Renormalisation Group Equations for 2+1 clover Fermions

K. U. Can, R. Horsley, P. E. L. Rakow, G. Schierholz, H. Stüben, R. D. Young, J. M. Zanotti

– QCDSF-UKQCD Collaboration –

Adelaide – Edinburgh – Liverpool – DESY – Hamburg

Lattice 2024, Liverpool, UK

2 August 15:35

QCDSF related talks with $2+1$ flavours:

- Utku Can — 2 August 15:55
  Updates on the parity-odd structure function of the nucleon from the Compton amplitude
- Joshua Crawford — 1 August 10:20
  Transverse Force Distributions in the Proton from Lattice QCD

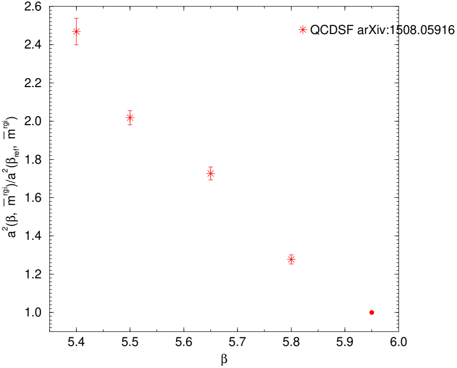
Introduction

Problem:

Usually make individual fits at a particular lattice spacing and then extrapolate to the continuum limit – noisy procedure

Here:

Try to use Renormalisation Group (RG) to 'smooth' data more globally before determining (eg) a



- Scaling:
  Lines of constant physics (eg constant mass ratios) passing through our parameter space
- Constant physics trajectory:
  Measure relative rate at which lattice spacing a changes by monitoring rate at which correlation lengths change as we move along the trajectory
- Overall scale: $\Lambda^{\rm lat}$
- Work in progress

Deriving the RG equation – Wilson type fermions

In g_0, κ_q space

$$a\frac{\partial}{\partial a}\bigg|_{\mathrm{physics}} = U(g_0,\{\kappa\})\,\frac{\partial}{\partial g_0}\bigg|_{\kappa} + \sum_q V_q(g_0,\{\kappa\})\,\frac{\partial}{\partial \kappa_q}\bigg|_{g_0}$$

- Re-write in terms of am_q

$$am_q \equiv \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_{0c}(g_0)}\right)$$

- Taylor expand U and V_q

This gives

$$
\begin{aligned}
a\,\frac{\partial}{\partial a}\bigg|_{\rm physics} = {} & B_0(g_0)\,\frac{\partial}{\partial g_0}\bigg|_{\{am\}} + B_1(g_0)a\overline{m}\,\frac{\partial}{\partial g_0}\bigg|_{\{am\}} \\
& + G_0(g_0)\sum_q am_q\,\frac{\partial}{\partial am_q}\bigg|_{g_0} + H_0(g_0)a\overline{m}\sum_q\,\frac{\partial}{\partial am_q}\bigg|_{g_0} \\
& + G_1(g_0)a\overline{m}\sum_q am_q\,\frac{\partial}{\partial am_q}\bigg|_{g_0} + G_2(g_0)\sum_q (am_q)^2\,\frac{\partial}{\partial am_q}\bigg|_{g_0} \\
& + H_1(g_0)(a\overline{m})^2\sum_q\,\frac{\partial}{\partial am_q}\bigg|_{g_0} + H_2(g_0)a^2\overline{m^2}\sum_q\,\frac{\partial}{\partial am_q}\bigg|_{g_0}
\end{aligned}
$$

where

$$
a\overline{m} \equiv \frac{1}{n_f}\sum_q am_q \qquad a^2\overline{m^2} \equiv \frac{1}{n_f}\sum_q a^2 m_q^2
$$

- Coefficients: B_0, G_0, H_0 – ‘renormalisation’
- Coefficients: B_1, G_1, G_2, H_1, H_2 – ‘improvement’
- In principle treats renorm/improv terms in unified fashion

