Coordinate-space calculation of the UV finite isospin breaking QED correction to the hadronic vacuum polarization contribution to $(g-2)_\mu$

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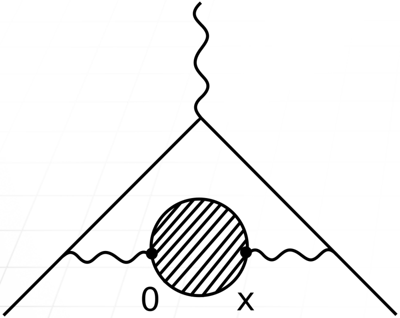
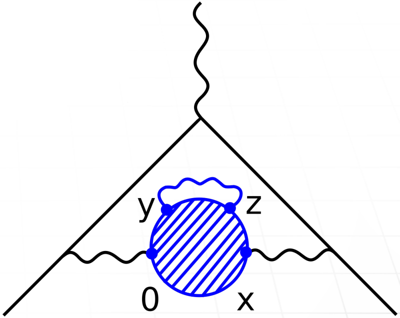
Lattice 2024, August 1st, 2024

Hadronic contributions to $(g-2)_\mu$





- $O(\alpha^2)$: Hadronic vacuum polarization (HVP) $a_\mu^{HVP} \sim 700 \cdot 10^{-10}$, desirable accuracy $\sim 0.5\%$

- $O(\alpha^3)$ QED corrections are of order 1%
- Need to calculate QCD four point function $\langle j_\mu(z) j_\nu(y) j_\rho(x) j_\sigma(0) \rangle_{\mathrm{QCD}}$

QED corrections to the HVP

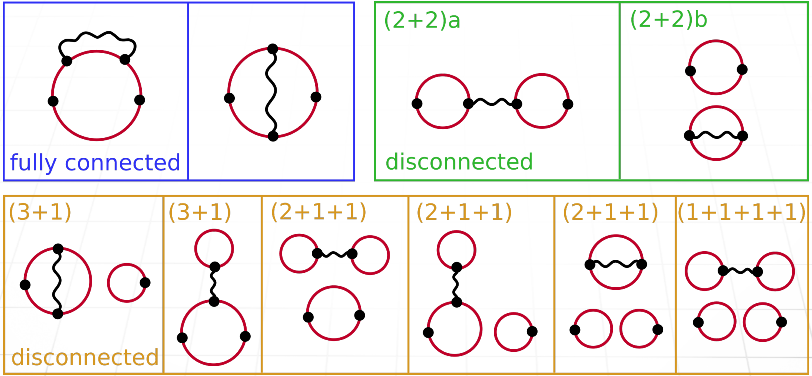
- Covariant coordinate-space (CCS) formulation [arXiv:1706.01139], [arXiv:2211.15581]
- QED_∞ : Photon propagator in the continuum and infinite volume [arxiv:2209.02149]

$$a_\mu^{HVP,NLO} = -\frac{e^2}{2}\int_{x,y,z} H_{\mu\sigma}(z)\delta_{\nu\rho}\Big[G_0(y-x)\Big]_\Lambda \langle j_\mu(z) j_\nu(y) j_\rho(x) j_\sigma(0)\rangle_{\mathrm{QCD}}$$
$$+\text{counterterms}$$

- with CCS kernel $H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{|x|^2}\mathcal{H}_2(|x|) + \partial_\mu\Big(x_\nu f(|x|)\Big)$
- and Pauli-Villars regularization of UV divergence $\Big[G_0(y-x)\Big]_\Lambda = \frac{1}{4\pi^2|y-x|^2} - \frac{\Lambda K_1(\Lambda|y-x|)}{4\pi^2|y-x|^2}$
- After including the counterterms and continuum limit $a_{\mathrm{lattice}} \to 0$, take the limit $\Lambda \to \infty$
- No power law finite-size effects

QED corrections to the HVP

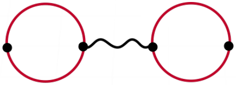
- $\langle j_\mu(z) j_\nu(y) j_\rho(x) j_\sigma(0)\rangle_{\rm QCD}$ involves computation of many different Wick contractions
- Diagrams with self contracted valence quark loop "$(X+1)$" are suppressed



