Left-hand cut and the HAL QCD method

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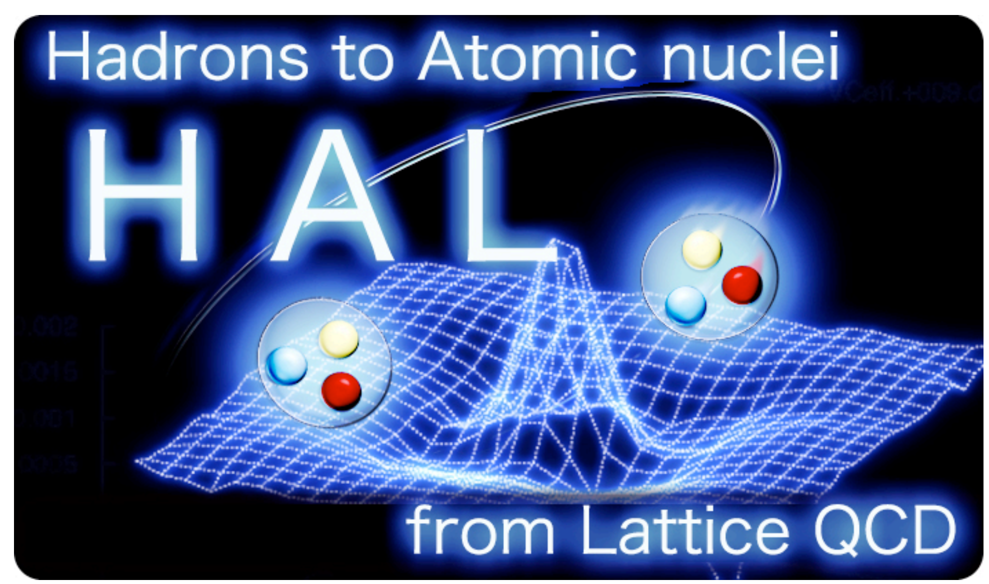

41st International Symposium on Lattice Field Theory (LATTICE 2024)
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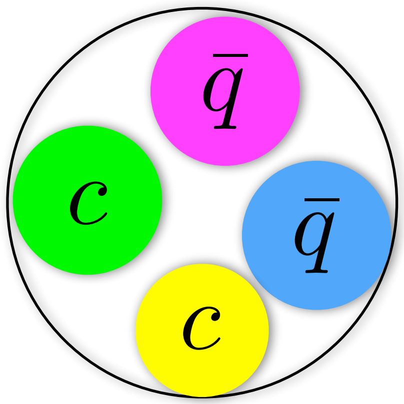
I. Introduction

Heavy tetra-quark states T_{cc}

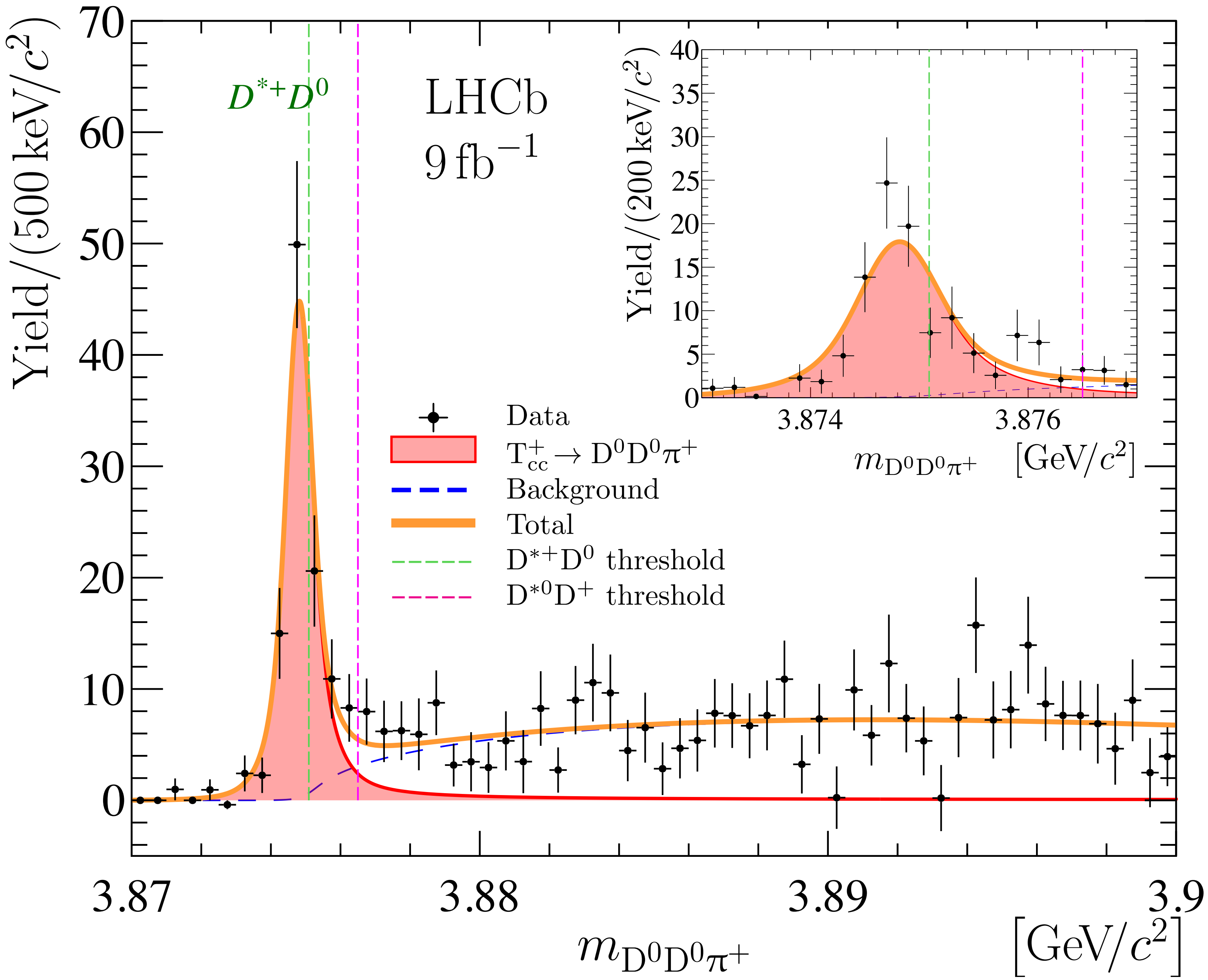


$\bar{q}$: light anti-quark

genuine tetra-quark states

$T_{cc}(cc\bar{u}\bar{d})$ observation by LHCb.

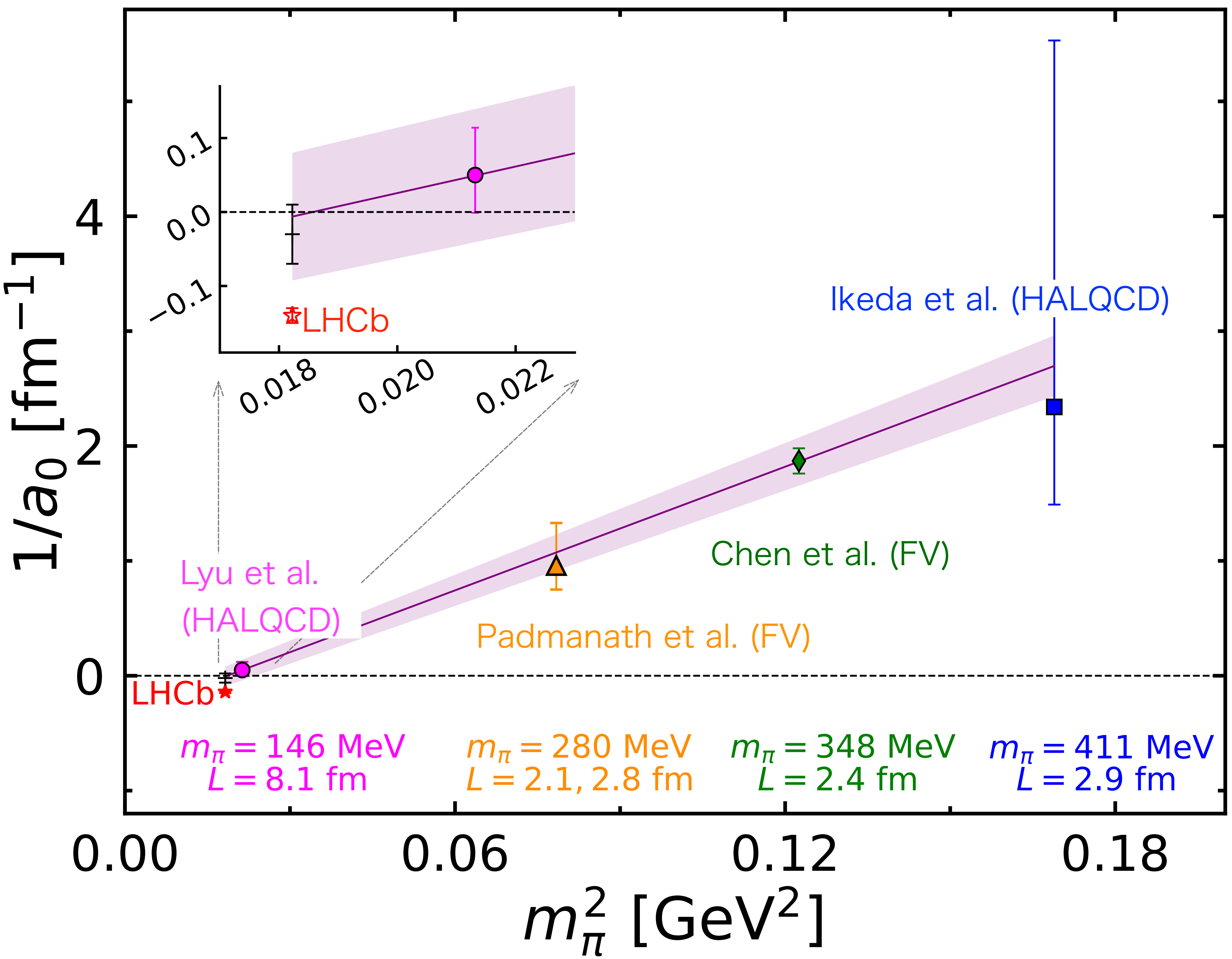
Aaij et al. (LHCb Collaboration), Nature Phys. (2022)



