Gluon nonlocal operator mixing in lattice QCD

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Related publication

Gavriel D., Panagopoulos H., Spanoudes G.,

``Renormalization of nonlocal gluon operators on the lattice'',

arXiv:2406.15446 [hep-lat], to appear in PRD next week

Renormalization of nonlocal gluon operators on the lattice

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We study the renormalization of a complete set of gauge-invariant gluon nonlocal operators in lattice perturbation theory. We determine the mixing pattern under renormalization of these operators using symmetry arguments, which extend beyond perturbation theory. Additionally, we derive the renormalization factors of the operators within the modified Minimal Subtraction ($\overline{\text{MS}}$) scheme up to one-loop. To enable a non-perturbative renormalization procedure, we investigate a suitable version of the modified regularization-invariant (RI$'$) scheme, and we calculate the conversion factors from that scheme to $\overline{\text{MS}}$. The computations are performed by employing both dimensional and lattice regularizations, using the Wilson gluon action. This work is relevant to nonperturbative studies of the gluon parton distribution functions (PDFs) on the lattice.

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I. INTRODUCTION

The QCD factorization theorems [1] allow the expression of the cross-sections for various high-energy hard processes as a convolution of a process-dependent hard scattering coefficient, computable in perturbation theory, and a parton distribution function (PDF). PDFs characterize the non-perturbative aspect of these processes, offering insights into

Gluon quasi-PDFs

Studies of gluon PDFs has been relatively limited compared to quark PDFs, however:

- Gluonic contributions make a significant impact on the proton's spin [C. Alexandrou et al., Phys. Rev. D 101, 094513 (2020)].
- Phenomenological data suggest that gluon PDFs dominate over quark PDFs in the small-x region [S. Alekhin, J. Blümlein, S. Moch, Phys. Rev. D 89, 054028 (2014)].
- Global analysis finds that accurate calculations of gluon-dependent quantities are essential for the cross-section of Higgs boson production, heavy quarkonium and jet production [J. Butterworth et al., J. Phys. G 43, 023001 (2016)].

- Motivation
- Theoretical setup
- Results
- Conclusion

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- Global analysis finds that accurate calculations of gluon-dependent quantities are essential for the cross-section of Higgs boson production, heavy quarkonium and jet production [J. Butterworth et al., J. Phys. G 43, 023001 (2016)].

One approach for the calculation of the x-dependence of PDFs using lattice QCD is the quasi-distribution method [X. Ji, Phys. Rev. Lett. 110, 262002 (2013), X. Ji, Sci. China Phys. Mech. Astron. 57, 1407 (2014)]; employs the large momentum effective theory (LaMET).

- Quasi-PDFs are defined as matrix elements of momentum-boosted hadrons coupled to gauge-invariant nonlocal operators, which includes a finite length Wilson line.

An important aspect of calculating quasi-PDFs is their **renormalization.**

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Gluon nonlocal operators

The nonlocal gluon operators under study are defined in the fundamental representation as:

$$\boxed{O_{\mu\nu\rho\sigma}(x+z\hat{\tau},x) \equiv 2\ \mathrm{Tr}\Big(F_{\mu\nu}(x+z\hat{\tau})W(x+z\hat{\tau},x)F_{\rho\sigma}(x)W(x,x+z\hat{\tau})\Big)}$$

- $F_{\mu\nu}$ is the gluon field strength tensor
- $W(\ x, x+z\hat{\tau})$ is the straight Wilson line with length z. Its expression is given by the path-ordered ($\mathcal{P}$) exponential of the gauge field A_μ as follows:

$$\boxed{W(x,x+z\hat{\tau}) \equiv \mathcal{P}\ \exp\left[ig\int_0^z A_\mu(x+\zeta\hat{\tau})\,d\zeta\right]}$$

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Due to the antisymmetry of $F_{\mu\nu}$, by selecting the indices of $O_{\mu\nu\rho\sigma}$ to be in any direction:

→There are 36 nonlocal operators in total

Renormalization of operators

In general, the nonlocal gluon operators may undergo mixing under renormalization.
→Their mixing pattern can be determined by the symmetries of the theory.

We define the renormalization mixing matrix Z:

$$O^R_{(i)} = \sum_j \left(Z^{-1}\right)_{ij} O_{(j)}$$

we use i and j as generic indices, to list operators within a mixing set.

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Renormalization schemes:

- Modified minimal subtraction ($\overline{MS}$) scheme, typically employed in phenomenological studies. Cannot be used in nonperturbative studies due to its perturbative nature.
- Modified regularization-invariant (RI′) scheme, applicable in both nonperturbative and perturbative studies.

→ Nonperturbative renormalization factors are calculated in a suitably defined variant of the RI′ scheme.

Then they can be converted to the $\overline{MS}$ scheme through conversion factors which can only be determined using perturbation theory and are regularization independent:

$$C^{\overline{\mathrm{MS}},\mathrm{RI}'} \equiv \left(Z^{\mathrm{LR},\overline{\mathrm{MS}}}\right)^{-1}\left(Z^{\mathrm{LR},\mathrm{RI}'}\right) = \left(Z^{\mathrm{DR},\overline{\mathrm{MS}}}\right)^{-1}\left(Z^{\mathrm{DR},\mathrm{RI}'}\right)$$

Convenient to use dimensional regularization (DR) instead of lattice regularization (LR).

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Lattice action

We consider a nonabelian gauge theory of $SU(N_c)$ group and N_f multiplets of fermions.

Full action:

$$\boxed{S = \frac{2}{g^2} \sum_{\text{plaq.}} \mathrm{Re}\,\mathrm{Tr}\left\{1 - U_{\text{plaq.}}\right\} + S_F}$$

Terms :

- Gluons: Wilson plaquette gauge action.
- Fermions: clover-improved Wilson action (c_{SW} term).

$F_{\mu\nu}$ is determined by the standard clover discretization.

Standard lattice discretization of the Wilson line, using gluon links $U_\tau\,(x)$:

$$W(x, x + z\hat{\tau}) = \prod_{\ell=0}^{n\mp1} U_{\pm\tau}(x + \ell a\hat{\tau})\,, \qquad n \equiv z/a$$

where $U_{-\tau}\,(x) \equiv U_\tau^\dagger\,(x - a\hat{\tau})$ and upper (lower) signs correspond to $n > 0$ ($n < 0$).

Symmetry properties

Possible mixing with similar operators:

It can be shown that:

- No mixing with non-gauge invariant operators.
- No mixing with operators involving alternative paths for the Wilson line joining gluon field-strength tensor.
- No mixing with nonlocal fermion operators.

=> All mixing operators will necessarily be of the same form as the original operator, possibly with different values for the Lorentz indices μ, ν, ρ, σ.

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C, P, T transformations:

- Under charge conjugation the operators remain invariant.
- There are 4 "parity" transformations (reflections P_1, P_2, P_3 , P_4 about each of the 4 axes); one these alters the sign of z in the operator.

Taking advantage of the translation invariance of the Lagrangian and the cyclic permutations on the trace of the operators , we perform a change of basis:

$$\boxed{O^{\pm}_{\mu\nu\rho\sigma}(z,0) = \frac{1}{2}\left(O_{\mu\nu\rho\sigma}(z,0) \pm O_{\rho\sigma\mu\nu}(z,0)\right)}$$

These operators are eigenstates of parity transformations (performed with respect to the midpoint of the operators) with eigenvalues of +1 (even, E) or −1 (odd, O).

