the integral



- Heaviside function
  - ➢ Slight volume dependence

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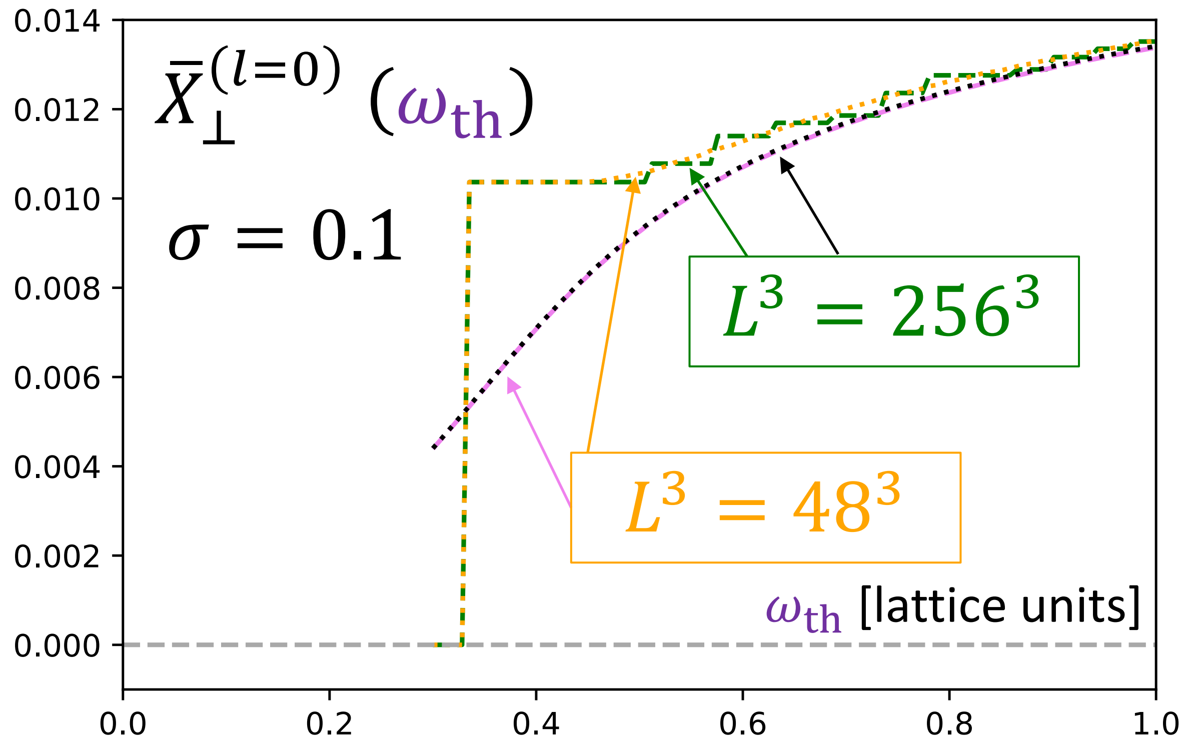
Test by (artificially) varying the upper limit of the integral