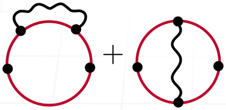
QED corrections to the HVP

Two main projects using coordinate-space approach

1. Computation of the UV finite $(2+2)a$ contribution at the physical point



2. Computation of the isospin violating contribution $G^{38}_{\mu\nu}(x)$ at the $SU(3)$ flavour symmetric point $\rightarrow$ Talk by Dominik Erb afterwards



$+$ counterterm

(2+2)a contribution

- QED corrections to the leading order disconnected contribution
- $(2+2)a$ is UV finite, from OPE calculation

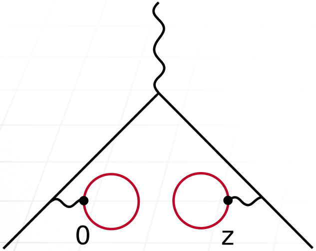


Fig. 1: Leading order disconnected contribution $a_\mu^{HVP,\mathrm{Disc}} \sim -20 \cdot 10^{-10}$

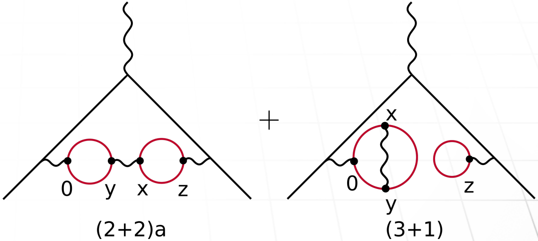
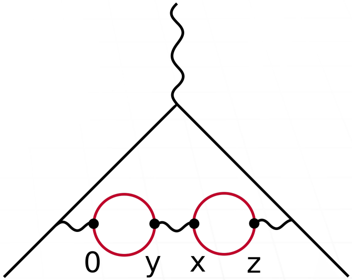


Fig. 2: QED corrections to disconnected contribution

(2+2)a contribution

- Result is UV finite → regulator can be dropped
- Benchmark quantity for lattice calculations of QED corrections
- Requires only calculation of 2-point functions



$$a_\mu^{(2+2)a} = -\frac{e^2}{2} 2C \int_{x,y,z} H_{\mu\sigma}(z)\delta_{\nu\rho} G_0(y-x) \langle \hat{\Pi}_{\mu\nu}(z,x)\hat{\Pi}_{\rho\sigma}(y,0)\rangle_U, \quad (1)$$

with a charge factor C ($C = 25/81$ for light quark contribution) and 2-pt function

$$\Pi_{\mu\nu}(x,y) = -Re\Big(Tr\Big[S(y,x)\gamma_\mu S(x,y)\gamma_\nu\Big]\Big), \quad \hat{\Pi}_{\mu\nu}(x,y) = \Pi_{\mu\nu}(x,y) - \langle \Pi_{\mu\nu}(x,y)\rangle_U \quad (2)$$

$$a_\mu^{(2+2)a} = -e^2 C 2\pi^2 \int_0^\infty d|x||x|^3 \left[\langle I^{(2)}_{\rho\sigma}(x) I^{(3)}_{\sigma\rho}(x)\rangle_U - \langle I^{(2)}_{\rho\sigma}(x)\rangle_U \langle I^{(3)}_{\sigma\rho}(x)\rangle_U \right] =: \int_0^\infty d|x| f(|x|),$$

$$I^{(2)}_{\rho\sigma}(x) = \int_y G_0(x-y)\Pi_{\rho\sigma}(y,0),$$

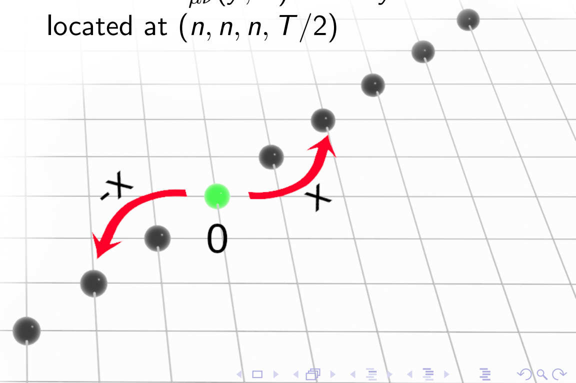
$$I^{(3)}_{\sigma\rho}(x) = \int_z H_{\nu\sigma}(z)\Pi_{\nu\rho}(z,x)$$