Symmetry properties

Rotational octahedral point group:

Symmetry group that describes the discrete rotational symmetries of an octahedron or a cube. [since the Wilson line is chosen to lie along the z direction]

Consists of 24 elements, corresponding to rotations by various angles with respect to different axes.

It possesses five irreducible representations.

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By exploring whether the operators support the same irreducible representations and combining our parity transformations findings, we arrange the 36 operators into **16 groups**, as shown in the Table.

Group	Operators	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	Irreducible Repr.
1	$O^+_{3131}+O^+_{3232}+O^+_{3434}$	E	E	E	E	A_1
2	$O^+_{1212}+O^+_{1414}+O^+_{2424}$	E	E	E	E	A_1
3	$\begin{pmatrix} 2O^+_{3434}-O^+_{3131}-O^+_{3232} \\ O^+_{3131}-O^+_{3232} \end{pmatrix}$	E	E	E	E	E
4	$\begin{pmatrix} 2O^+_{1212}-O^+_{1414}-O^+_{2424} \\ O^+_{1414}-O^+_{2424} \end{pmatrix}$	E	E	E	E	E
5	$\begin{pmatrix} O^-_{3132} \\ O^-_{3431} \\ O^-_{3234} \end{pmatrix}$	E E O	E O E	E E E	O E E	T_1
6	$\begin{pmatrix} O^-_{4142} \\ O^-_{2421} \\ O^-_{1214} \end{pmatrix}$	E E O	E O E	E E E	O E E	T_1
7	$\begin{pmatrix} O^+_{3132} \\ O^+_{3431} \\ O^+_{3234} \end{pmatrix}$	O O E	O E O	E E E	E O O	T_2
8	$\begin{pmatrix} O^+_{4142} \\ O^+_{2421} \\ O^+_{1214} \end{pmatrix}$	O O E	O E O	E E E	E O O	T_2
9	$\begin{pmatrix} O^+_{3212}+O^+_{3414} \\ O^+_{3121}+O^+_{3424} \\ O^+_{3141}+O^+_{3242} \end{pmatrix}$	O E E	E O E	O O O	E E O	T_1
10	$\begin{pmatrix} O^-_{3212}+O^-_{3414} \\ O^-_{3121}+O^-_{3424} \\ O^-_{3141}+O^-_{3242} \end{pmatrix}$	E O O	O E O	O O O	O O E	T_1
11	$\begin{pmatrix} O^+_{3212}-O^+_{3414} \\ O^+_{3121}-O^+_{3424} \\ O^+_{3141}-O^+_{3242} \end{pmatrix}$	O E E	E O E	O O O	E E O	T_2
12	$\begin{pmatrix} O^-_{3212}-O^-_{3414} \\ O^-_{3121}-O^-_{3424} \\ O^-_{3141}-O^-_{3242} \end{pmatrix}$	E O O	O E O	O O O	O O E	T_2
13	$O^+_{3124}+O^+_{3241}+O^+_{3412}$	O	O	O	O	A_1
14	$O^-_{3124}+O^-_{3241}+O^-_{3412}$	E	E	O	E	A_1
15	$\begin{pmatrix} 2O^+_{3412}-O^+_{3241}-O^+_{3124} \\ O^+_{3124}-O^+_{3241} \end{pmatrix}$	O	O	O	O	E
16	$\begin{pmatrix} 2O^-_{3412}-O^-_{3241}-O^-_{3124} \\ O^-_{3124}-O^-_{3241} \end{pmatrix}$	E	E	O	E	E

Symmetry properties

Key points:

- Operators in groups {1, 2}, {3, 4}, {5, 6}, {7, 8} have the potential to mix under renormalization. They share the same behavior under parity transformations and under the octahedral group.

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	O^-_{3431}	E	O	E	E	
	O^-_{3234}	O	E	E	E	
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	O^-_{2421}	E	O	E	E	
	O^-_{1214}	O	E	E	E	
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	O^+_{3431}	O	E	E	O	
	O^+_{3234}	E	O	E	O	
8	O^+_{4142}	O	O	E	E	T_2
	O^+_{2421}	O	E	E	O	
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	$O^-_{3121} + O^-_{3424}$	O	E	O	O	
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- Operators in groups 9-16 cannot possibly mix; they multiplicatively renormalize.

The above findings, **are valid beyond perturbation theory.** [based on symmetry properties alone].

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Perturbative calculation of Green's functions

To study the renormalization of the gluon nonlocal operators:

We calculate the following one-particle-irreducible (1-PI) two-point bare amputated Green's functions using dimensional regularization (DR) and lattice regularization (LR):

$$\boxed{\delta^{(4)}(q+q')\ \Lambda_O(q,z) = \langle A^a_\alpha(q)\ \left(\int d^4x\, O_{\mu\nu\rho\sigma}(x+z\hat{\tau},x)\right)\ A^b_\beta(q')\rangle_{\rm amp}}$$

Calculations are more complicated than local operators due to the presence of both the external momentum (q) and the length of the Wilson line (z) in the integrands.

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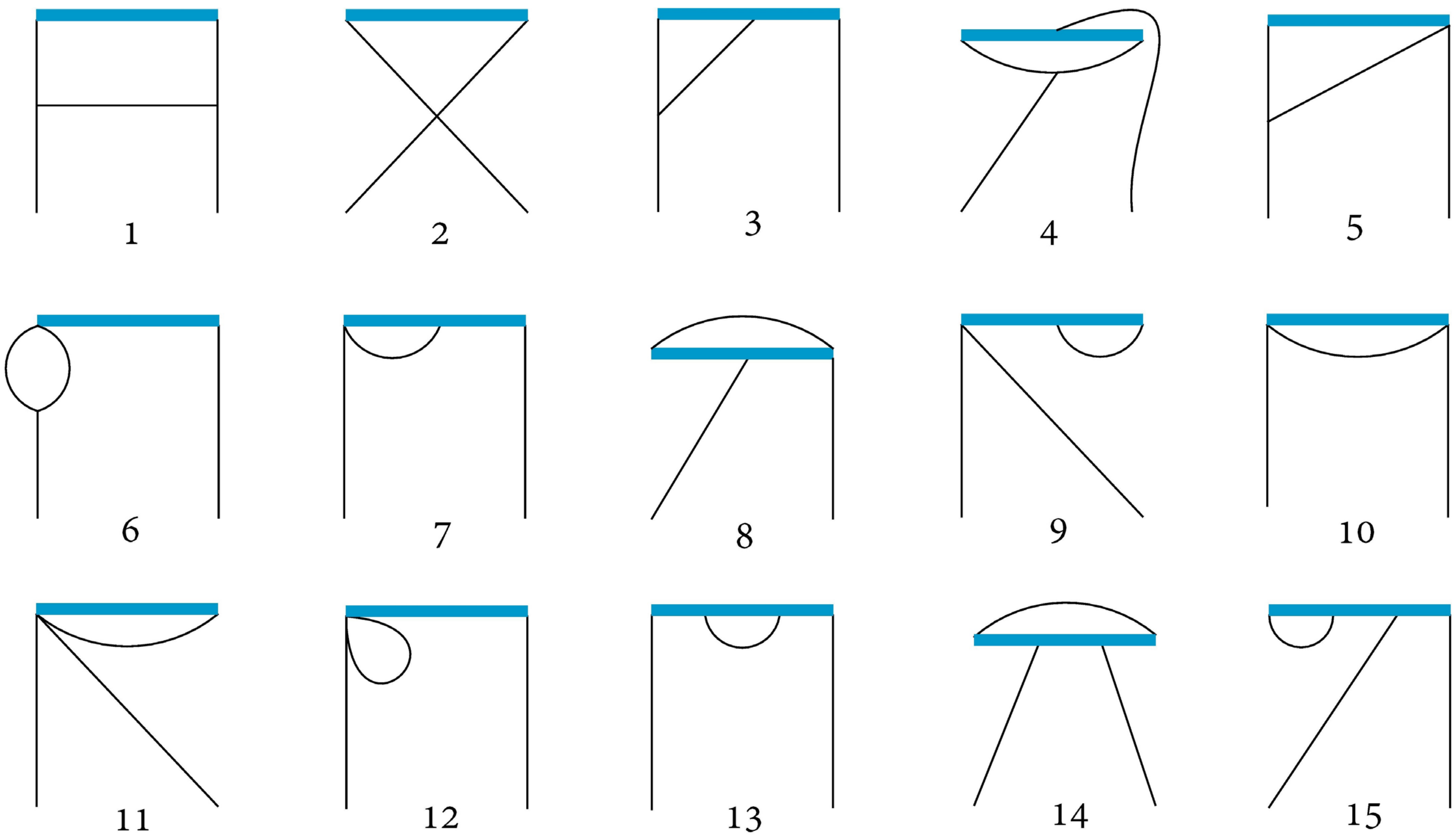