Drop $O(m^2)$ terms

$$a\,\frac{\partial}{\partial a}\Big|_{\rm physics} = B_0(g_0)\,\frac{\partial}{\partial g_0}\Big|_{\{am\}} + B_1(g_0)a\overline{m}\,\frac{\partial}{\partial g_0}\Big|_{\{am\}} + G_0(g_0)\sum_q am_q\,\frac{\partial}{\partial am_q}\Big|_{g_0} + H_0(g_0)a\overline{m}\sum_q \frac{\partial}{\partial am_q}\Big|_{g_0}$$

Leading functions related to usual renormalisation coefficients

$$\begin{aligned} B_0(g_0) &= -\beta(g_0) \\ G_0(g_0) &= 1-\gamma_m^{\rm NS}(g_0) \\ H_0(g_0) &= \gamma_m^{\rm NS}(g_0)-\gamma_m^{\rm S}(g_0) \end{aligned}$$

- $\beta(g_0) = -b_0 g_0^3 - b_1 g_0^5 - b_2^{\rm lat} g_0^7 + O(g_0^9)$
- $\gamma_m^{\rm NS}(g_0) = d_0 g_0^2 + d_1^{\rm NS\,lat} g_0^4 + O(g_0^6)$; $\gamma_m^{\rm S}(g_0) = d_0 g_0^2 + d_1^{\rm S\,lat} g_0^4 + O(g_0^6)$
- $B_1(g_0) = b_{10}^{\rm lat} g_0^3 + O(g_0^5)$

Solving the RG equation

(I) Renormalised masses

Wilson-like fermions (no chiral symmetry), so Singlet/Non-Singlet pieces evolve differently

$$m_q^{\mathrm{rgi}} = (am_q - a\overline{m})\, v(g_0) + a\overline{m}\, u(g_0) + O(m^2)$$

Solve the RG equation

$$a \left.\frac{\partial}{\partial a}\right|_{\mathrm{physics}} m_q^{\mathrm{rgi}} = 0$$

(I) Renormalised masses

Singlet and Non-Singlet pieces

$$\overline{m}^{\rm rgi} = a\overline{m}\, u(g_0) \qquad m_q^{\rm rgi} - \overline{m}^{\rm rgi} = (am_q - a\overline{m})\, v(g_0)$$

$$u(g_0) = C(g_0) \exp\left[\int_0^{g_0} d\xi \left\{ \frac{1}{b_0 \xi^3} - \frac{b_1 + b_0 d_0}{b_0^2 \xi} - \frac{G_0(\xi) + H_0(\xi)}{B_0(\xi)} \right\}\right]$$

and

$$v(g_0) = C(g_0) \exp\left[\int_0^{g_0} d\xi \left\{ \frac{1}{b_0 \xi^3} - \frac{b_1 + b_0 d_0}{b_0^2 \xi} - \frac{G_0(\xi)}{B_0(\xi)} \right\}\right]$$

where

$$C(g_0) = \Lambda^{\rm lat} \left[2 b_0 g_0^2\right]^{\frac{d_0}{2b_0}} \left[b_0 g_0^2\right]^{\frac{b_1}{2b_0^2}} \exp\left[\frac{1}{2 b_0 g_0^2}\right]$$

(II) Coupling constant

(IIA) Changing lattice spacing

[ie include $a\overline{m}$ in definition]

Setting

$$a = s(g_0)\,\{1 + a\overline{m}\,r(g_0)\}$$

and solving

$$a\,\frac{\partial}{\partial a}\bigg|_{\text{physics}} a = a$$

gives

$$\begin{aligned} s(g_0) &= A(g_0)\exp\left(\int_0^{g_0} d\xi\left\{\frac{1}{B_0(\xi)} - \frac{1}{b_0\xi^3} + \frac{b_1}{b_0^2\xi}\right\}\right), \\ r(g_0) &= -u(g_0)\int_0^{g_0} d\xi\,\frac{B_1(\xi)}{u(\xi)B_0^2(\xi)} \underbrace{=}_{g_0^2\to 0} -\frac{b_{10}^{\mathrm{lat}}}{b_0}g_0^2 + O(g_0^4) \end{aligned}$$

with

$$A(g_0) = \frac{1}{\Lambda^{\mathrm{lat}}}\left[b_0 g_0^2\right]^{-\frac{b_1}{2b_0^2}}\exp\left[-\frac{1}{2b_0g_0^2}\right]$$