Compute for two different CCS kernels

$$H^{\mathrm{XX}}_{\mu\nu}(x) = \frac{x_\mu x_\nu}{|x|^2}\left(\mathcal{H}_2(|x|) + |x|\frac{d}{d|x|}\mathcal{H}_1(|x|)\right),$$

$$H^{\mathrm{TL}}_{\mu\nu}(x) = \left(-\delta_{\mu\nu} + 4\frac{x_\mu x_\nu}{|x|^2}\right)\mathcal{H}_2(|x|)$$

- Calculate $\Pi_{\mu\nu}(y,P)$ for all y with P located at $(n,n,n,T/2)$



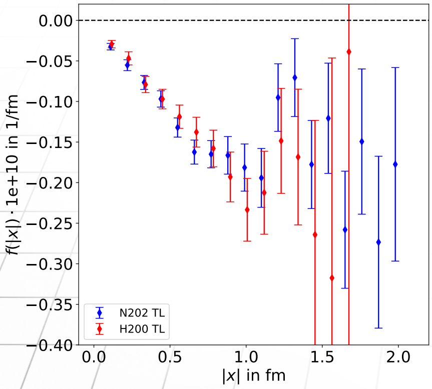
Strategy

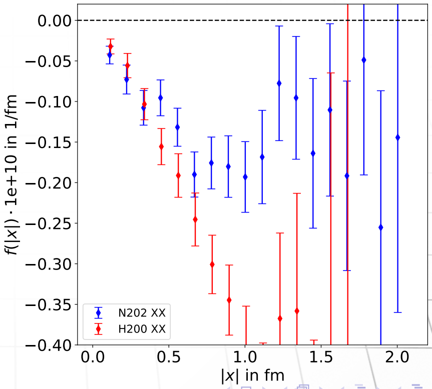
- On 8 CLS ensembles with $N_f = 2 + 1$ dynamical flavors of non-perturbatively $O(a)$ improved Wilson quarks and tree-level $O(a^2)$ improved Lüscher-Weisz gauge action

Id	β	$N_L^3 \times N_T$	a [fm]	m_π [MeV]	m_K [MeV]	$m_\pi L$	L [fm]	#confs light/strange
C101	3.4	$48^3 \times 96$	0.0849(9)	222(3)	478(5)	4.6	4.1	200
N451	3.46	$48^3 \times 128$	0.0751(8)	291(4)	468(5)	5.3	3.6	200 /200
D450		$64^3 \times 128$		219(3)	483(5)	5.3	4.8	200
H200	3.55	$32^3 \times 96$	0.0635(6)	423(5)	423(5)	4.4	2.0	200
N202		$48^3 \times 128$		418(5)	418(5)	6.5	3.0	200
N203		$48^3 \times 128$		349(4)	447(5)	5.4	3.0	200
E250		$96^3 \times 192$		132(2)	495(6)	4.1	6.1	140
E300	3.7	$96^3 \times 192$	0.0491(5)	177(2)	497(6)	4.2	4.7	200

Finite-volume effects

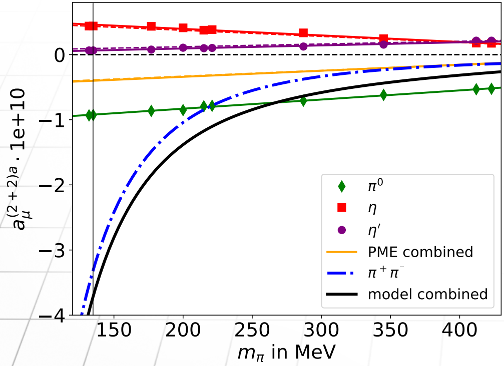
- Compare results on N202 ($m_\pi L = 6.5$) and H200 ($m_\pi L = 4.4$) with same m_π and a_{lattice}
- Good agreement between 'TL' (left) and 'XX' (right) kernel on N202
- For 'TL' kernel only small deviation between H200 and N202 is observed





Phenomenological description: π^0, η, η', $\pi^+\pi^-$

$$a_\mu^{(2+2)a-II,\mathrm{model}} = -\frac{25}{9} a_\mu^{\pi^0}(m_\pi, m_{V,\pi}, F_\pi) + \hat{c}_\eta^{(II)}(m_\pi, m_K, m_\eta, m_{\eta'}, \theta) a_\mu^{\eta}(m_\eta, m_{V,\eta}, F_\eta) + \hat{c}_{\eta'}^{(II)}(m_\pi, m_K, m_\eta, m_{\eta'}, \theta) a_\mu^{\eta'}(m_{\eta'}, m_{V,\eta'}, F_{\eta'}) + \frac{50}{81} a_\mu^{\pi^+\pi^-}(m_\pi, m_{V,\pi})$$