Fig.4: One-loop Feynman diagrams of the Green's functions of the gluon nonlocal operators. [Solid lines represent gluons. The operator insertion is denoted by a blue solid box.]

One loop calculations in DR

The renormalization matrix of the operators turns out to be diagonal, both in the original basis ($O_{\mu\nu\rho\sigma}$) and in the basis of symmetries table (16 groups).

Using the $\overline{MS}$ renormalization condition and the 1-loop results, we find:

$$\boxed{Z_{ij}^{\mathrm{DR},\overline{\mathrm{MS}}} = \delta_{ij}\left[1 + \frac{g^2}{16\epsilon\pi^2}\left(\left(\frac{5}{3}+\omega_i\right)N_c - \frac{2}{3}N_f\right)\right]}$$

where $\omega_i = \begin{cases} 0 & \text{for } i = 2,4,6,8 \\ 1 & \text{for } i = \text{9-16} \\ 2 & \text{for } i = 1,3,5,7 \end{cases}$

[Groups containing multiplets share the same renormalization factor for each component within the multiplet]

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where $\omega_i = \begin{cases} 0 & \text{for } i = 2,4,6,8 \\ 1 & \text{for } i = 9\text{-}16 \\ 2 & \text{for } i = 1,3,5,7 \end{cases}$

[Groups containing multiplets share the same renormalization factor for each component within the multiplet]

<u>This result:</u>

- is independent of gauge parameter (β) . It is expected from gauge invariance in $\overline{MS}$.
- is independent of the length of the Wilson line (z).
- indicates no operator mixing of equal or lower dimension.

RI′ renormalization prescription

Significant flexibility in defining renormalization conditions in a RI′ scheme, particularly in cases of operator mixing.

A practical choice for a RI′-like scheme suitable for nonperturbative studies is to consider:

- four 2×2 mixing matrices (for each of the four 2-element mixing groups).
- eight 1×1 matrices (for operators that are multiplicatively renormalizable).

→We need to impose several conditions to identify the elements of these matrices.

In our proposed conditions:

- We select certain values for the Lorentz indices α, β of the external gluons.
- We set specific components of the RI′ renormalization scale ($\bar{q}$) to zero.

With this choice, we aim to create a solvable system of conditions and achieve simpler expressions. Other options can be tested by using our results on the full Green's functions.

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Using the renormalization conditions of the RI′ scheme and the $\overline{MS}$ -renormalized Green's functions we derive the conversion factors of the operators.

→ They are complex expressions including integrals of modified Bessel functions of the second kind, K_n, over a Feynman parameter and/or over variables running along the Wilson line.

Insights by plotting the conversion factors for the parameters used in lattice simulations.

For the $N_f = 2+1+1$ ensemble of twisted-mass clover-improved fermions and Iwasaki-improved gluons (cA211.30.32) [C. Alexandrou et al., Phys. Rev. D 98, 054518 (2018)]:

- $\overline{MS}$ scale is fixed at $\bar{\mu} = 2\ GeV$
- lattice volume is $L^3 \times T$ with $L = 32$ and $T = 64$ (in lattice units)
- lattice spacing is $a = 0.0938\ fm$
- $g^2 = 3.47625$
- $\beta = 1$ (Landau gauge)

RI′ scale is defined as $a\bar{q} = (\frac{2\pi}{L}\ n_1, \frac{2\pi}{L}\ n_2, \frac{2\pi}{L}\ n_3, \frac{2\pi}{L}\ (n_4 + \frac{1}{2}))$, where n_i are integers.

Conversion factors

The plots:

- Show the real part of the conversion factors as a function of z/a [imaginary part is strictly zero or negligible (10^{-5})]
- Highlight data points at integer values of z/a ranging from 1 to $L/2 = 16$, while dashed lines connecting these points display the conversion factors for arbitrary noninteger values of z/a (value at $z/a = 0$ has been excluded)
- Do not show negative values of z/a as the conversion factors are symmetric with respect to $z = 0$.
- Depending on the choice of $\bar{q}$ the numerical values of the conversion factors can be excessively large (must tune $\bar{q}$ accordingly).