(IIB) Alternatively: Changing coupling constant

eg ALPHA arXiv:2401.00216

Setting

$$\tilde{g}_0^2 = g_0^2 \{1 + b_g(g_0) a\overline{m}\}$$

and solving

$$a \left. \frac{\partial}{\partial a} \right|_{\text{physics}} \tilde{g}_0 = B_0(\tilde{g}_0)$$

gives

$$b_g(g_0) = 2 \frac{B_0(g_0)}{g_0} \, r(g_0)$$

QCDSF strategy:

[arXiv:1102.5300]

- Consider a flavour singlet quantity

$$X_S(m_u, m_d, m_s)$$

For example

$$X_\pi^2 = \frac{1}{3}\left(2M_K^2 + M_\pi^2\right)\,, \quad X_{t_0}^2 = \frac{1}{t_0}\,, \quad X_N^2 = \frac{1}{3}\left(M_N^2 + M_\Sigma^2 + M_\Xi^2\right)$$

- Invariant under u, d, s permutations (by definition)
- Stationary point about the $SU(3)$ flavour symmetric line

$$\overline{m} = m_u = m_d = m_s$$

has the simple property:

$$X_s(\overline{m} + \delta m_u, \overline{m} + \delta m_d, \overline{m} + \delta m_s) = X_s(\overline{m}, \overline{m}, \overline{m}) + O((\delta m_q)^2)$$

Lattice

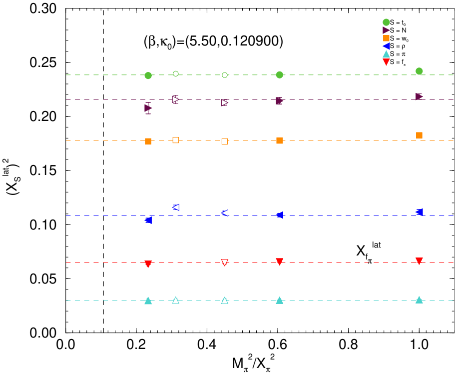
- $O(a)$ NP improved clover action
  - tree level Symanzik glue
  - mildy stout smeared $2+1$ clover fermion
  - $\beta = 10/g_0^2 = 5.40, 5.50, 5.65, 5.80, 5.95$ $[24^3 \times 48,\ 32^3 \times 64,\ 48^3 \times 96]$

- $$am_q = \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_{0c}}\right)$$

  κ_{0c} is chiral limit along symmetric line

- $$\delta am_q = am_q - a\overline{m} = \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_0}\right)$$

- Various $X_s^{\mathrm{lat}\,2}$



We have

$$X_s^{\rm lat}(g_0,\overline{m}^{\rm rgi}) = a(g_0,\overline{m}^{\rm rgi})X_s(\overline{m}^{\rm rgi}) \qquad [M_\pi^{\rm lat}(g_0,\overline{m}^{\rm rgi}) = a(g_0,\overline{m}^{\rm rgi})M_\pi(\overline{m}^{\rm rgi})]$$