360 keV below $D^{*+}D^{0}$ threshold

$$(I, J^P) = (0, 1^+)$$

Inverse scattering length from Lattice QCD



extrapolation in $1/a_0$

$$\frac{1}{a_0}\,[\mathrm{fm}^{-1}] = -0.01(9)$$

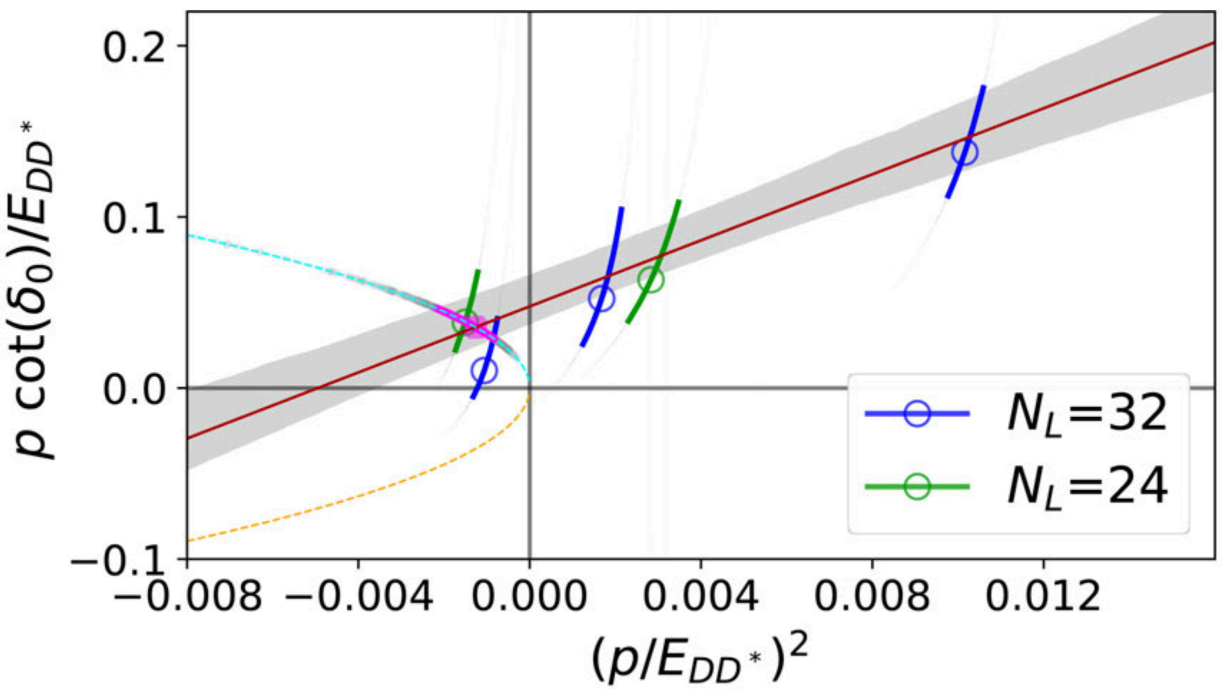
extrapolation in potential

$$\frac{1}{a_0}\,[\mathrm{fm}^{-1}] = -0.03(4)$$

T_{cc} seems to appear as a shallow bound state at physical pion mass.

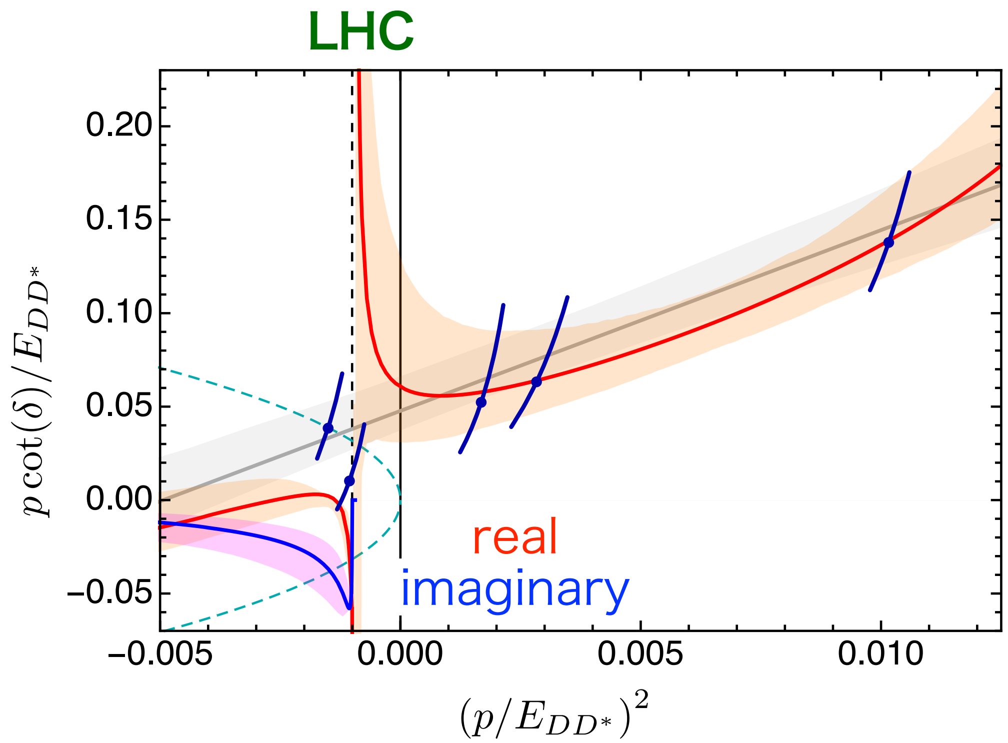
Padmanath-Prelovsek, PRL 129 (2022) 032002.

$m_\pi \simeq 280$ MeV



T_{cc} seems a “virtual” state.

Du et al. PRL 131 (2023) 131903.



Left-hand cut(LHC) invalidates the analysis.

“Effective theory” should be applied to data above LHC.

See also

Meng et al. PRD 109 (2024) L071506.