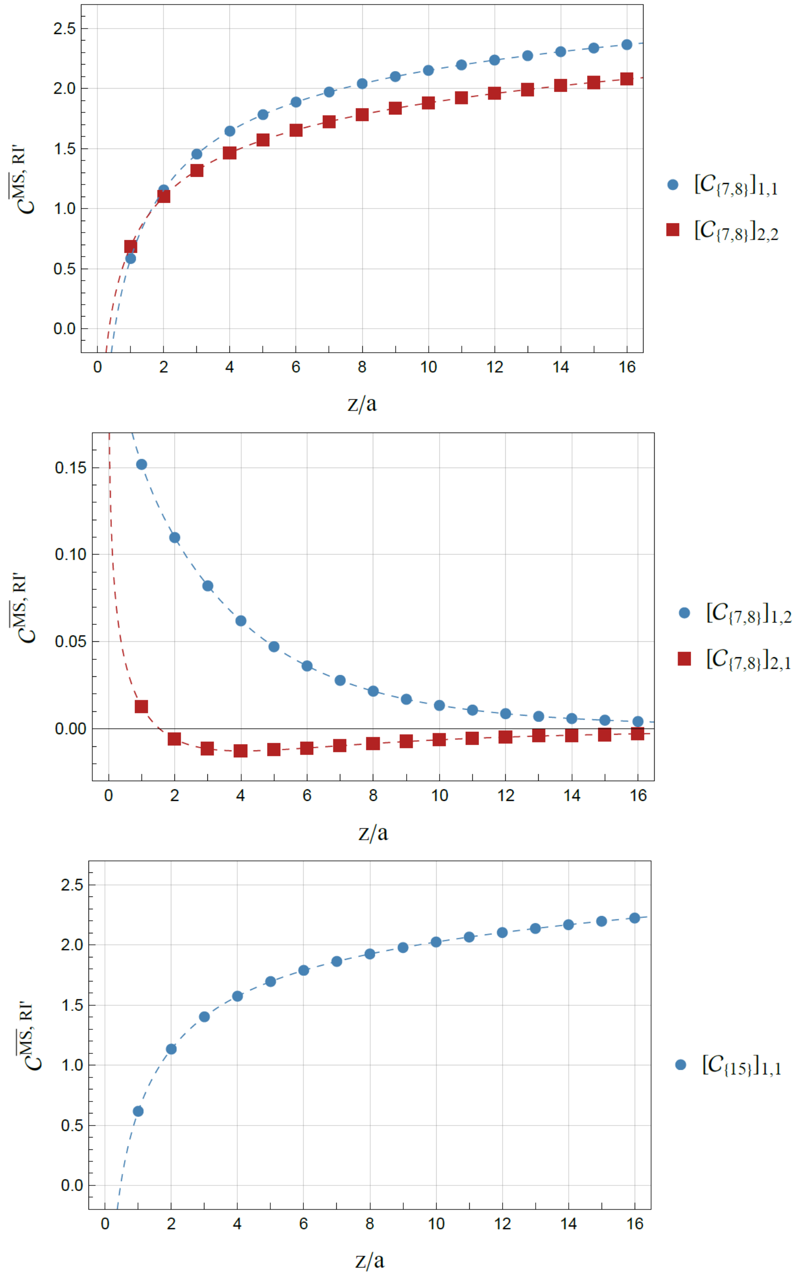

Fig.5: Conversion factor of ‘plus-type’ operators as a function of z/a

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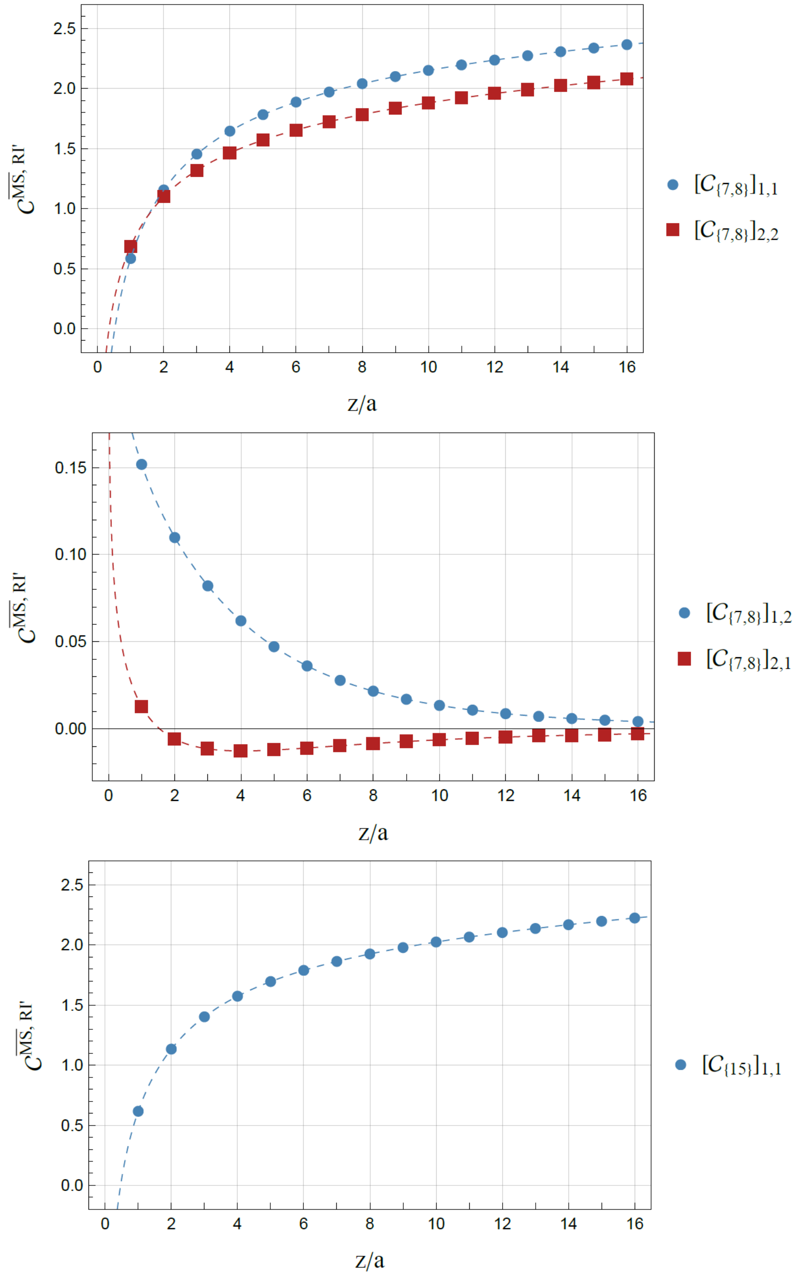
As an example, for the ‘plus-type’ operators (mixing pairs {1, 2}, {3, 4}, {7, 8}, and operators 9,11, and 15):

- We set $n_1 = n_2 = 3$, $n_3 = 0$, and $n_4 = -1/2$.



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Conversion factors

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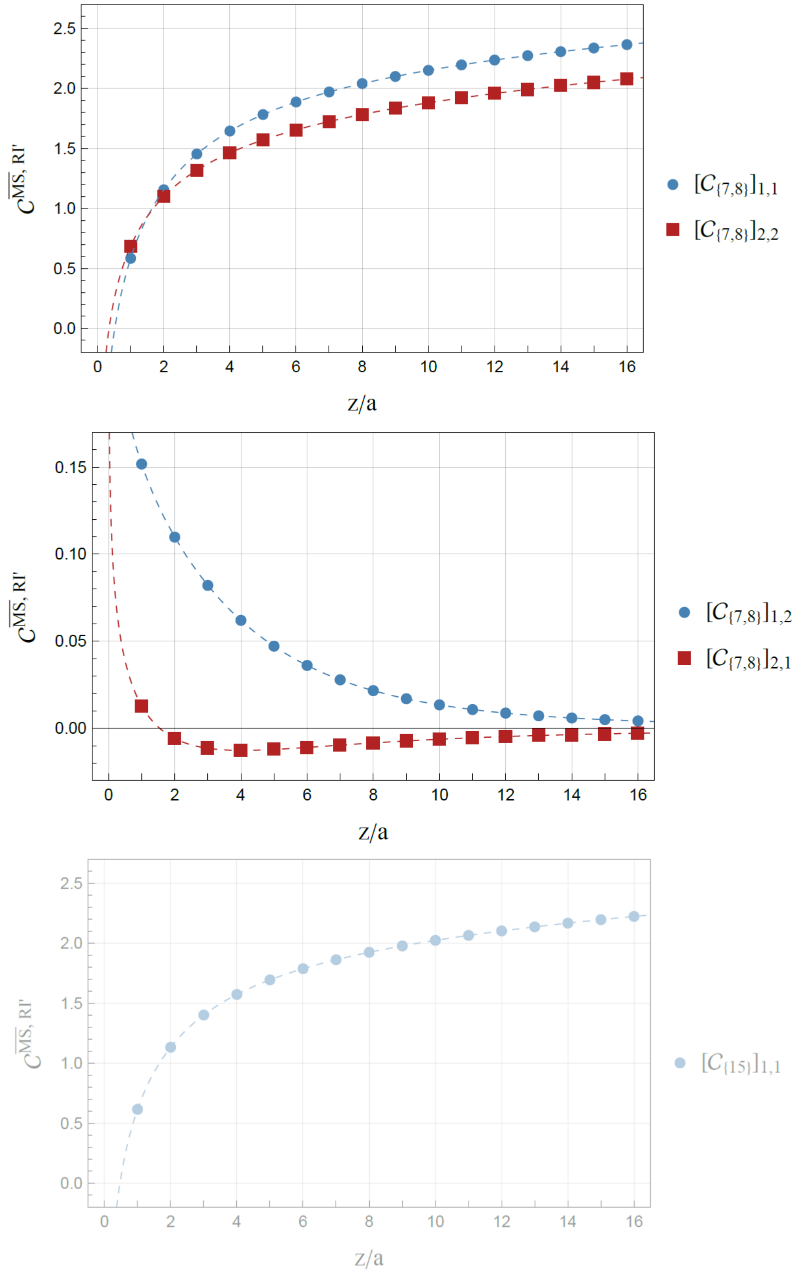
Conversion matrix elements for pair {7, 8}:

- Diagonal elements (figure above)
- Nondiagonal elements (figure below)

→mixing pairs, {1, 2} and {3, 4}, have similar qualitative behavior.