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\+ apply smearing

  - ➢ Volume dependence washes out

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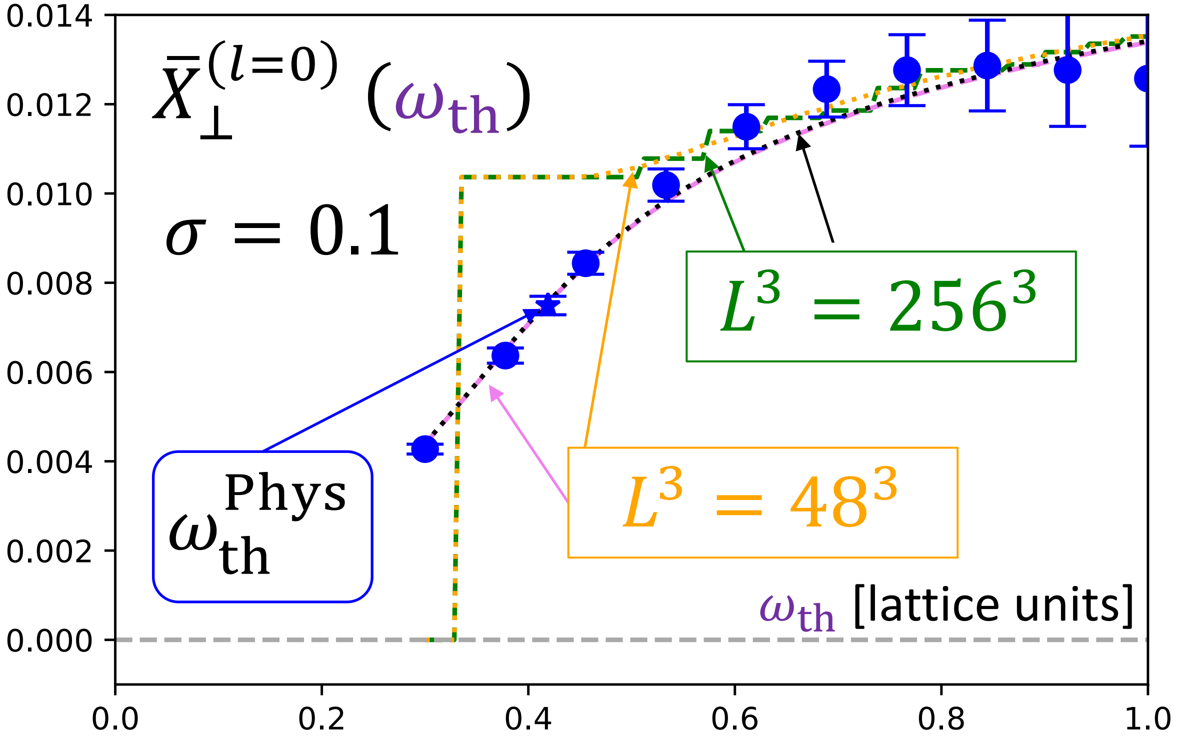


- Heaviside function
  - ➢ Slight volume dependence

+ apply smearing
  - ➢ Volume dependence washes out

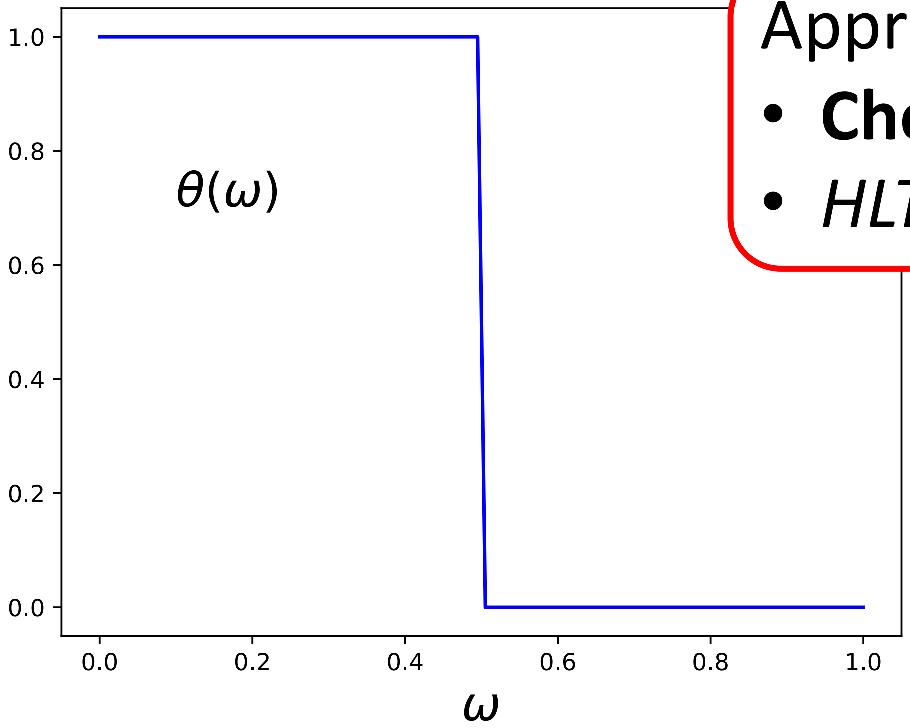
+ include lattice data
  - ➢ Nicely follows model prediction