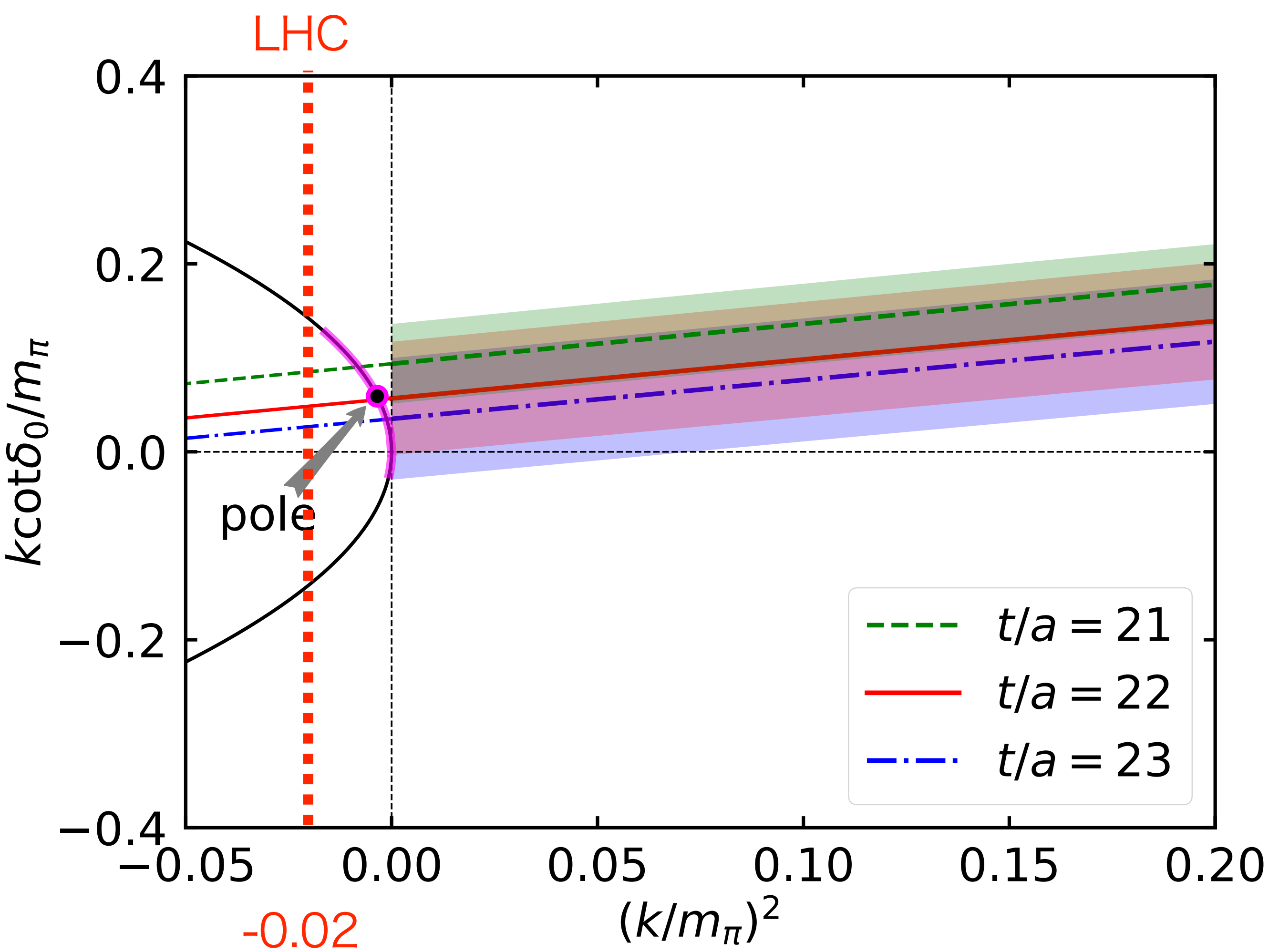
Collins et al. PRD 109 (2024) 094509.

Rapposo-Hansen, 2311.18793[hep-lat]

A modified finite volume formula in the presence of the LHC using an effective theory.

Does left-hand cut also affect the HAL QCD method ?

Lyu et al. PRL 131 (2023) 161901.



The virtual pole appears above the LHC in this lattice setup.

↓

The analysis in this paper is valid.

However we need to understand what happens if a bound state appears below the left-hand cut in the HAL QCD method.