- PME shows only mild chiral dependence
- Charged pion loop increases drastically when approaching the physical point

$$a_\mu^{(2+2)a,\pi^+\pi^-} \propto m_\pi^{-3}$$

for $m_\pi^{\mathrm{phys}} \le m_\pi \le m_\pi^{SU(3)}$

Approximation of the tail

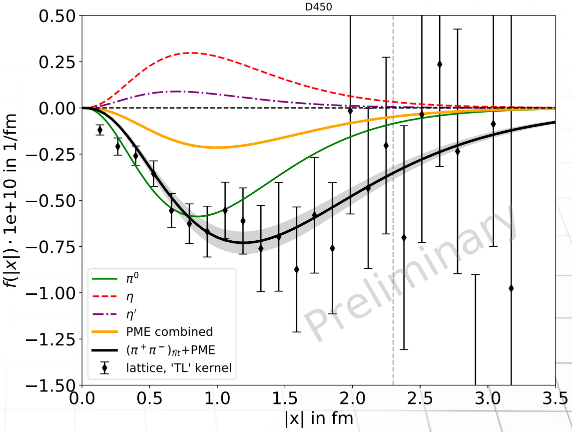
- Integrand for the PME calculated

$$f^{PME}(|x|)$$

- Use ansatz for $\pi^+\pi^-$ integrand

$$f^{\pi^+\pi^-}(|x|) = A|x|^n e^{-2m_\pi |x|}$$

- Exponent n obtained from global fit, A fitted on each ensemble
- Use model $f^{PME}(|x|) + f^{\pi^+\pi^-}(|x|)$ for reconstruction of the tail for $|x| > |x|_{\text{cut}}$

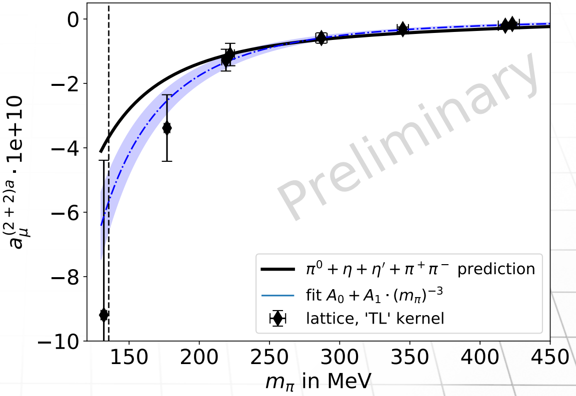


Extrapolation to the physical point

- Use model inspired fit ansatz

$$a_\mu^{(2+2)a-II} = A_0 + A_1 m_\pi^{-3} + A_2 g(m_\pi)$$

- No $a_{\rm lattice}$ dependance can be observed within error
- Compute Model average with AIC to weight different fits and cuts in m_π
- Tail computed from pheno model is taken with 100% uncertainty at this stage of the analysis



Discussion of result

Result from model average:

$$a_\mu^{(2+2)a-ll} = \Big(-5.94 \pm (0.45)_{\mathrm{stat}} \pm (0.41)_{\mathrm{extr}} \pm (0.79)_{\mathrm{tail}} \Big) \cdot 10^{-10} = -5.94(0.99) \cdot 10^{-10}$$

- Uncertainty of the tail will be reduced with direct computation of the integrand of the $\pi^+\pi^-$ contribution
- So far, no dedicated study of finite-volume effects, although comparison of H200 and N202 shows only minor FV effects for the 'TL' kernel
- No lattice spacing dependance can be observed within uncertainty, but in general continuum limit needs to be taken
- In agreement with RBC/UKQCD 2018 result [arxiv:1801.07224]: $-6.9(2.9) \cdot 10^{-10}$

