41st International Symposium on Lattice Field Theory
28th July-3rd August 2024, Liverpool

Lüscher formalism on the left-hand cut: *an update*

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in collaboration with
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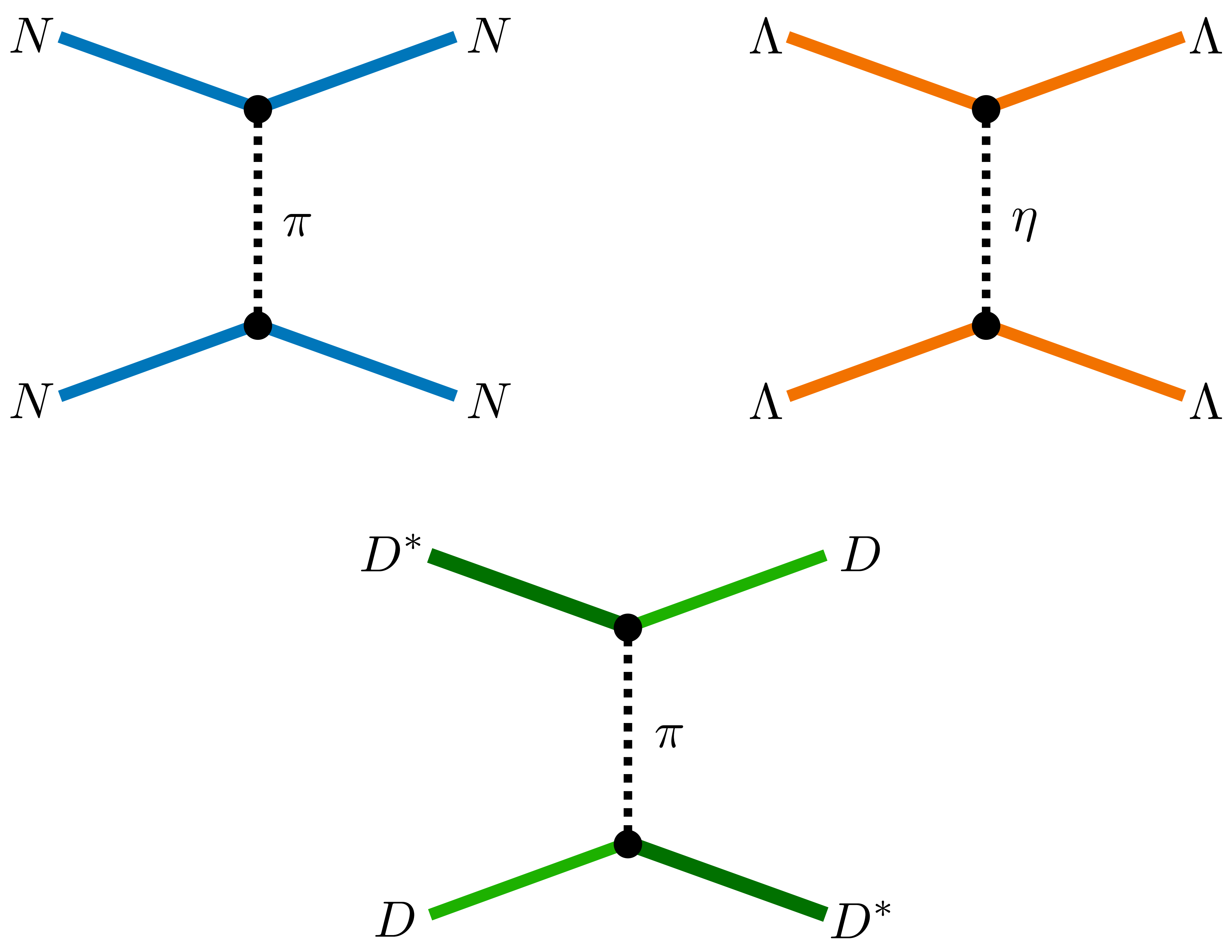
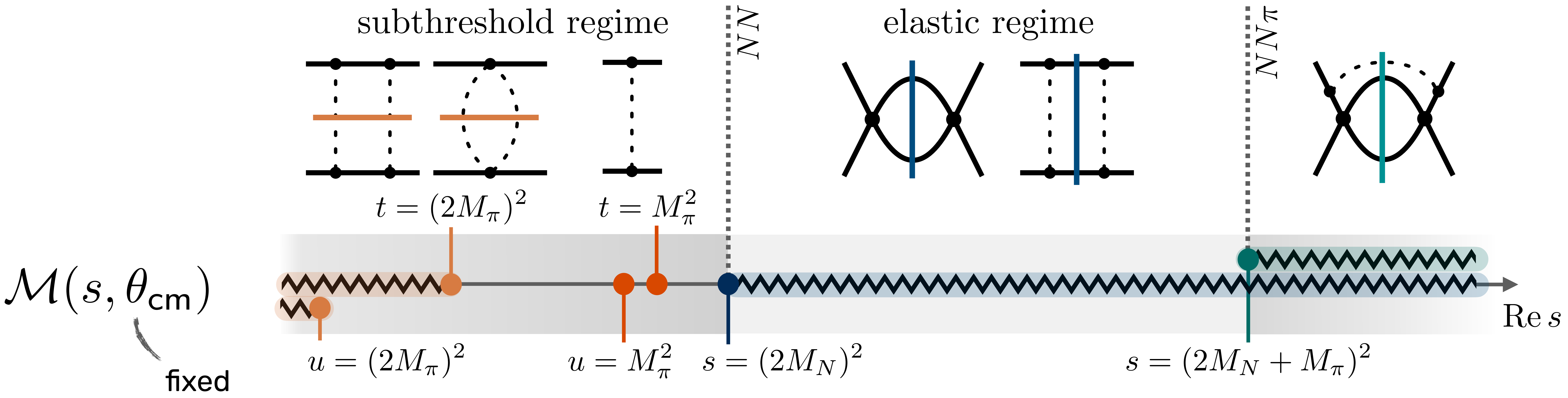
Motivation