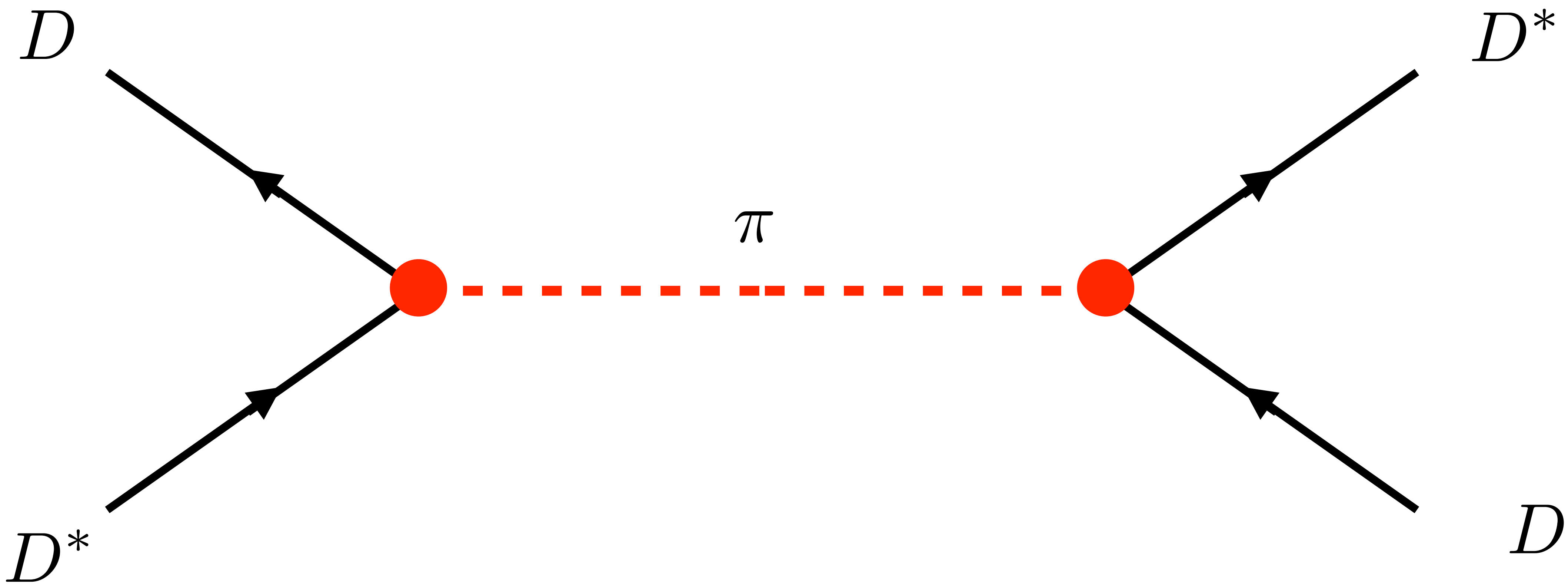
I. Introduction

II. Bound state below LHC

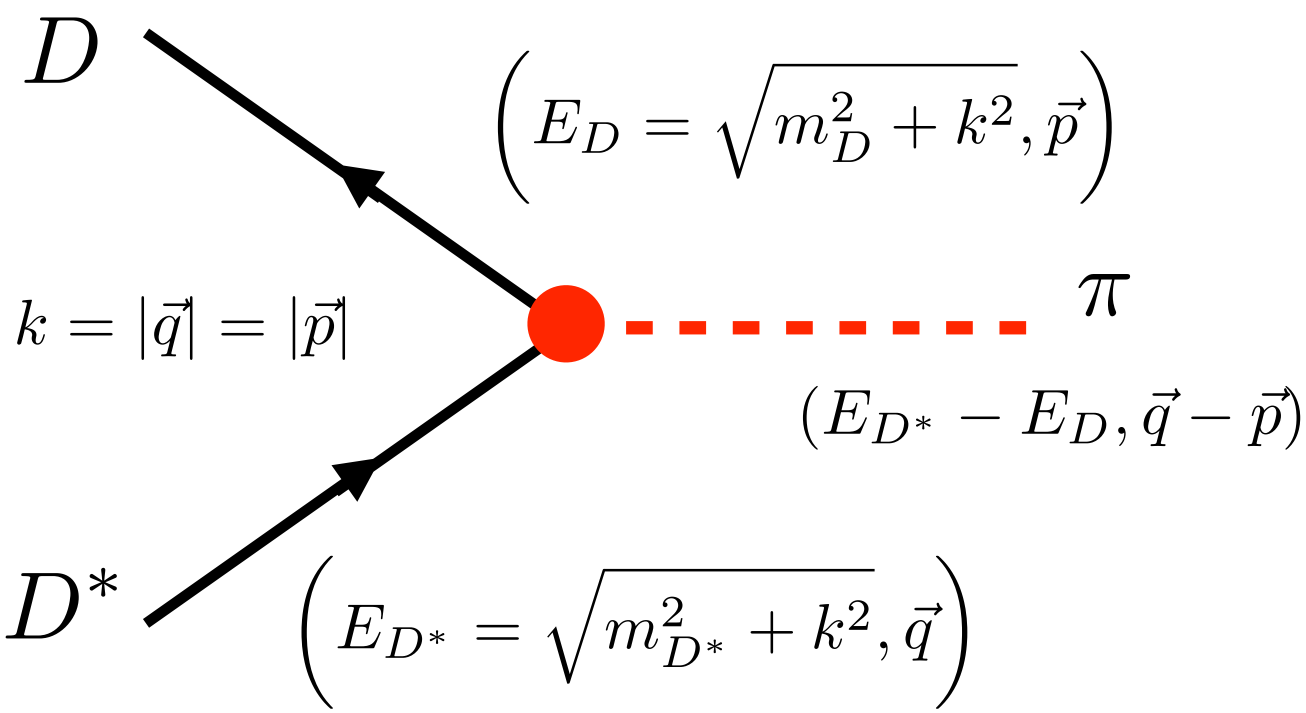
III. HAL QCD method with LHC

IV. Conclusion and discussion

II. Bound state below LHC



Left-hand cut



pion propagator

$$\frac{1}{(E_{D^*} - E_D)^2 - (\vec{q} - \vec{p})^2 - m_\pi^2}$$

$\downarrow$ S-wave projection

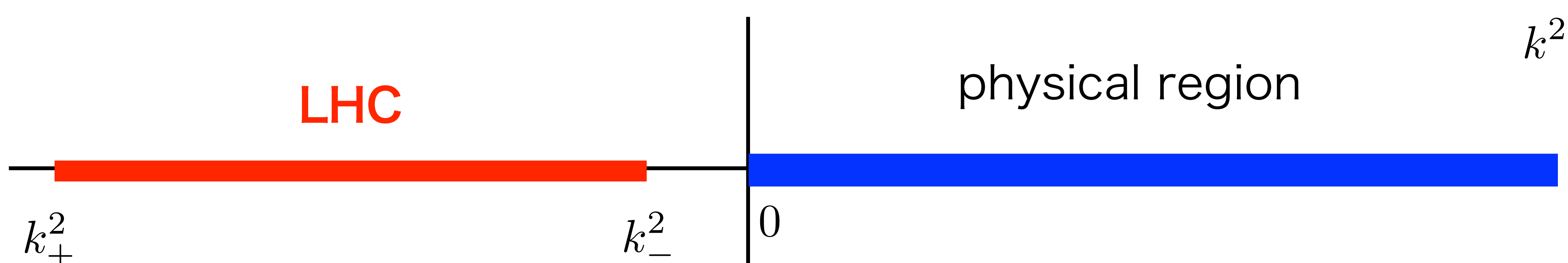
$$\propto \frac{1}{k^2} \log \left| \frac{m_{D^*}^2 + m_D^2 - m_\pi^2 - 2E_{D^*}E_D + 2k^2}{m_{D^*}^2 + m_D^2 - m_\pi^2 - 2E_{D^*}E_D - 2k^2} \right|$$

Branch points appear at

$$k_\pm^2 = \frac{\left[(m_{D^*} - m_D)^2 - m_\pi^2\right]\left[(m_{D^*} + m_D)^2 - m_\pi^2\right]}{4A_\pm}$$

$$A_+ = m_\pi^2 << \; A_- = 2(m_{D^*}^2 + m_D^2) - m_\pi^2$$

Therefore, if $(m_{D^*} - m_D)^2 - m_\pi^2 < 0$, the cut appears at negative k^2 .



Left-hand cut in the potential

S-wave Schrödiner equation $$\left[\frac{d^2}{dr^2} - U(r) + k^2\right]\varphi(k,r) = 0$$

ex. virtual "pion" exchange $\longrightarrow$ Yukawa potential $U(r) = g\dfrac{e^{-mr}}{r}$

regular solution $\varphi(k,0) = 0, \quad \dfrac{d}{dr}\varphi(k,0) = 1$

$\longrightarrow$ $\varphi(k,r) \xrightarrow{r\to\infty} ae^{ikr} - be^{-ikr}$ $\qquad S(k) := \dfrac{a}{b}$ S-matrix

$$\boxed{\varphi(k,r) = \frac{1}{2ik}\left[e^{ikr}\mathcal{F}(-k,r) - e^{-ikr}\mathcal{F}(k,r)\right] \qquad \mathcal{F}(k,r) = 1 + \int_0^r dr' e^{ikr'}U(r')\varphi(k,r')}$$

If $U(r)$ is cutoff at R as $U(r>R) = 0$, $S(k)$ is well-defined for $^\forall k \in \mathbb{C}$. $\quad S(k) = \dfrac{\mathcal{F}(-k,R)}{\mathcal{F}(k,R)}$

No LHC (Who care the tail of the potential behind the moon ?)

How different is $S(k)$ in the large R limit from an analytic continuation w/o IR cutoff ?

$R \to \infty$ limit

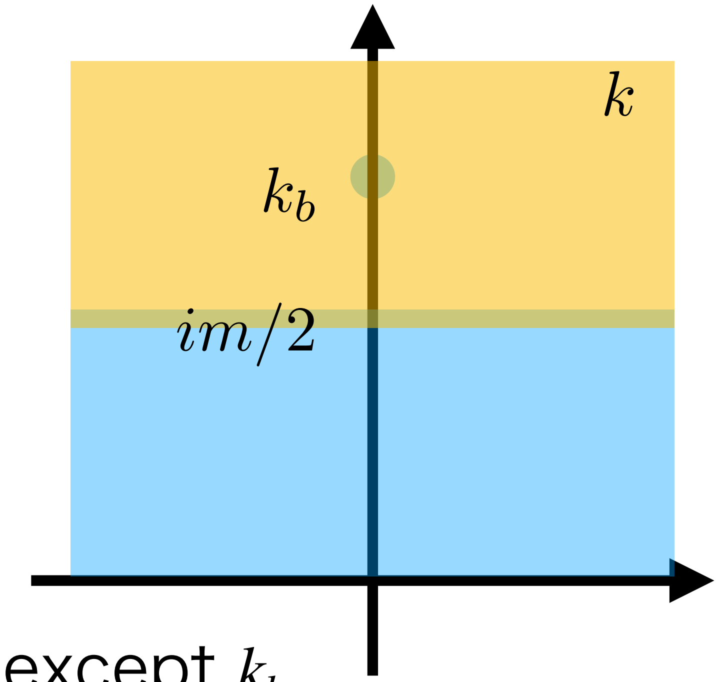
In the upper k plane ($\mathrm{Im}k \geq 0$), $\mathcal{F}(k,R)$ is convergent as $R \to \infty$, while

$$\mathcal{F}(-k,R) = 1 + \int_0^R dr' e^{-ikr'} U(r')\varphi(k,r') = \mathcal{F}(-k,\bar{R}) + \int_{\bar{R}}^R dr' e^{-ikr'} U(r')\varphi(k,r')$$

$$(2\,\mathrm{Im}\,k - m)r$$

$$\simeq \mathcal{F}(-k,\bar{R}) - g\frac{\mathcal{F}(k,\infty)}{2ik}\int_{\bar{R}}^R dr' e^{-2ikr'}\frac{e^{-mr'}}{r'} \qquad \varphi(k,r) \simeq -\frac{\mathcal{F}(k,\infty)}{2ik}e^{-ikr}$$

Therefore $\mathcal{F}(-k,R)$ is divergent (convergent) as $R \to \infty$ for $\mathrm{Im}\,k > m/2$ ($\mathrm{Im}\,k < m/2$). At k_b for a bound state, where $\mathcal{F}(k_b,\infty) = 0$, $\mathcal{F}(-k_b,R)$ is convergent.