Systematic error – kernel approximation

Upper limit of the energy integral

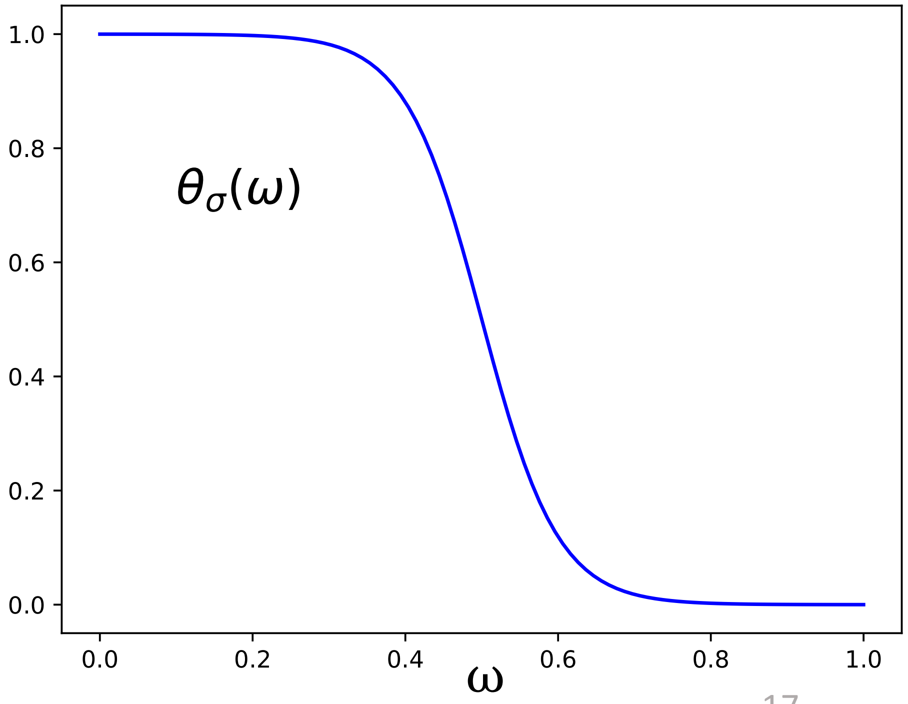
Approximation strategies [1903.06476, 2305.14092] :

- **Chebyshev approximation**
- *HLT approach (A. De Santis, Tue. Jul. 30th, 3:05 PM)*



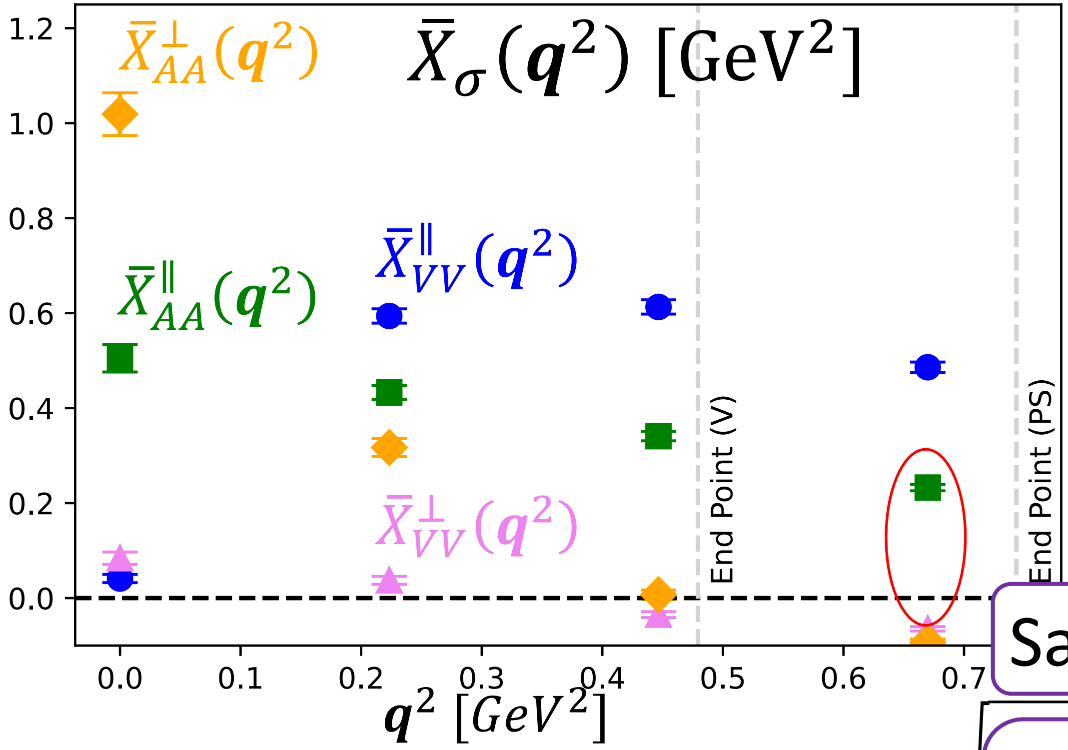
Direct approximation with $e^{-\omega(t_2-t_1)}$ not possible