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Conversion factors

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- Nondiagonal elements (figure below)

  →mixing pairs, {1, 2} and {3, 4}, have similar qualitative behavior.

Conversion matrix element for operator 15:

- single element as it is multiplicatively renormalized

  →non mixing operators, 9 and 11, have similar graphical representation.

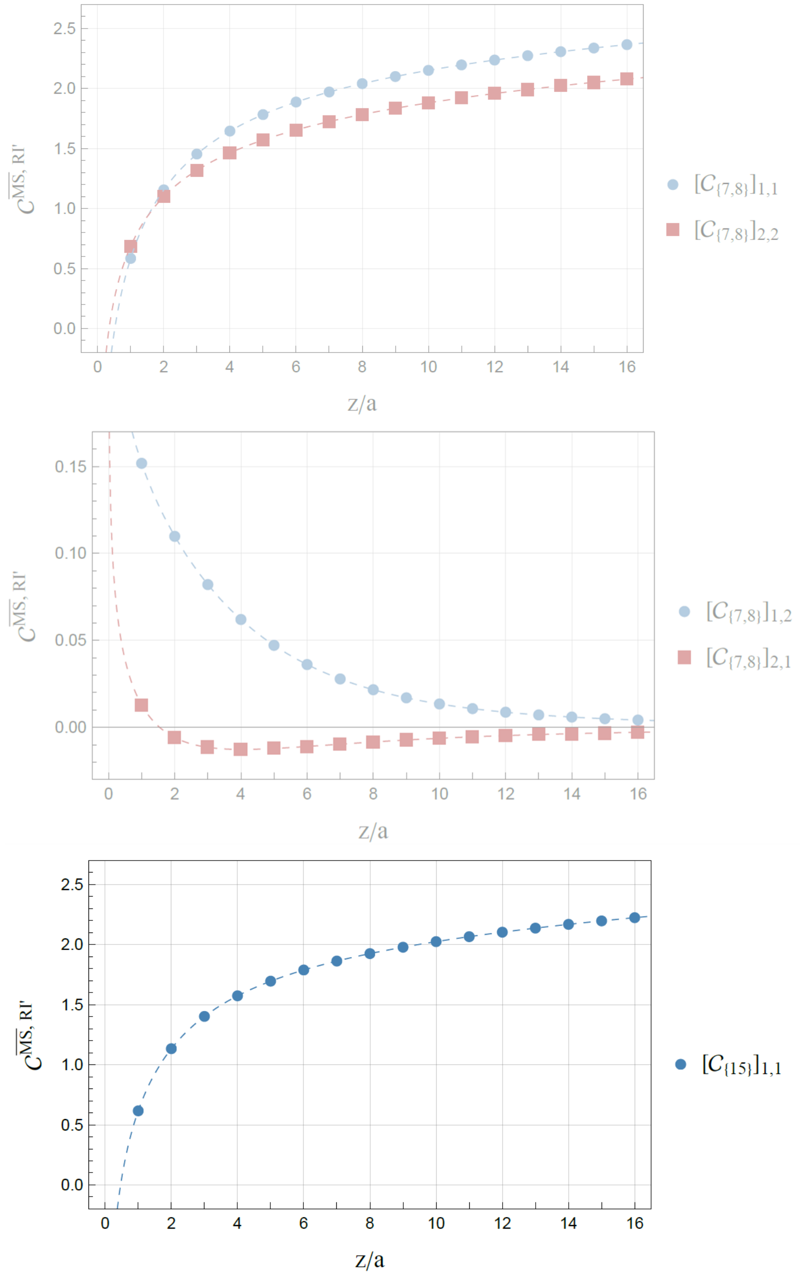

Fig.5: Conversion factor of 'plus-type' operators as a function of z/a

Motivation

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Conversion factors

For the 'minus-type' operators (mixing pair {5,6}, and operators 10, 12, 14, and 16):

- We set n_1=0, n_2= n_3 = 3, and n_4 = 5 for mixing pair {5,6}
- and n_1= n_2= 0, n_3 = 3, and n_4 = 5 for operators 10,12,14, and 16

[thus tree-level Green's functions will be invertible for all integer values]

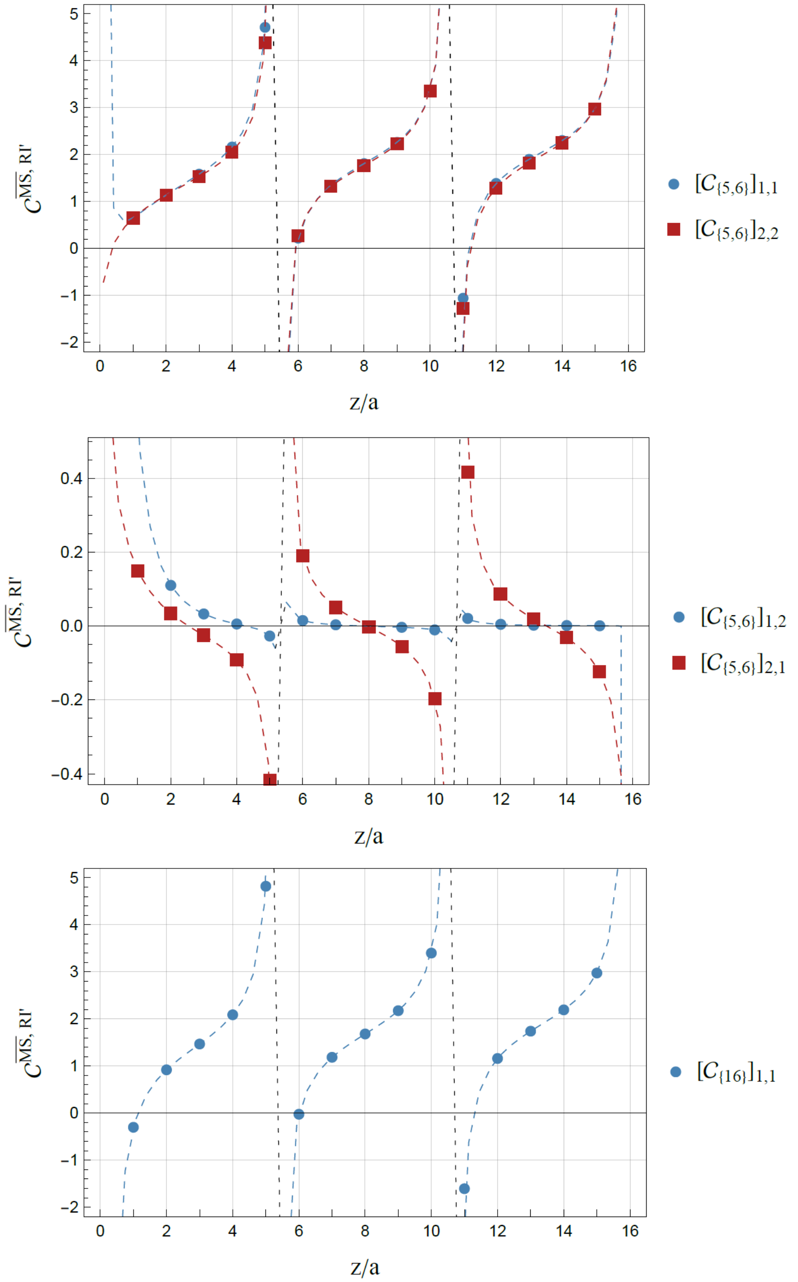

Fig.6: Conversion factor of 'minus-type' operators as a function of z/a

Conversion factors

For the ‘minus-type’ operators (mixing pair {5,6}, and operators 10, 12, 14, and 16):