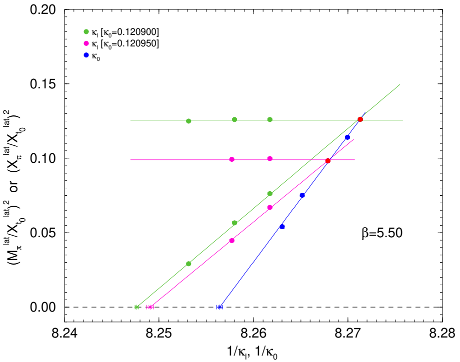
Write: $\overline{m}^{\rm rgi}/\Lambda^{\rm lat} \to \overline{m}^{\rm rgi}$ Conversion: $\overline{m}^{\rm rgi} = u(g_0)a\overline{m}$

$$\frac{X_\pi^{\rm lat\,2}}{X_{t_0}^{\rm lat\,2}} = \frac{X_\pi^2(\overline{m}^{\rm rgi})}{X_{t_0}^2(\overline{m}^{\rm rgi})} = \frac{X_\pi^2(0)\overline{m}^{\rm rgi} + O(m^2)}{X_{t_0}^2(0) + O(m)} = \frac{X_\pi^2(0)}{X_{t_0}^2(0)}\overline{m}^{\rm rgi} + \ldots = \frac{X_\pi^2(0)}{X_{t_0}^2(0)}u(g_0)a\overline{m} + \ldots$$

- $X_\pi^{\rm lat\,2}/X_{t_0}^{\rm lat\,2}$ proxy for $\overline{m}^{\rm rgi}$
- Can measure

$$D_{t_0}(g_0) = \frac{1}{2}\frac{X_\pi^2(0)}{X_{t_0}^2(0)}u(g_0)$$

as gradient of $a\overline{m}$ or $1/\kappa_0$



Lattice scaling

- Compare lattice spacings as a function of β with matching physics (ie same $\overline{m}^{\rm rgi}$)
- Plot y-axis – proxy for $\overline{m}^{\rm rgi}$

$$y_{t_0} = \frac{X_\pi^{\rm lat\,2}}{X_{t_0}^{\rm lat\,2}} = \frac{X_\pi^2(\overline{m}^{\rm rgi})}{X_{t_0}^2(\overline{m}^{\rm rgi})}$$

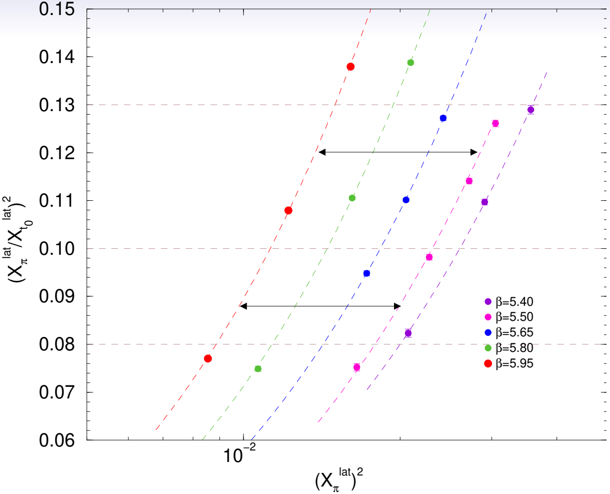
- Against x-axis

$$x_\pi = X_\pi^{\rm lat\,2} = a^2(g_0, \overline{m}^{\rm rgi}) X_\pi^2(\overline{m}^{\rm rgi})$$

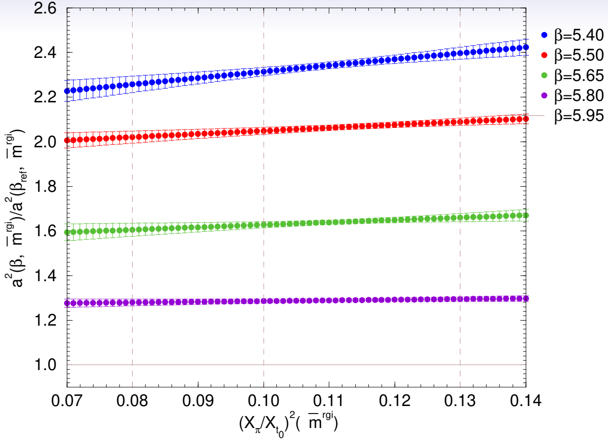
- Interpretation

  For two β-values with common $\overline{m}^{\rm rgi}$ (ie same height for y-axis) then ratio of x-axes is ratio of lattice spacings only

$$\ln x_\pi(\beta, \overline{m}^{\rm rgi}) - \ln x_\pi(\beta_{\rm ref}, \overline{m}^{\rm rgi}) = \ln\left(\frac{a^2(\beta, \overline{m}^{\rm rgi})}{a^2(\beta_{\rm ref}, \overline{m}^{\rm rgi})}\right)$$