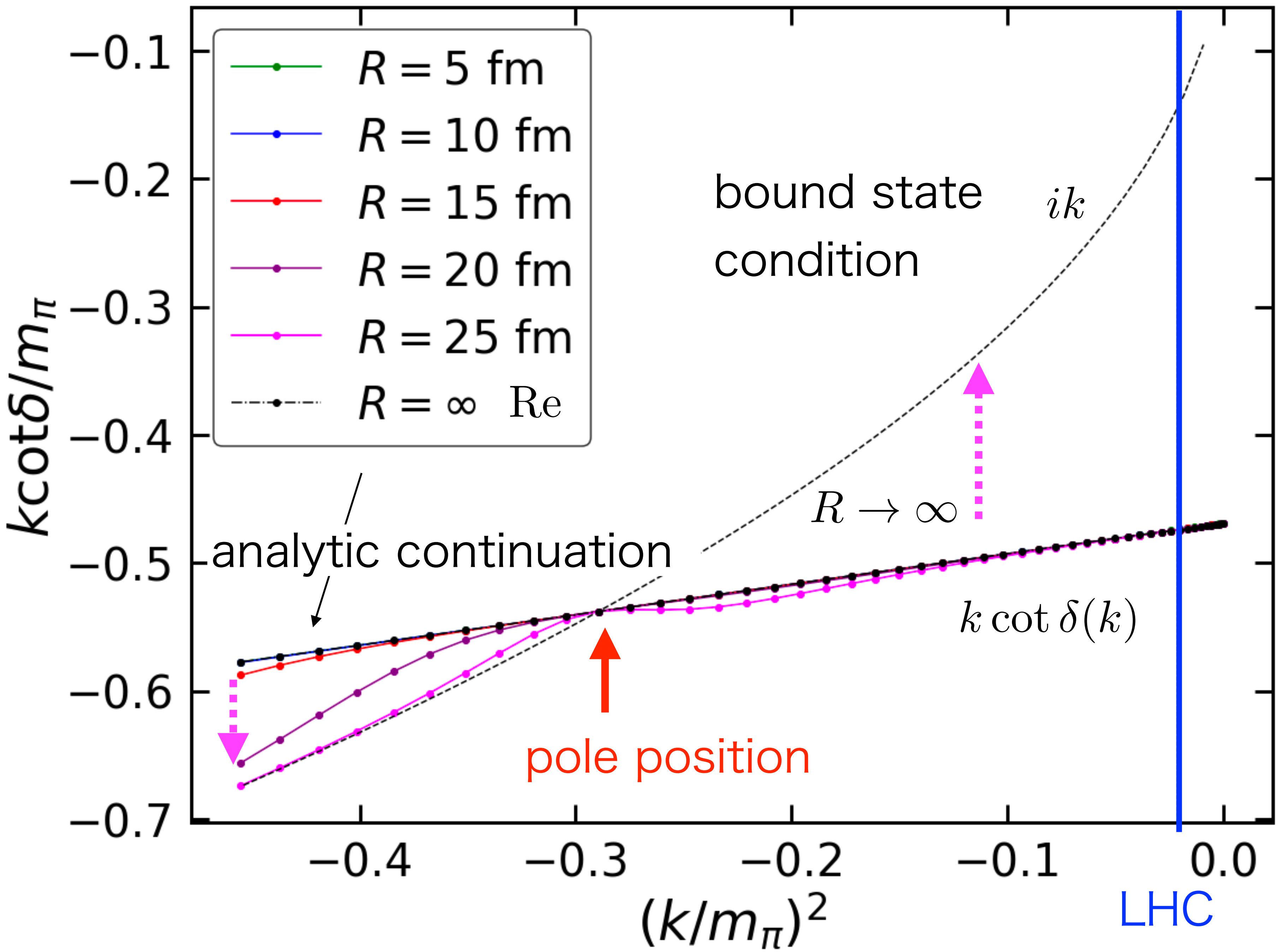
$S(k)$ is convergent at $\mathrm{Im}\,k < m/2$.

$S(k)$ is divergent at $\mathrm{Im}\,k > m/2$ due to $\mathcal{F}(-k,\infty)$ except k_b .

At finite R, $S(k)$ has a pole at $k = k_b(R)$, which converges to $k_b := k_b(\infty)$.

$S(k_b(R))$ diverges since the denominator becomes zero.

Gaussian + Yukawa potential

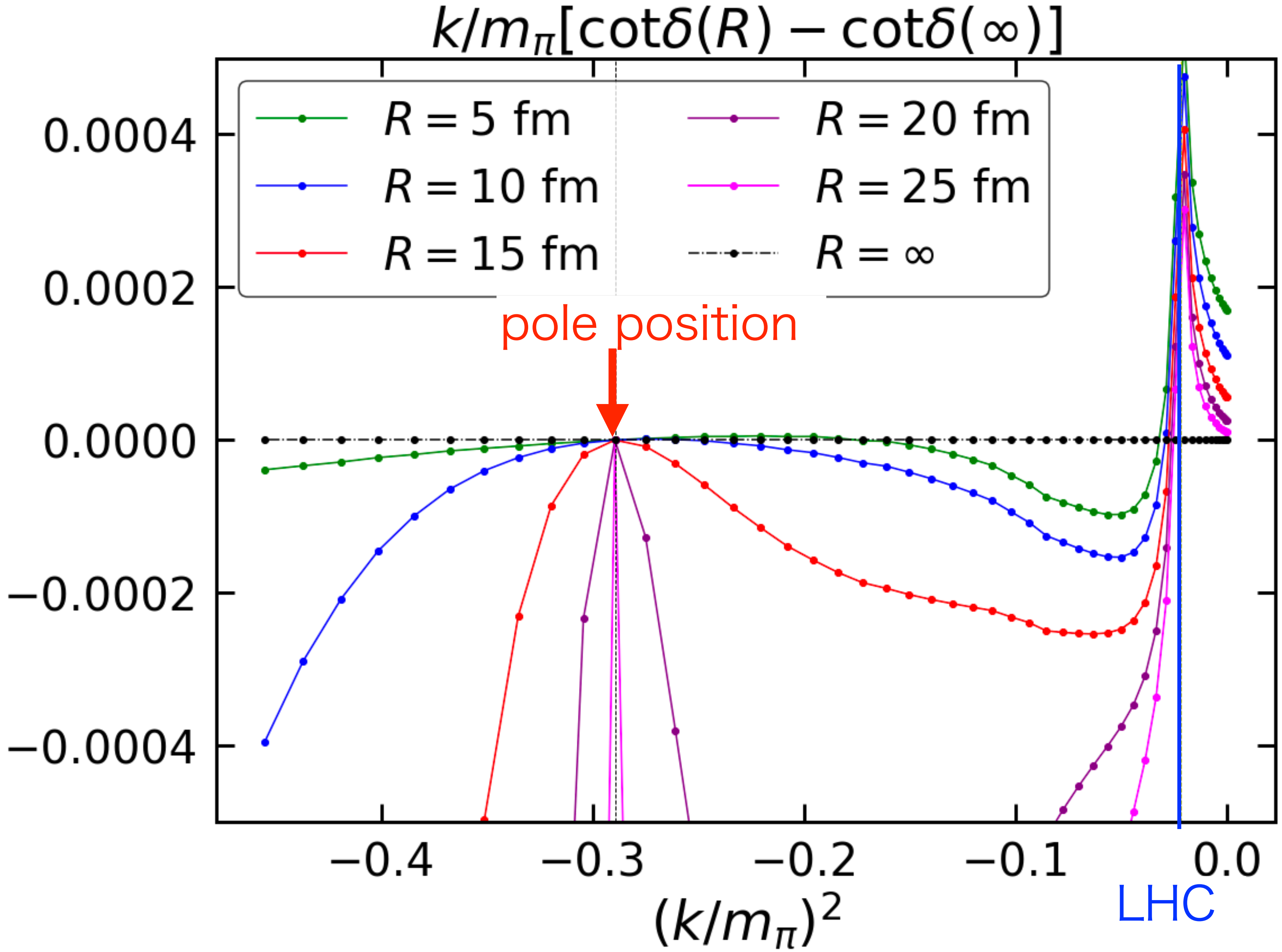


$$k\cot\delta(k) = ik\frac{S(k)+1}{S(k)-1}$$

At $\mathrm{Im}\,k < m/2$, $S(k)$ converges to $S_{\mathrm{anal}}(k)$, an analytic continuation of the physical S-matrix, as $R\to\infty$.

At $\mathrm{Im}\,k > m/2$, $S(k)$ diverges as $R\to\infty$, so that $k\cot\delta(k)$ converges to the bound state condition ik .

As $R\to\infty$, a pole position $k_b(R)$ of $S(k)$ converges to the pole k_b of $S_{\mathrm{anal}}(k)$, where $S(k_b(R))$ is always real.



At $\mathrm{Im}\, k < m/2$, $S(k)$ converges to $S_{\mathrm{anal}}(k)$, an analytic continuation of the physical S-matrix, as $R \to \infty$.

As $R \to \infty$, a pole position $k_b(R)$ of $S(k)$ converges to the pole k_b of $S_{\mathrm{anal}}(k)$, where $S(k_b(R))$ is always real.