- **Lüscher formalism** is widely used for extraction of 2-to-2 scattering amplitudes from finite-volume energies
- formalism breaks down at energies near/on **left-hand cuts** in partial-wave-projected amplitudes, as seen in
  - $\Lambda\Lambda$ [Green, Hanlon et al. '21] arXiv 2103.01054
  - DD^* (relevant for $T_{cc}(3875)^+$ ($cc\bar{u}\bar{d}$) tetraquark)
    [Padmanath, Prelovšek '22] arXiv 2202.10110
    [Du, Filin, Baru et al. '23] arXiv 2303.09441
    … many talks at Lattice 2023 and 2024
- left-hand cuts in the projected amplitudes due to light meson exchanges
- extension of standard formalism needed for resolving these issues



Amplitude and partial-wave projection

- We focused on the case of identical particles, e.g. NN scattering
- General **structure of the amplitude** at fixed CM scattering angle:
  - right-hand **2-particle cut** above NN threshold and **3-particle cut** above $NN\pi$ threshold
  - sub-threshold **poles** due to single exchanges and lower **left-hand cuts** due to multiple exchanges

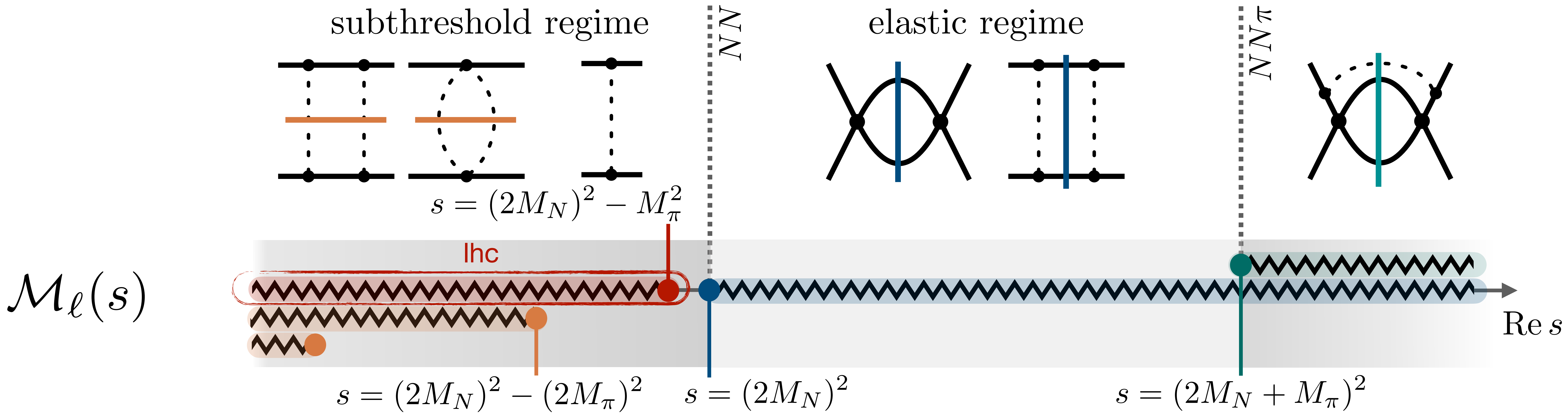


Amplitude and partial-wave projection

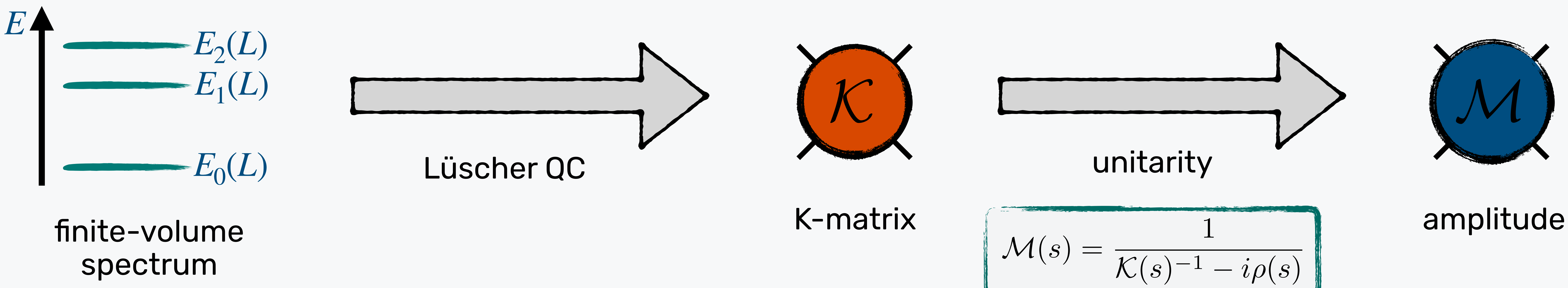
- Projecting to definite angular momentum $\mathcal{M}(s,\theta_{\text{cm}}) \longrightarrow \mathcal{M}_\ell(s) = \frac{1}{2}\int d\cos\theta_{\text{cm}}\, P_\ell(\cos\theta_{\text{cm}})\, \mathcal{M}(s,\theta_{\text{cm}})$

Legendre polynomial

… sub-threshold poles become the **nearest left-hand cut**



Lüscher formalism breakdown

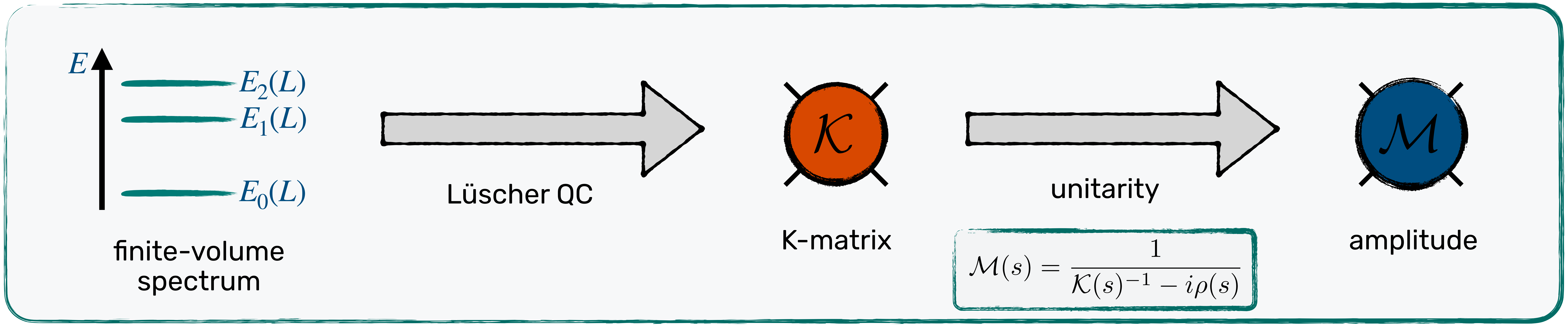


$$\mathcal{M}(s) = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

Lüscher quantisation condition:

$$\det_{\ell m}\left[\mathcal{K}(E,\boldsymbol{P})^{-1} + F(E,\boldsymbol{P},L)\right] = 0$$ at the FV energies

Lüscher formalism breakdown



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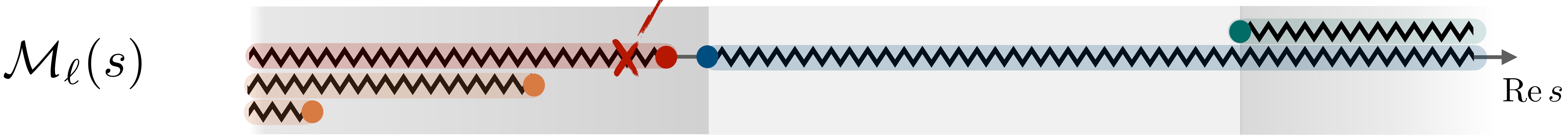

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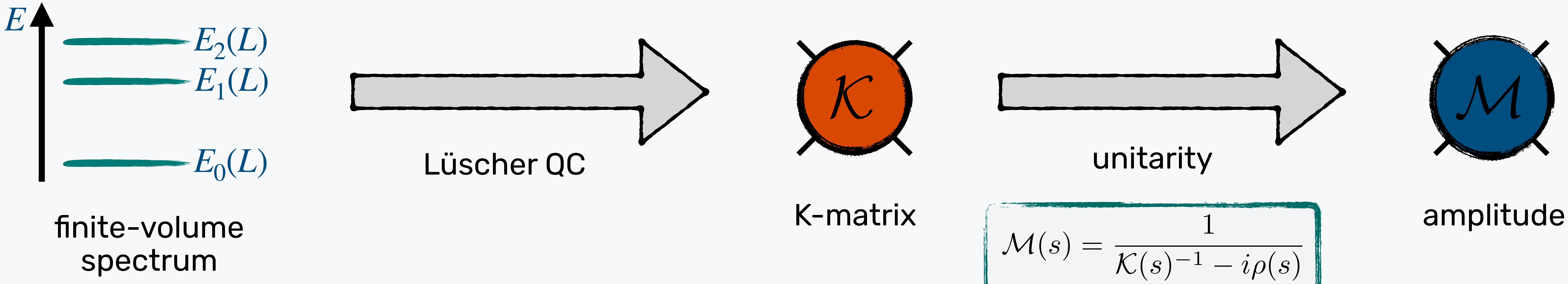


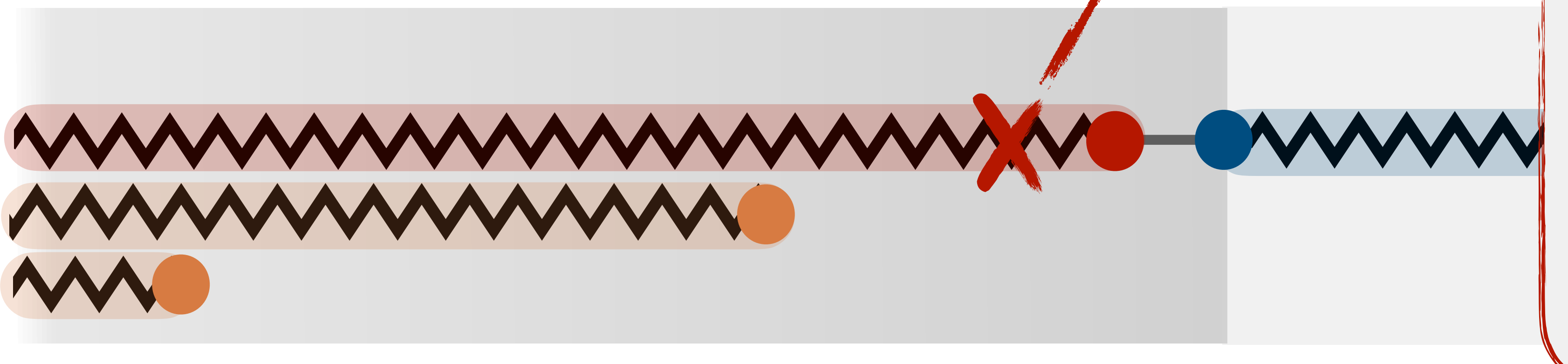
$$\mathcal{M}(s) = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