Apply smearing



Systematic error - Approximation

$N = 10, \sigma = 0.1$



Create estimate [2211.16830]

- $N \to \infty$; frequency component
- $\sigma \to 0$; width

$$\sigma = \frac{1}{N}$$

Property of Chebyshev polynomials

$$\left|\langle \tilde{T}_k(\omega) \rangle\right| \leq 1$$

Estimate error from

$$\sqrt{\mathrm{var}\left(\sum_{j=N_{\mathrm{cut}}}^{N} \tilde{c}_j^{(l)} \tilde{T}_j\right)}$$

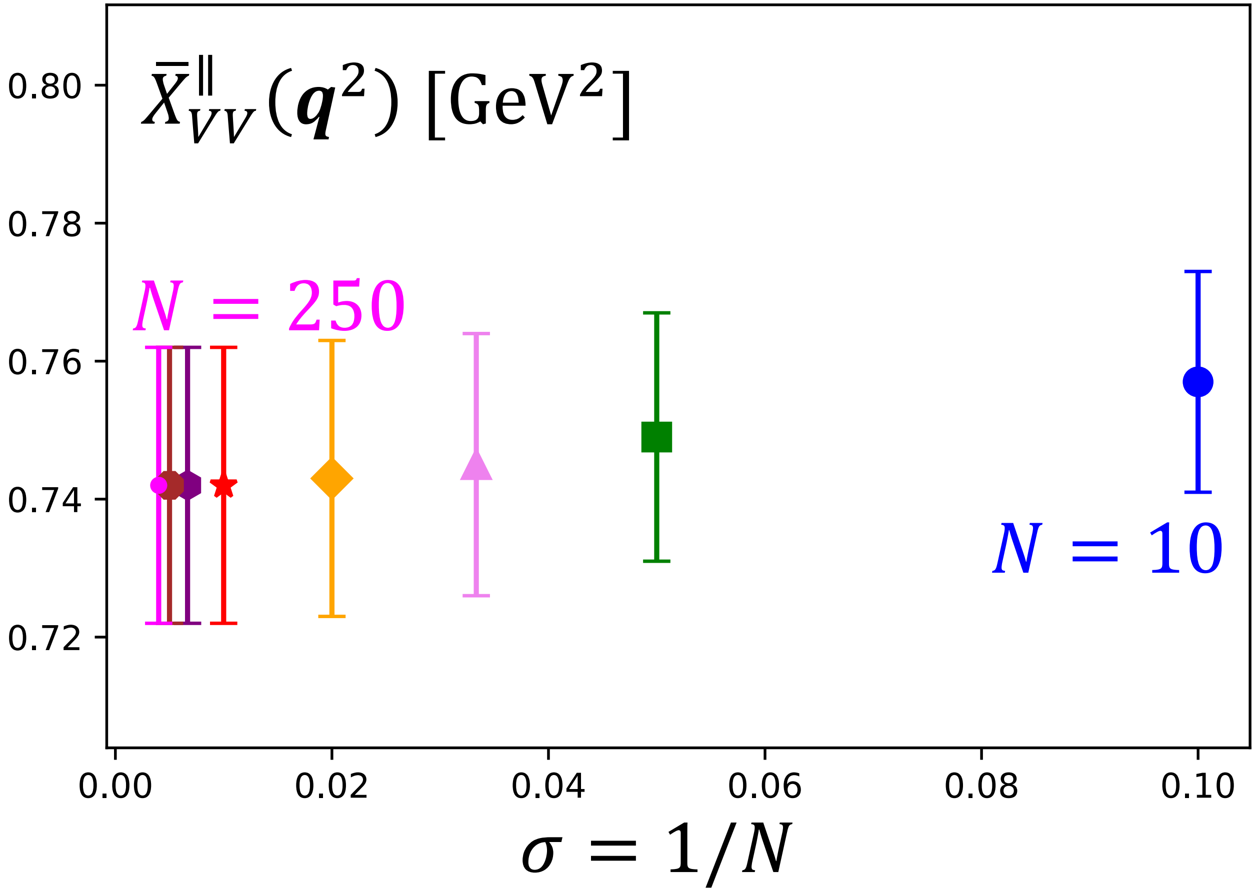