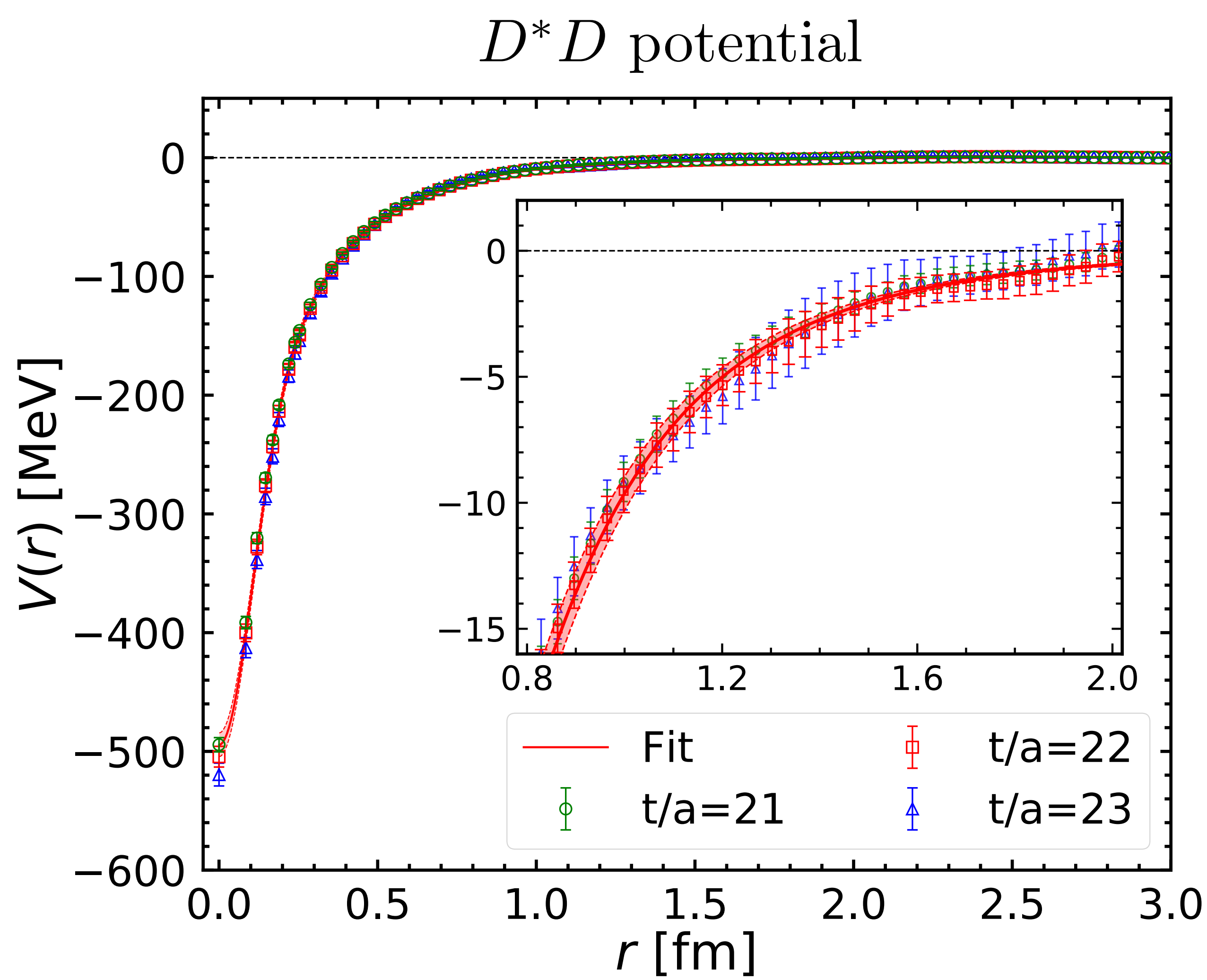
III. HAL QCD method with LHC

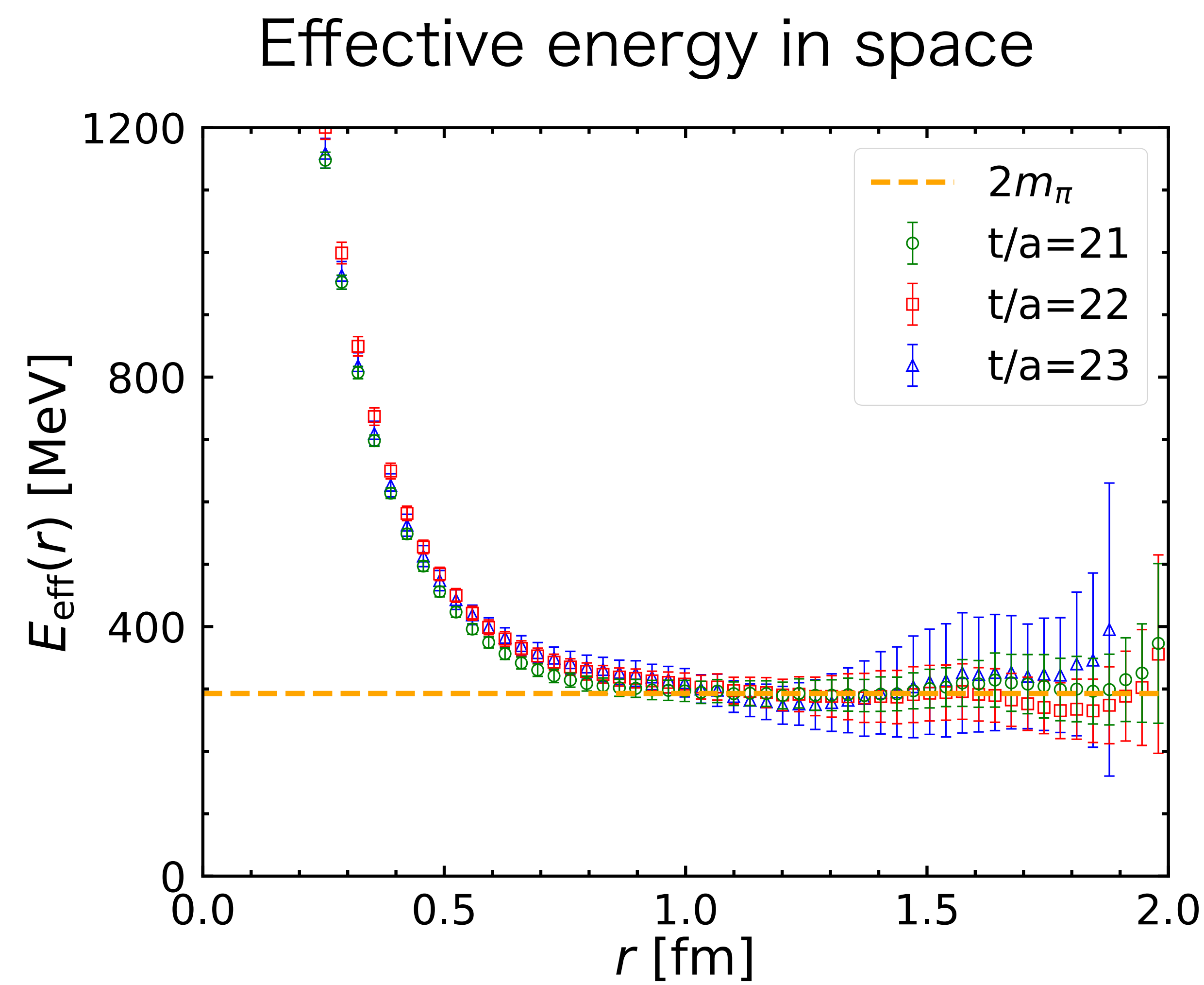
How can we treat the LHC in the HAL QCD method ?

1. Investigate the long distance behavior of the potential numerically

Ex. D^*D Potentials

Lyu et al. PRL 131 (2023) 161901.





consistent with Yukawa^2 at large r

$$E_{\mathrm{eff}}(r) = -\frac{\ln[V(r)r^2/a_3]}{r}$$

2-pion rather than 1-pion exchange is observed.