Lüscher quantisation condition:

$$\det_{\ell m}\left[\mathcal{K}(E,\boldsymbol{P})^{-1} + F(E,\boldsymbol{P},L)\right] = 0$$ at the FV energies

- F is real on the cut → solutions for $\mathscr{K}$ are real
- $\mathscr{K}$ should be complex on the cut!
- **cannot apply the usual QC!!!**

$\mathcal{M}_\ell(s)$



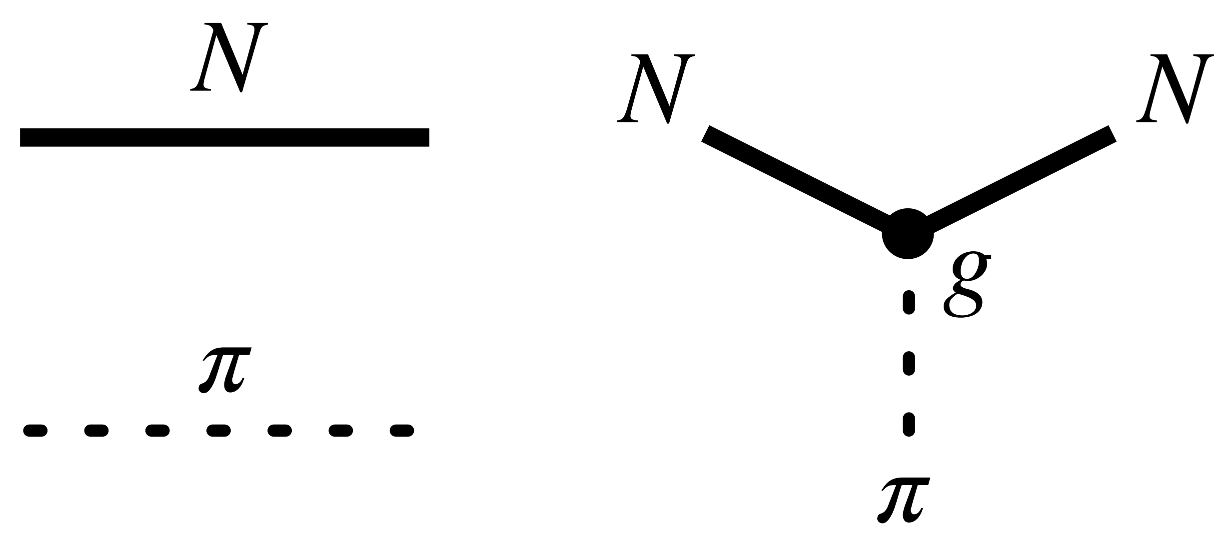
[Green, Hanlon et al. '21] arXiv 2103.01054

Other approaches to the left-hand cut problem

- Use a plane-wave basis (no projection to angular momentum) [Meng, Epelbaum '21]
- HAL QCD method [Lyu et al. '23] (see Sinya Aoki's talk just before)
- Apply the 3-particle RFT formalism [Hansen et al. '24] – applied to $DD\pi$ system (talk by Sebastian M. Dawid on Monday)
- NREFT-based approach [Bubna et al. '24] (see the next talk by Akaki Rusetski)