- $y_{t_0} = X_\pi^2(\overline{m}^{\rm rgi})/X_{t_0}^2(\overline{m}^{\rm rgi})$ versus $x_\pi = X_\pi^{\rm lat\,2}$
- 2-parameter fit: $x_{t_0} = A_{t_0} + B_{t_0} y_{t_0}$
- Sample lines when the y-axis height $y_{t_0} = 0.08,\ 0.10,\ 0.13$
- Physical region: $y_0 \sim 0.09 - 0.10$



- $a^2(\beta, \overline{m}^{\rm rgi})/a^2(\beta_{\rm ref}, \overline{m}^{\rm rgi})$ [ie previous x_π/x_π]
  versus $X_\pi^2(\overline{m}^{\rm rgi})/X_{t_0}^2(\overline{m}^{\rm rgi})$ [ie previous y_{t_0}, proxy $\overline{m}^{\rm rgi}$]
- Range of previous sample lines $y_{t_0} = 0.08 - 0.13$
- Slowly varying:
  eg 10% change in $(X_\pi^2/X_{t_0})^2(\overline{m}^{\rm rgi})$ gives $\sim$ 1-2% change in $a^2(5.40, \overline{m}^{\rm rgi})/a^2(5.95, \overline{m}^{\rm rgi})$
  So effectively only overall shift in scale if miss a $\overline{m}^{\rm rgi}$ *

$$a(g_0, \overline{m}^{\mathrm{rgi}}) = s(g_0)\left\{1 + a\overline{m}\, r(g_0)\right\}$$

Forming

$$\underbrace{\frac{a^2(\beta, \overline{m}^{\mathrm{rgi}})}{a^2(\beta_{\mathrm{ref}}, \overline{m}^{\mathrm{rgi}})}}_{x_\pi / x_\pi} = \underbrace{\frac{s^2(\beta)}{s^2(\beta_{\mathrm{ref}})}}_{\frac{A_{t_0}(\beta)}{A_{t_0}(\beta_{\mathrm{ref}})}} \left\{ 1 + \left(\underbrace{\frac{r(\beta)}{D_{t_0}(\beta)} - \frac{r(\beta_{\mathrm{ref}})}{D_{t_0}(\beta_{\mathrm{ref}})}}_{\frac{B_{t_0}(\beta)}{A_{t_0}(\beta)} - \frac{B_{t_0}(\beta_{\mathrm{ref}})}{A_{t_0}(\beta_{\mathrm{ref}})}} \right) \underbrace{\frac{X_\pi^2(\overline{m}^{\mathrm{rgi}})}{X_{t_0}^2(\overline{m}^{\mathrm{rgi}})}}_{y_{t_0}} \right\}$$

$s^2(\beta)/s^2(\beta_{\rm ref})$

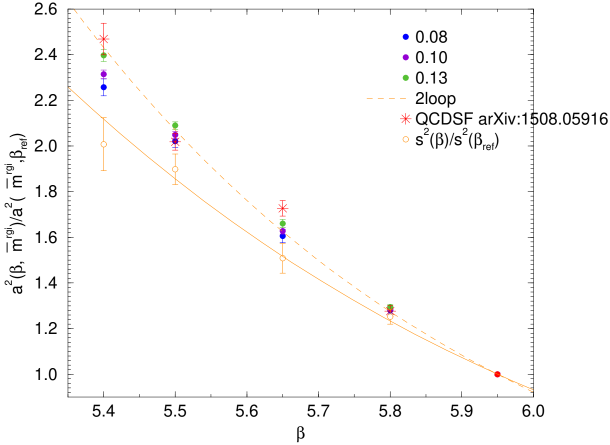
Recall solution for $s(g_0)$ and forming ratio

$$\frac{s^2(g_0)}{s^2(g_{0\,\rm ref})} = \exp\left[-2\int_{g_{0\,\rm ref}}^{g_0} \frac{d\xi}{B_0(\xi)}\right]$$