2-Gauss + Yukawa^2

$$V_{\mathrm{fit}}(r; m_\pi) = \sum_{i=1,2} a_i e^{-(r/b_i)^2} + a_3(1 - e^{-(r/b_3)^2})^2 \left(\frac{e^{-m_\pi r}}{r}\right)^2$$

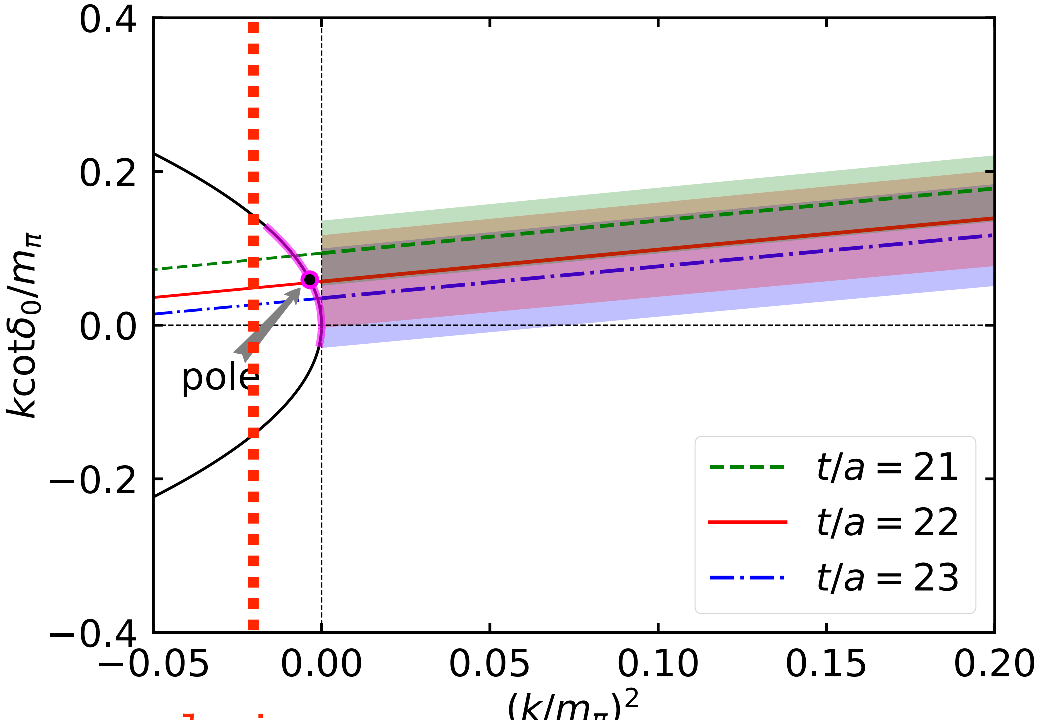
Caution: 1-pion exchange is NOT a dominant contribution for T_{cc} .

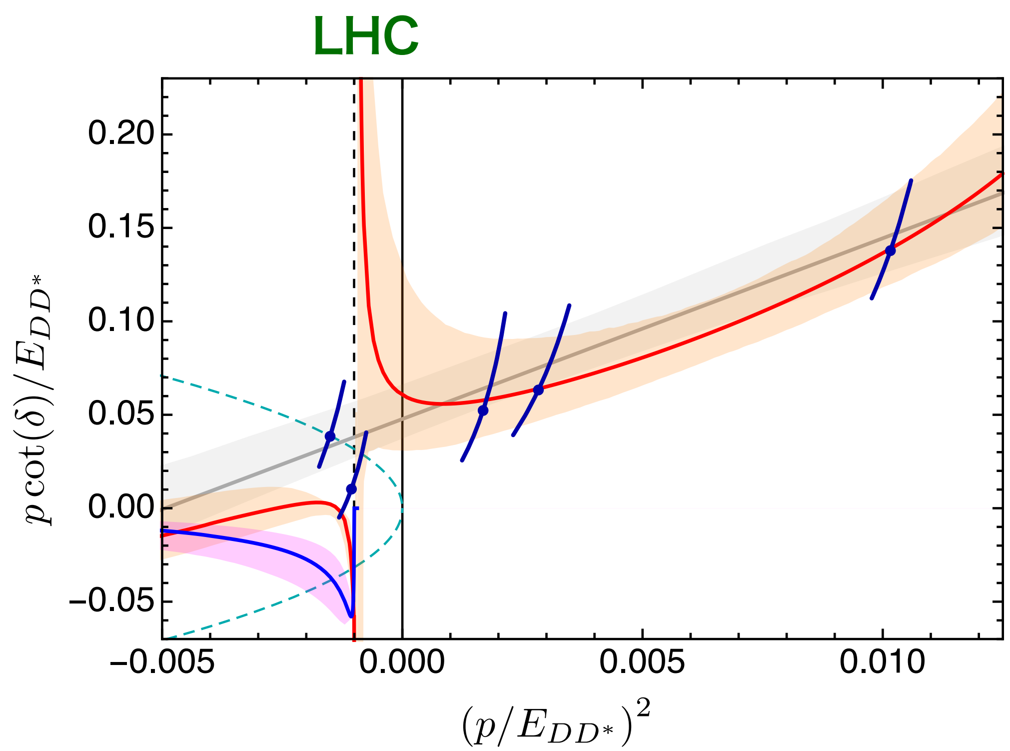
2. Estimate the position of the LHC.

2-pion $\dfrac{k_-^2}{m_\pi^2} \simeq \dfrac{(m_{D^*} - m_D)^2 - (2m_\pi)^2}{4m_\pi^2} \simeq -0.77$

1-pion $\dfrac{k_-^2}{m_\pi^2} \simeq -0.02$

3. Compare the analytic continuation with the position of the LHC





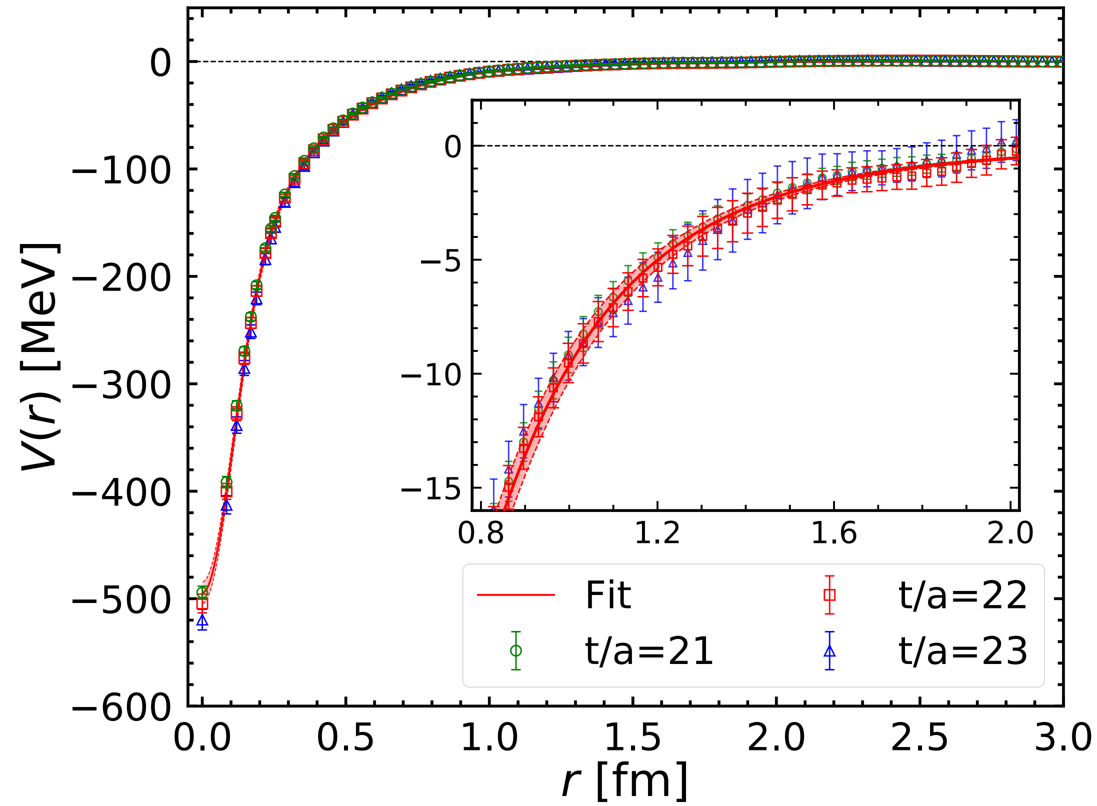
If a bound (virtual) state appears above the LHC, the conclusion from the analytic continuation is valid.

If a virtual state appears below the LHC, the analysis should be made including the LHC through the potential.

If a bound state appears below the LHC, we should directly extract the binding energy w/o analytic continuation.

4. If the LHC is expected to exist, we may include it in the potential fit.

Ex. Even though the one-pion exchange is not seen in D^*D potential, we include it as an alternative fit to estimate its effect.



Gauss + (Yukawa^2) + Yukawa

$$V_{\mathrm{fit}}(r; m_\pi) = \sum_i a_i e^{-(r/b_i)^2} + a_3(1 - e^{-(r/b_3)^2})^2 \left(\frac{e^{-m_\pi r}}{r}\right)^2 + a_4(1 - e^{-(r/b_4)^2}) \left(\frac{e^{-m_\pi r}}{r}\right)$$

The analysis with the alternative fit gives an estimate for systematic errors.