- We set n_1=0, n_2= n_3 = 3, and n_4 = 5 for mixing pair {5,6}
- and n_1= n_2= 0, n_3 = 3, and n_4 = 5 for operators 10,12,14, and 16

[thus tree-level Green’s functions will be invertible for all integer values]

Conversion matrix elements for pair {5, 6}:

- Diagonal elements (figure above)
- Nondiagonal elements (figure below)

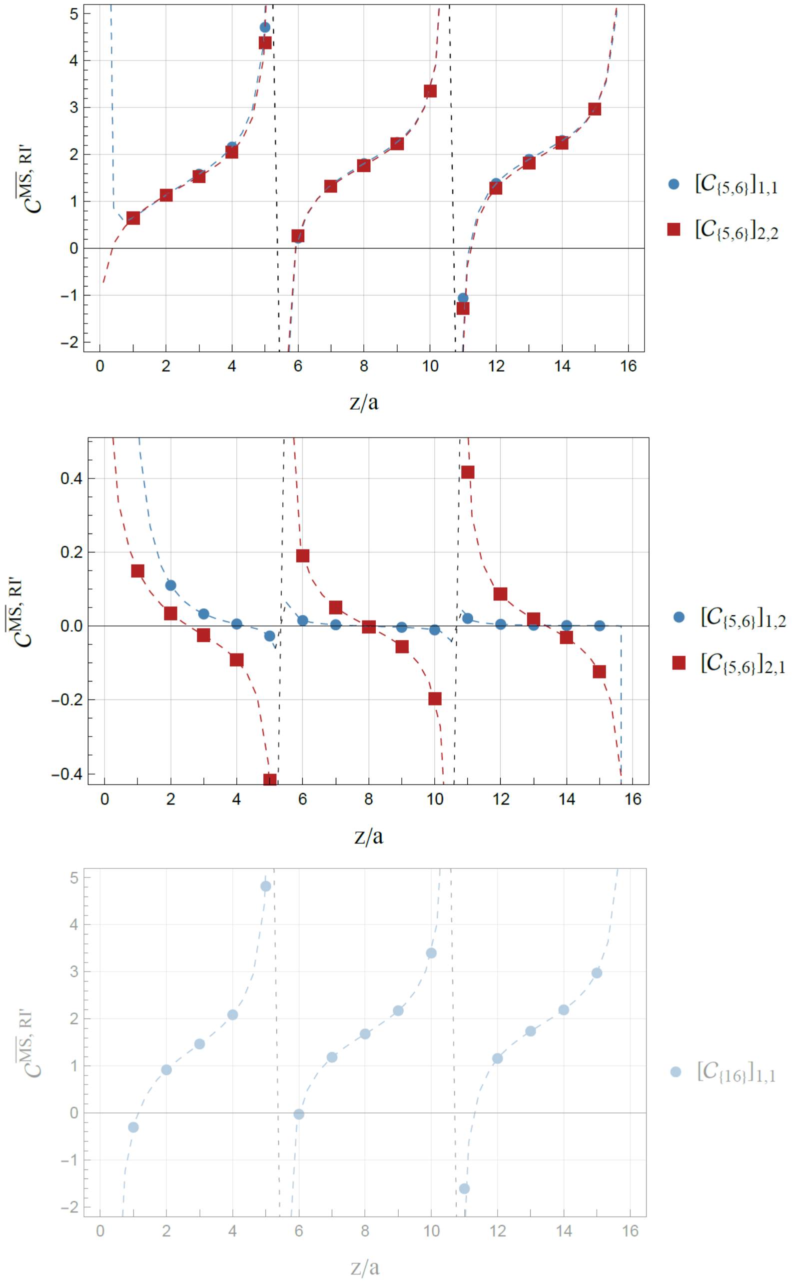

Fig.6: Conversion factor of ‘minus-type’ operators as a function of z/a

Conversion factors

For the ‘minus-type’ operators (mixing pair {5,6}, and operators 10, 12, 14, and 16):

- We set n_1=0, n_2= n_3 = 3, and n_4 = 5 for mixing pair {5,6}
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[thus tree-level Green’s functions will be invertible for all integer values]

Conversion matrix elements for pair {5, 6}:

- Diagonal elements (figure above)
- Nondiagonal elements (figure below)

Conversion matrix element for operator 16:

- single element as it is multiplicatively renormalized

→non mixing operators, 10,12, and 14, have similar graphical representation.