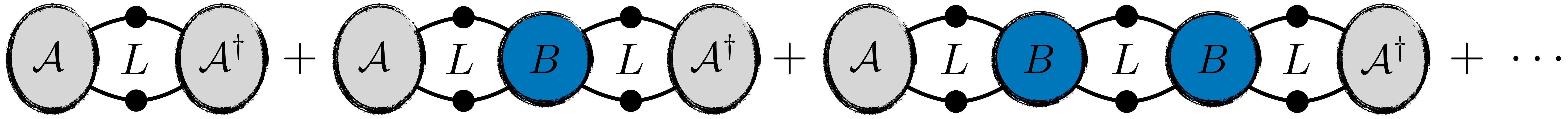
Finite-volume analysis

- focus on two-nucleon elastic scattering $NN \to NN$ as model system which includes a left-hand cut



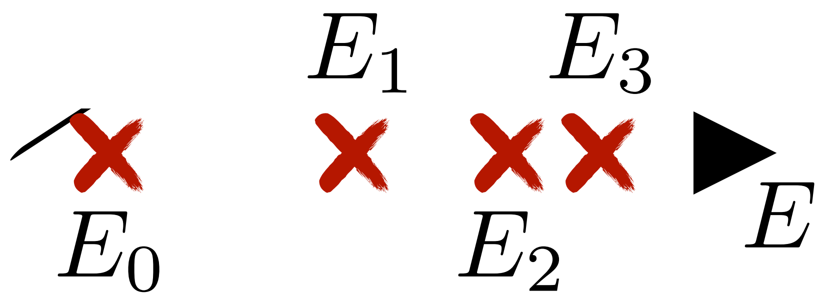
- use generic, EFT-independent, all-orders diagrammatic expansion in periodic cubic finite volume
- considered a **"skeleton expansion"** for a finite-volume correlation function…

$C_L(P) =$



[Kim, Sachrajda, Sharpe '05] arXiv 0507006

has poles at the FV energies



… and track the volume dependence of the different building blocks to derive a QC

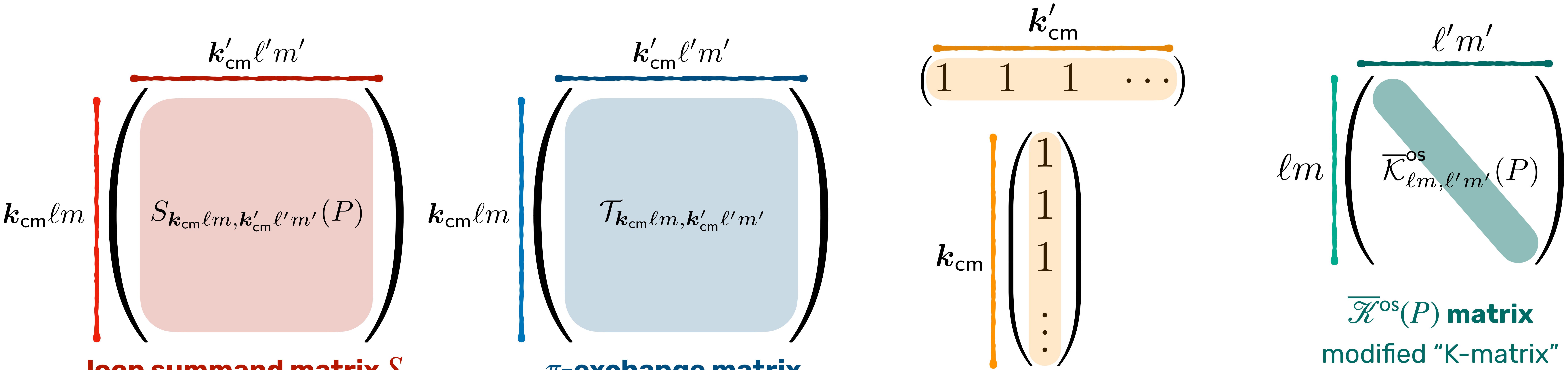
Adapted quantisation condition

Revisiting derivation step-by-step, we derive the following **modified QC** for identical spinless particles:

$$\det_{\boldsymbol{k}_{\mathrm{cm}}\ell m}\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}^{\mathrm{os}}(P)\xi + 2g^2\mathcal{T}\right] = 0$$

used to constrain $\overline{\mathcal{K}}^{\mathrm{os}}(P)$ from the finite-volume spectrum