Sample size: 1000

Random variable taken from uniform distribution in $[-1; 1]$

Analytically known

Systematic error - Approximation

Application for $\bar{X}_{VV}^{\parallel}(\boldsymbol{q}^2)$ for $\boldsymbol{q} = (0,0,1)$



In the $\sigma \to 0$ limit:

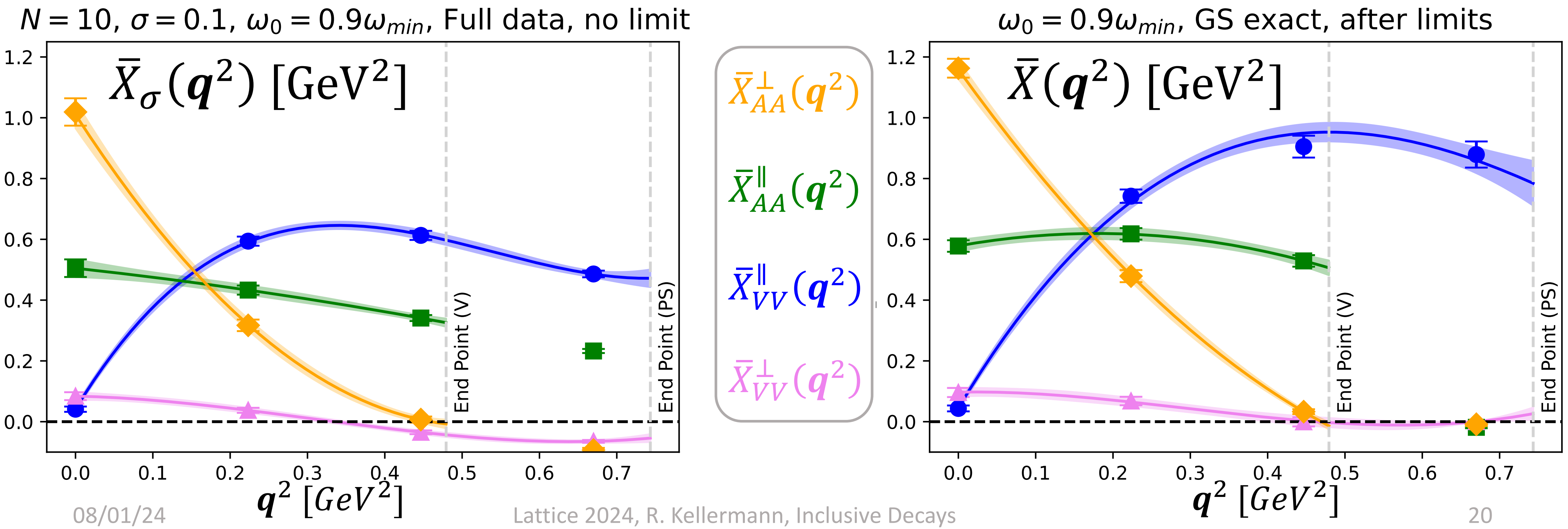
1. Slight shift in central values
   - Due to dependence of $\tilde{c}_j^{(l)}$ on σ
2. Minor increase in errors that nicely converges