Backup Slides

Covariant coordinate-space representation

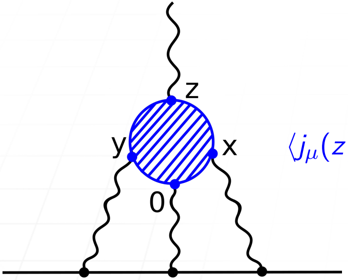
$$a_\mu^{HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty f(t, m_\mu) G(t)\, dt \quad \longrightarrow \quad a_\mu^{HVP} = \int H_{\mu\nu}(x) G_{\mu\nu}(x)\, d^4x$$

- with CCS kernel

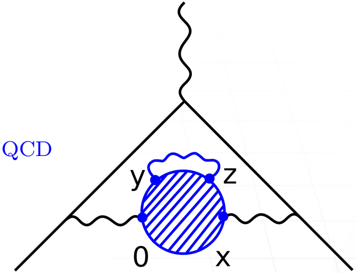
$$H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{|x|^2}\mathcal{H}_2(|x|)$$

- and the vector-vector correlator $G_{\mu\nu}(x) = \langle j_\mu(0) j_\nu(x)\rangle_{\text{QCD}}$, $\qquad G(t) = 1/3 \sum_{\mathbf{x},i} G_{ii}(x)$
- Integral is invariant under $H_{\mu\nu}(x) \to H_{\mu\nu}(x) + \partial_\mu \Big(x_\nu f(|x|)\Big)$
  - family of kernelfunctions, e.g. traceless ('TL'), transverse ('TR'), ('XX'), etc.
- Successful calculation window quantity a_μ^W in CCS formulation [arxiv:2211.15581]

Hadronic contributions to $(g-2)_\mu$ at $O(\alpha^3)$



$\langle j_\mu(z) j_\nu(y) j_\rho(x) j_\sigma(0) \rangle_{\mathrm{QCD}}$



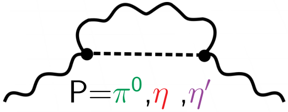
- Isospin breaking correction to HVP (right) requires the same lattice four-point function as hadronic light-by-light (Hlbl) contribution (left)
- Propose calculation similar to Mainz calculation of Hlbl [arxiv:2210.12263]

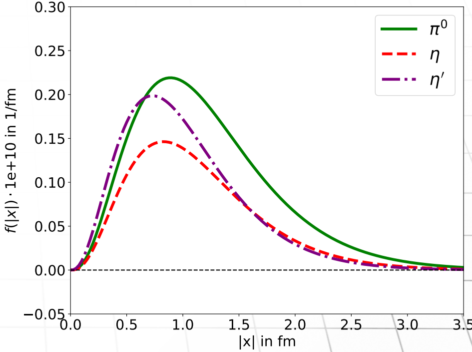
Phenomenological description: $\pi^0, \eta\ , \eta'$

- Pseudoscalar meson exchange (PME)
- With VMD form factor

$$F_{P\gamma\gamma}(q^2, k^2) = \frac{m^4_{V,P} F_{P\gamma\gamma}}{(m^2_{V,P}+q^2)(m^2_{V,P}+k^2)}$$

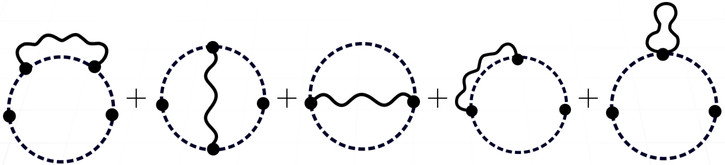
- Parameters: $(m_\pi, m_{V,\pi}, F_{\pi^0\gamma\gamma})$, $(m_\eta, m_{V,\eta}, F_{\eta\gamma\gamma})$, $(m'_\eta, m_{V,\eta'}, F_{\eta'\gamma\gamma})$
- Meson masses measured on CLS ensembles [arxiv:2112.06696], [arxiv:2106.05398], [arxiv:2211.03744]
- Study $F_{P\gamma\gamma}$ in large N_c chiral perturbation theory [arxiv:1612.05473], [arxiv:2005.08550]





Phenomenological description: $\pi^+\pi^-$

- Charged pion loop: $\pi^+\pi^-$



- $a_\mu^{NLO,\pi^+\pi^-}$ computed from the light-by-light scattering amplitude $\mathcal{M}(\nu, K^2, Q^2)$ similar to QED calculation in [arxiv:2209.02149]
- Using scalar QED approach with VMD form factor for the $\pi\pi\gamma$ vertex: Parameters $(m_\pi, m_{V,\pi})$