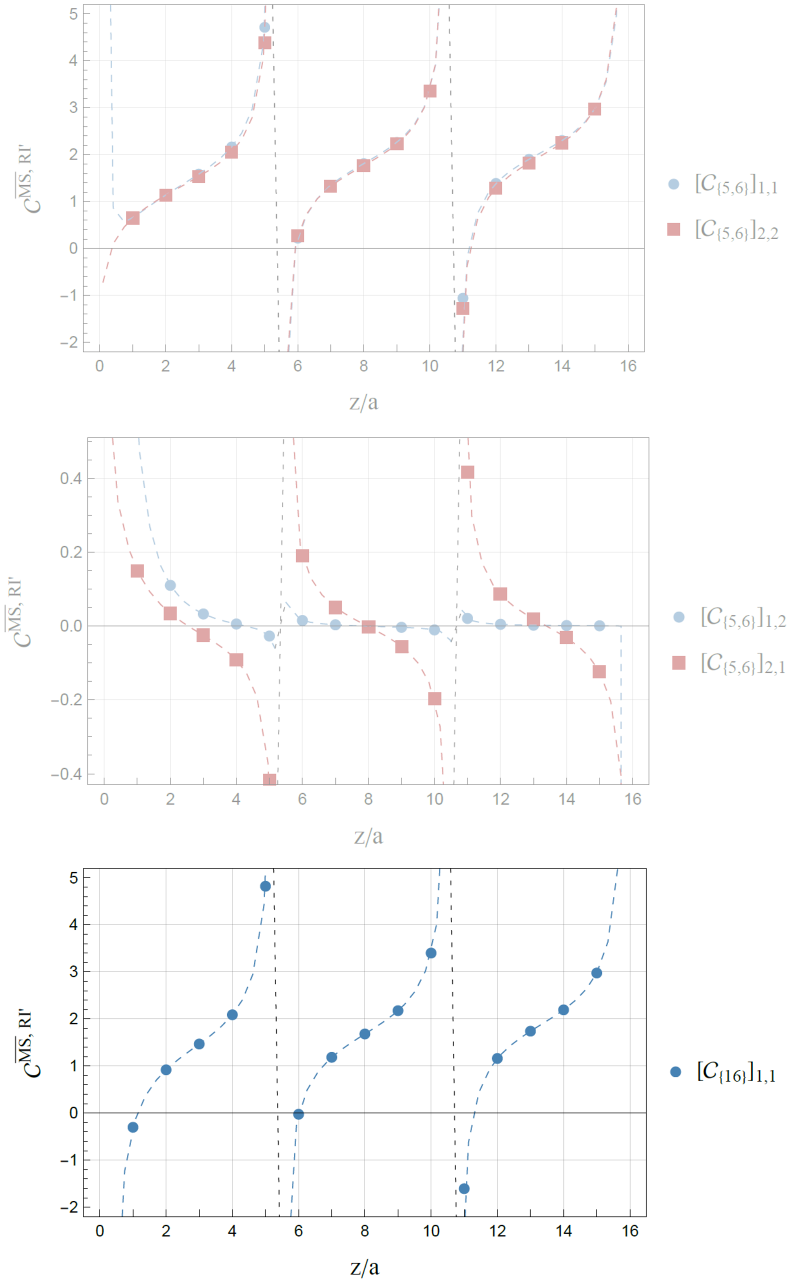

Fig.6: Conversion factor of ‘minus-type’ operators as a function of z/a

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One loop calculations in LR

We find the renormalization factors in LR [$i, j \in \{1, \ldots, 16\}$]:

$$Z_{ij}^{\mathrm{LR},\overline{\mathrm{MS}}} = \delta_{ij}\left[1 + \frac{g^2}{16\pi^2}\left\{\frac{2\pi^2}{N_c} + N_f\left(e_1 + e_2\, c_{SW} + e_3\, c_{SW}^2 + \frac{2}{3}\log(a^2\bar{\mu}^2)\right)\right.\right.$$
$$\left.\left. + N_c\left(e_4 + e_5\frac{|z|}{a} - \frac{5}{3}\log(a^2\bar{\mu}^2) - \left(e_6 + \log(a^2\bar{\mu}^2)\right)\omega_i\right)\right\}\right]$$

where $e_1 = -1.05739$, $e_2 = 0.79694$, $e_3 = -4.71269$, $e_4 = -17.81504$, $e_5 = -19.95484$, and $e_6 = -8.37940$.

- The presence of c_{SW} is inherited from lattice gluon field renormalization factor.
- The expression is gauge independent (numerically confirmed up to $\mathcal{O}(10^{-5})$).
- Coefficient e_5 has the same value as in the quark nonlocal operators of an arbitrary Wilson line's shape (linear divergence arises only from Wilson-line self-energy).

One loop calculations in LR

Motivation

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The renormalization matrix turns out to be diagonal, both in the original basis ($\mathcal{O}_{\mu\nu\rho\sigma}$) and in the basis of symmetries table (16 groups), as observed in the case of DR.

=>in lattice theory at the 1-loop level, the nonlocal gluon operators under investigation are multiplicatively renormalized.

One loop calculations in LR

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Even though the one-loop lattice calculation shows a multiplicative renormalization for all the gluon nonlocal operators under study:

- We expect that mixing among pairs of operators, as dictated by the symmetries of QCD, will be revealed at higher orders.
- Provides a valuable input to the nonperturbative studies regarding the size of mixing contributions expected to arise in lattice simulations.
- Although a multiplicatively renormalizable operator is a better candidate to explore the hadron matrix elements of gluon PDFs, in practice, other operators, which can mix only at higher orders of perturbation theory, can be possible alternatives.

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Summary and Discussion

We have studied the renormalization of the gluon nonlocal operators.

- By analyzing the symmetry properties of these operators, we have identified their mixing pattern under renormalization; some undergo mixing into pairs ({1, 2}, {3, 4}, {5, 6}, {7, 8}), while others are multiplicatively renormalizable (9-16).
- We have computed the two-point bare Green's functions of gluon nonlocal operators using both DR and LR.
- We have evaluated the renormalization factors in the $\overline{MS}$ scheme at one-loop. They are found to be diagonal, both in the continuum and on the lattice.
- In lattice theory, at the 1-loop level, the nonlocal gluon operators undergo multiplicative renormalization. Mixing is expected to occur at higher orders of perturbation theory.
- We determined the conversion factors of these operators between the RI$'$ and $\overline{MS}$ renormalization schemes. The RI$'$ scheme was defined to be compatible with the mixing pattern of the operators and be practical for nonperturbative studies.
- The outcomes of this study are essential for exploring potential paths for investigations of gluon PDFs through lattice QCD.

Acknowledgements:

D.G., H.P. and G.S. acknowledge financial support from the project (EXCELLENCE/0421/0025) which is implemented under the programme of social cohesion ``THALIA 2021-2027" co-funded by the European Union through the Research and Innovation Foundation (RIF). The results are generated within the FEDILA software (project: CONCEPT/0823/0052), which is also implemented under the same ``THALIA 2021-2027" programme, co-funded by the European Union through RIF.

Thank you!

Backup slides

Symmetry properties

<u>Possible mixing with similar operators:</u>

- No mixing with operators involving alternative paths for the Wilson line joining gluon field-strength tensor.

  → Wilson lines renormalize multiplicatively

- No mixing with nonlocal fermion operators:

$$\boxed{\bar{\Psi}(x + z\hat{\tau})\, \Gamma\, W(x + z\hat{\tau}, x)\, \Psi(x)}$$

where Ψ generally stands for a fermion field, and Γ is a Dirac γ-matrix (or product thereof).

→Ψ transforms under the fundamental, rather than the adjoint, representation of the global gauge group.

Symmetry properties

Possible mixing with similar operators:

- No mixing with non-gauge invariant operators, in particular:
  - BRST variations of other operators [Class A]
  - operators which vanish by the equations of motion [Class B]
  - finite mixing with any other operator having the same symmetry properties [Class C].

→substitution of the field strength tensor $F_{\mu\nu}$, on either side of the Wilson line, by any combination of elementary fields, would violate one or more of the symmetries. [e.g., local BRST symmetry.]

Perturbative calculation of Green's functions

We calculate the following one-particle-irreducible (1-PI) two-point bare amputated Green's functions using dimensional regularization (DR) and lattice regularization (LR):

$$\boxed{\delta^{(4)}(q+q')\,\Lambda_O(q,z) = \langle A^a_\alpha(q)\,\left(\int d^4x\, O_{\mu\nu\rho\sigma}(x+z\hat{\tau},x)\right)\,A^b_\beta(q')\rangle_{\rm amp}}$$

More complicated calculations than local operators due to the presence of both the external momentum (q) and the length of the Wilson line (z) in the integrands.

The amputated tree-level Green's functions for both DR and LR:

$$\begin{aligned}\Lambda^{\rm tree}_{O_{\mu\nu\rho\sigma}} = \delta^{ab}\big(&+q_\mu q_\rho\;\delta_{\alpha\nu}\delta_{\beta\sigma}\;e^{-izq_3} + q_\mu q_\rho\;\delta_{\alpha\sigma}\delta_{\beta\nu}\;e^{izq_3}\\ &- q_\nu q_\rho\;\delta_{\alpha\mu}\delta_{\beta\sigma}\;e^{-izq_3} - q_\nu q_\rho\;\delta_{\alpha\sigma}\delta_{\beta\mu}\;e^{izq_3}\\ &- q_\mu q_\sigma\;\delta_{\alpha\nu}\delta_{\beta\rho}\;e^{-izq_3} - q_\mu q_\sigma\;\delta_{\alpha\rho}\delta_{\beta\nu}\;e^{izq_3}\\ &+ q_\nu q_\sigma\;\delta_{\alpha\mu}\delta_{\beta\rho}\;e^{-izq_3} + q_\nu q_\sigma\;\delta_{\alpha\rho}\delta_{\beta\mu}\;e^{izq_3}\big)\end{aligned}$$

- the expression is antisymmetric in {μ, ν} and {ρ, σ}
- the expression is symmetric under (μ, ν) ↔ (ρ, σ) and under (α, β, q) ↔ (β, α,−q).