- Estimate the β-function using the [1/1] Padé approximation

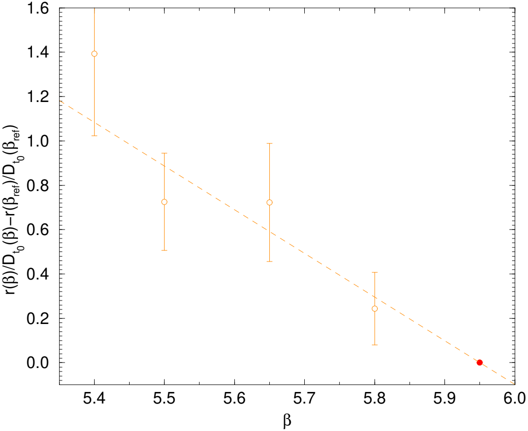
$$B_0(g_0) = -\beta_{[1/1]}(g_0) = b_0 g_0^3 \, \frac{1 + \left(\frac{b_1}{b_0} - \frac{b_2^{\rm eff}}{b_1}\right) g_0^2}{1 - \frac{b_2^{\rm eff}}{b_1} g_0^2}$$

- [Can integrate analytically]



- Range of previous sample lines $y_{t_0} = (X_\pi/X_{t_0})^2(\overline{m}^{\rm rgi}) = 0.08 - 0.13$
- $b_2^{\rm eff} \sim 0.00105$

$$r(\beta)/D_{t_0}(\beta) - r(\beta_{\rm ref})/D_{t_0}(\beta_{\rm ref}) \propto r(\beta)/u(\beta) - r(\beta_{\rm ref})/u(\beta_{\rm ref})$$



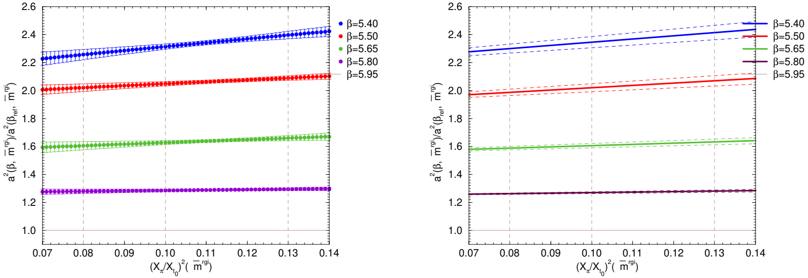
Try

$$\frac{r(\beta)}{D_{t_0}(\beta)} - \frac{r(\beta_{\rm ref})}{D_{t_0}(\beta_{\rm ref})} \propto \int_{g_{0\,\rm ref}}^{g_0} d\xi \frac{B_1(\xi)}{B_0^2(\xi)u(\xi)} \propto \beta - \beta_{\rm ref}$$

as $u(g_0)$ appears not to vary much in small g_0 interval

Re-construct $a^2(\beta, \overline{m}^{\rm rgi})/a^2(\beta, \overline{m}^{\rm rgi})$

Preliminary:



- Use previously determined fitted values
- Reasonable agreement
- Errors to be improved (!)

Conclusions

- RG equations given for $2+1$ Clover fermions
- Attempting to use this as guide on constructing some (effective) RG functions
- Find scaling, with little dependence on $\overline{m}^{\mathrm{rgi}}$
- Hope this can help with determination of a and give smoother continuum extrapolations
- All very preliminary (!)