The binding energy is not affected at all by the LHC, if the Schrödinger equation is directly solved.

IV. Conclusion and discussion

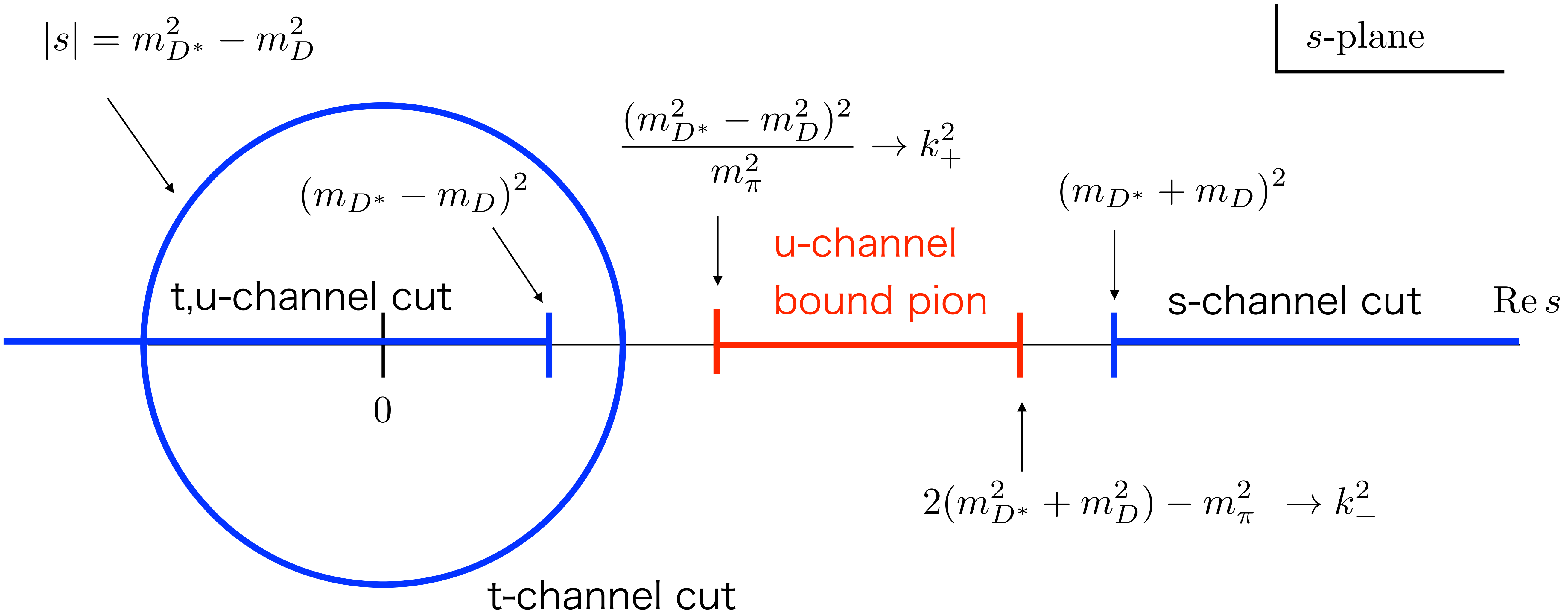
- The analytic continuation and the result in the limit of the infra-red cutoff $R \to \infty$ differ below the LHC, except the binding energy k_b .
- Thus, the binding energy in the HAL QCD method is not affected by the LHC.
- For the virtual state below the LHC, the analytic continuation can be performed through the HAL QCD potential to include an effect of the LHC.
    - The long distance behavior which causes the LHC may be included in the fit of the HAL QCD potential.
- A doubly charmed tetra-quark T_{cc}^{+} in the previous HAL QCD method appears as a virtual state above the LHC, and thus remains valid.

In order to control the effect of LHC, it is essential to determine the long distance behavior of the potential: The HAL QCD method is a suitable framework to do this explicitly.

A relation between the LHC and the analytic structure of the partial wave amplitude in the S-matrix theory.

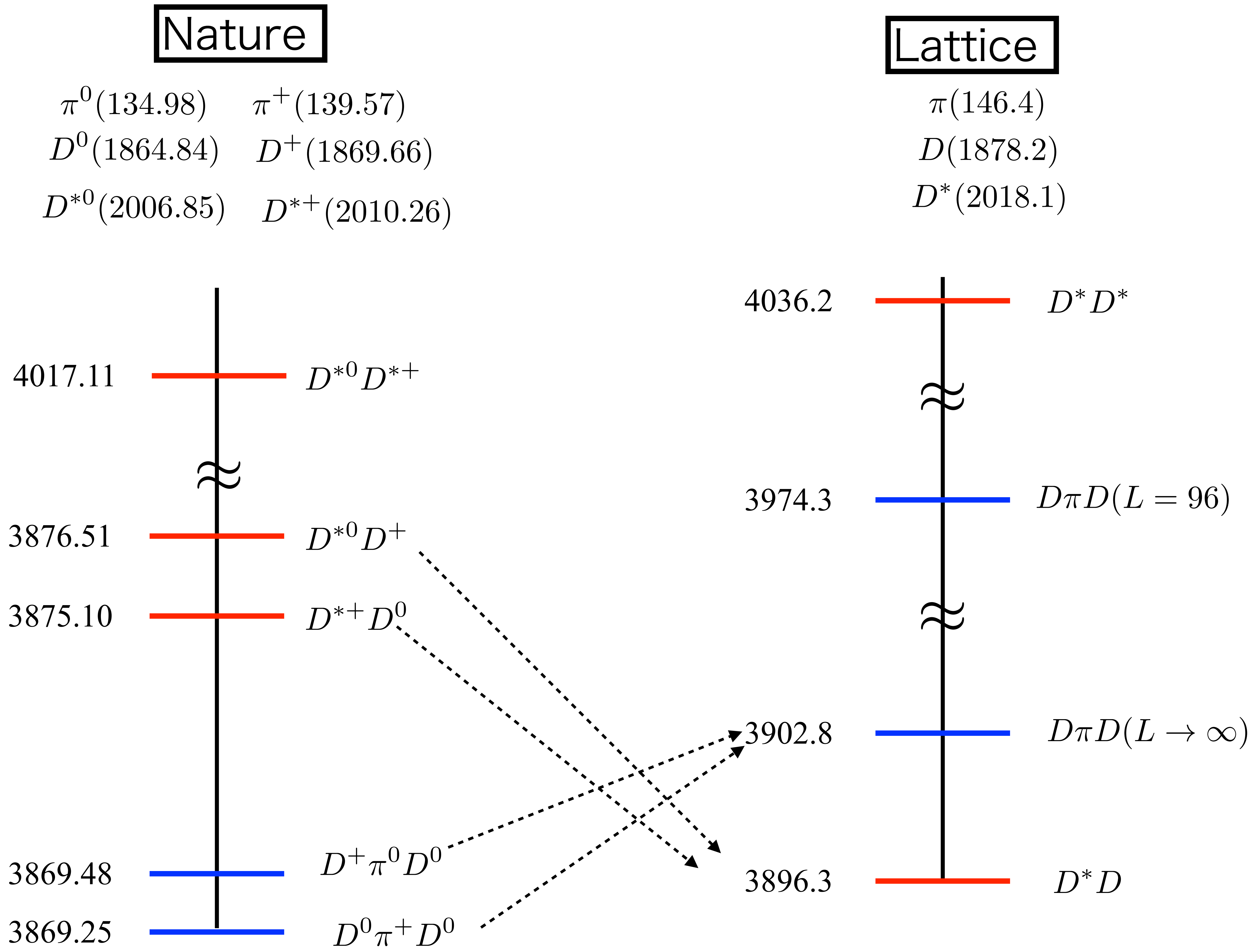
Cut from the u-channel bound pion = Left-hand cut ?

$$D^* + \bar{D} \longrightarrow \pi \longrightarrow D^* + \bar{D}$$



Thank you for your attention !

Backup



Nature
$\pi^0(134.98)$ $\pi^+(139.57)$
$D^0(1864.84)$ $D^+(1869.66)$
$D^{*0}(2006.85)$ $D^{*+}(2010.26)$
4017.11 $D^{*0}D^{*+}$
3876.51 $D^{*0}D^+$
3875.10 $D^{*+}D^0$
3869.48 $D^+\pi^0D^0$
3869.25 $D^0\pi^+D^0$
Lattice
$\pi(146.4)$
$D(1878.2)$
$D^*(2018.1)$
4036.2 D^*D^*
3974.3 $D\pi D(L=96)$
3902.8 $D\pi D(L\to\infty)$
3896.3 D^*D