Estimating the systematic corrections

Channels:

1. AA: infinite-volume limit
2. VV: finite-volume corrections expected small; only $\sigma \to 0$ limit

+ subtr. Ground state from correlator and assume as exact



Summary & Outlook

Summary

- Study into systematic effects in the inclusive analysis of semileptonic decays on the lattice
  - Error from Chebyshev polynomial approximation
    - Obtained a better estimate following the first idea
  - Finite volume corrections
    - Work out further details; supplement with data
- Publication in work (hopefully this year)

Outlook

- Discretization effects & continuum limit need to be addressed
- Extend towards a full analysis in the bottom sector
- Extend to different observables, e.g. moments
  - Increase pool for comparison to experiment and continuum theory predictions, e.g. OPE
- P-wave form factors from inclusive lattice simulation (Zhi Hu’s talk, 08/01 11:30)

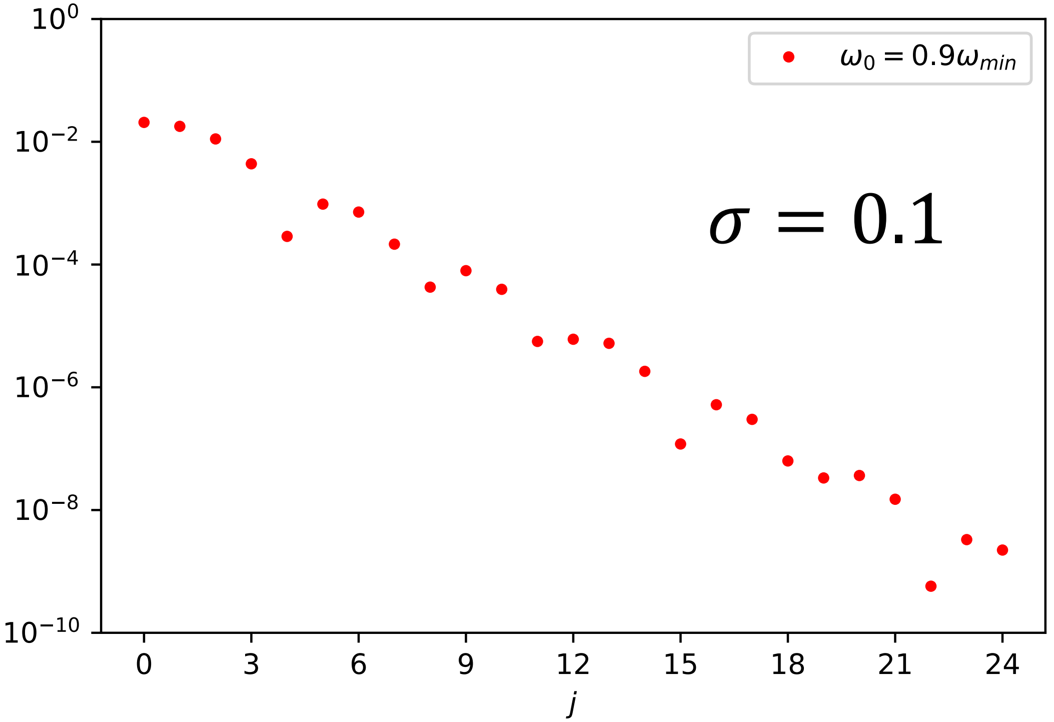
Back-up

Systematic errors - Approximation

$\boldsymbol{q}^2 = 0.66\ \mathrm{GeV}^2$ $\omega_0 = 0.9\omega_{\mathrm{min}},$

Coefficients for kernel with $l = 0$

$\left|\tilde{c}^{(0)}_{j,0.1}\right|$



$\left|\tilde{c}^{(0)}_{j,0.01}\right|$