Phenomenological description: Full light-light contribution

- Matching factor for the $(2+2)a$ contribution analogous to Hlbl [arxiv:2104.02632]
- Full model prediction for the light quark (ll) component

$$
\begin{aligned}
a_\mu^{(2+2)a-ll,\mathrm{model}} \quad &= \quad -\frac{25}{9} a_\mu^{\pi^0}(m_\pi, m_{V,\pi}, F_\pi) + \hat{c}_\eta^{(ll)}(m_\pi, m_K, m_\eta, m_{\eta'}, \theta) a_\mu^{\eta}(m_\eta, m_{V,\eta}, F_\eta) \\
&\quad + \hat{c}_{\eta'}^{(ll)}(m_\pi, m_K, m_\eta, m_{\eta'}, \theta) a_\mu^{\eta'}(m_{\eta'}, m_{V,\eta'}, F_{\eta'}) + \frac{50}{81} a_\mu^{\pi^+\pi^-}(m_\pi, m_{V,\pi})
\end{aligned}
$$

- 10 parameter, but meson masses are well known along chiral trajectory
- Solid description of F_P at physical and $SU(3)$ symmetric point
- In between: Interpolate in $m_K^2 - m_\pi^2$ for unknown parameters

Pseudoscalar exchange model

$$f(|x|) = \int_{z,y} H_{\sigma\lambda}(z) G_0(x-y) \int_{q,k,p} e^{i(p\cdot z+q\cdot y+k\cdot x)} \Pi_{\sigma\mu\mu\lambda}(p,q,k) \quad (3)$$

- Euclidean space polarization tensor from [arxiv:0111058]

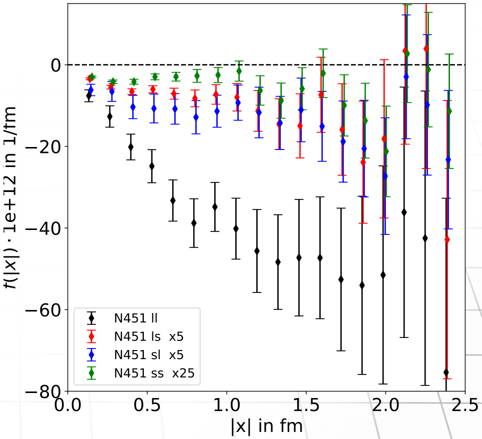
$$\Pi_{\sigma\mu\mu\lambda}(p,q,k) = \epsilon_{\sigma\mu\alpha\beta}\epsilon_{\mu\lambda\gamma\delta}\, p_\alpha \Bigg(\frac{\mathcal{F}(-p^2,-k^2)\,\mathcal{F}(-q^2,-(p+k+q)^2)}{(p+k)^2+m_\pi^2}\, k_\beta\, q_\gamma (p+k)_\delta + \frac{\mathcal{F}(-p^2,-q^2)\,\mathcal{F}(-k^2,-(p+k+q)^2)}{(p+q)^2+m_\pi^2}\, q_\beta\, k_\gamma (p+q)_\delta \Bigg). \quad (4)$$

- VMD form factor

$$\mathcal{F}(-p^2,-k^2) = \frac{m_{V,P}^4 F_{P\gamma\gamma}(0,0)}{(p^2+m_{V,P}^2)(k^2+m_{V,P}^2)} \quad (5)$$

light-strange and strange-strange contribution

- Comparison between light-light (ll), light-strange (ls) and (sl) and strange-strange (ss) component on the ensemble N451, with $m_\pi = 291(4)$ MeV
- (ls) and (sl) each contributes with chargefactor $C = 5/81$
- (ss) contributes with $C = 1/81$



Phenomenological description of the light-strange contribution

- For light-strange (ls) component η and η' are dominant, Kaon is neglected
- Comparison on N451, with $m_\pi = 291(4)$ MeV, $m_K = 468(5)$ MeV, $m_\eta = 525(6)$MeV, $m_{\eta'} = 930(6)$ MeV, mixing angle $\theta = 6.37°$

$$a_\mu^{(2+2)a-ls,\eta\eta'} = \hat{c}_\eta^{(ls)} a_\mu^\eta + \hat{c}_{\eta'}^{(ls)} a_\mu^{\eta'}$$

- due to cancelation between η and η', hard to make a prediction, but order of magnitude agrees.