RI′ renormalization prescription

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Renormalization conditions for mixing pairs:

- q_ν is the momentum of the external gluon fields
- $\bar{q}_\nu$ is the RI′ renormalization scale
- The trace is performed across color space.

$$\left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=\beta=3}} = \left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=\beta=3}} = \begin{cases} 2\bar{q}^2 & \text{for } i=1 \\ -2\bar{q}^2 & \text{for } i=3 \\ 2\bar{q}_1\bar{q}_2 & \text{for } i=7 \\ 0 & \text{for } i=2,4,8 \end{cases}$$

$$\left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=\beta=4}} = \left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=\beta=4}} = \begin{cases} 2\bar{q}^2 & \text{for } i=2 \\ -2\bar{q}^2 & \text{for } i=4 \\ 2\bar{q}_1\bar{q}_2 & \text{for } i=8 \\ 0 & \text{for } i=1,3,7 \end{cases}$$

$$\left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_1=0,\\ \alpha=1,\beta=3}} = \left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_1=0,\\ \alpha=1,\beta=3}} = \begin{cases} i\sin(z\bar{q}_3)\,\bar{q}_2\bar{q}_3 & \text{for } i=5 \\ 0 & \text{for } i=6 \end{cases}$$

$$\left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_1=0,\\ \alpha=1,\beta=4}} = \left.\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\right|_{\substack{\bar{q}_1=0,\\ \alpha=1,\beta=4}} = \begin{cases} 0 & \text{for } i=5 \\ i\sin(z\bar{q}_3)\,\bar{q}_2\bar{q}_4 & \text{for } i=6 \end{cases}$$

RI′ renormalization prescription

Renormalization conditions for multiplicative renormalized operators:

$$\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=1,\beta=3}} = \frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=1,\beta=3}} = \bar{q}_2^2 \qquad \text{for } i=9,11$$

$$\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_1=\bar{q}_4=0,\\ \alpha=1,\beta=3}} = \frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_1=\bar{q}_4=0,\\ \alpha=1,\beta=3}} = i\sin(z\bar{q}_3)\,\bar{q}_2^2 \qquad \text{for } i=10,12$$

$$\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_1=\bar{q}_4=0,\\ \alpha=4,\beta=1}} = \frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_1=\bar{q}_4=0,\\ \alpha=4,\beta=1}} = \begin{cases} 2i\sin(z\bar{q}_3)\,\bar{q}_2\bar{q}_3 & \text{for } i=14 \\ i\sin(z\bar{q}_3)\,\bar{q}_2\bar{q}_3 & \text{for } i=16 \end{cases}$$

$$\frac{\mathrm{Tr}\left[\Lambda^{\mathrm{RI}'}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=3,\beta=4}} = \frac{\mathrm{Tr}\left[\Lambda^{\mathrm{tree}}_{O_{(i)}}(\bar{q},z)\right]}{N_c^2-1}\Bigg|_{\substack{\bar{q}_3=\bar{q}_4=0,\\ \alpha=3,\beta=4}} = \frac{\bar{q}_1\bar{q}_2}{2} \qquad \text{for } i=15$$

- q_ν is the momentum of the external gluon fields
- $\bar{q}_\nu$ is the RI′ renormalization scale
- The trace is performed across color space.

For ‘minus-type’ operators (mixing pair {5, 6} and operators 10,12,14, and 16 with multiplicative renormalization) we cannot select $\bar{q}_3$=0 (nor $\bar{q}_3=\pi\ z\ n$, where n is an integer) because sin($\bar{q}_3\ z$) will vanish, thus making their expression noninvertible.

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Note for ‘minus-type’ operators:

Divergent behavior for noninteger values of z/a

→ due to the unavoidable factor of $\sin(\bar{q}_3\, z)$ in their tree-level expressions

But z/a is necessarily an integer in the lattice definition of the operators, making these divergences inconsequential.

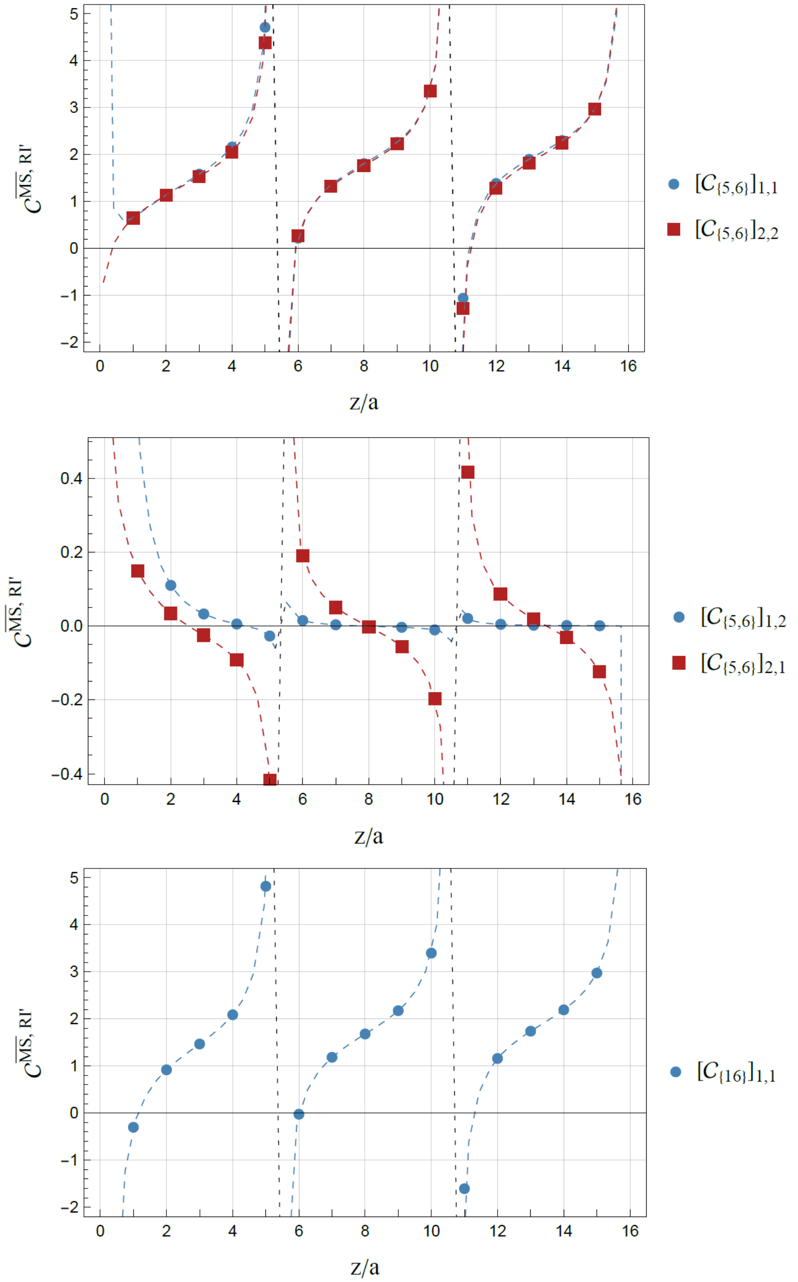

Fig.6: Conversion factor of ‘minus-type’ operators as a function of z/a