[ABR, Hansen '23] arXiv 2311.18793



loop summand matrix S
(known functions)

π-exchange matrix
(known off-shell logs)

$\xi, \xi^\dagger$ trivial projector

$\overline{\mathcal{K}}^{\mathrm{os}}(P)$ matrix
modified "K-matrix"

g – $NN\pi$ effective coupling

Adapted quantisation condition

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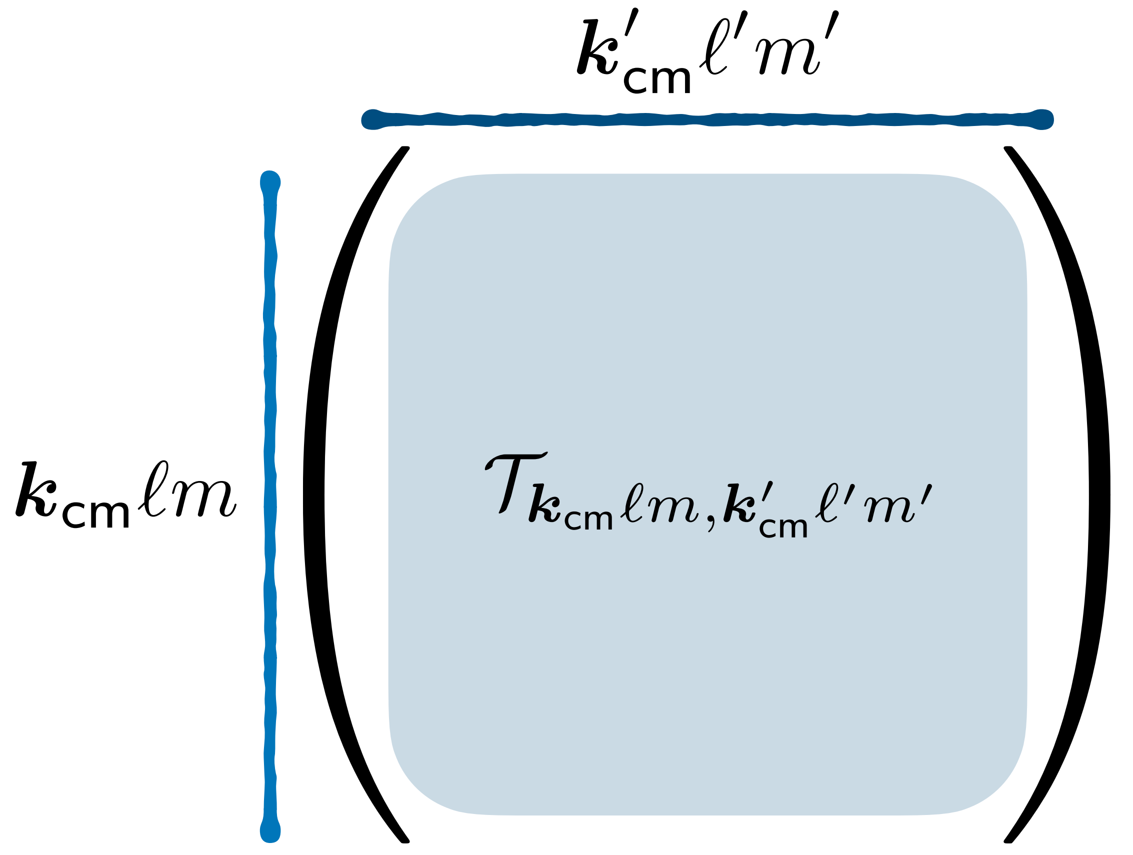

π-exchange matrix
(known off-shell logs)

e.g. S-wave result

$$\mathcal{T}_{\boldsymbol{k}_{\mathrm{cm}}00,\boldsymbol{k}'_{\mathrm{cm}}00} = \frac{1}{4|\boldsymbol{k}_{\mathrm{cm}}||\boldsymbol{k}'_{\mathrm{cm}}|}\log\left(\frac{2\omega_N(\boldsymbol{k}_{\mathrm{cm}})\omega_N(\boldsymbol{k}'_{\mathrm{cm}}) + 2|\boldsymbol{k}_{\mathrm{cm}}||\boldsymbol{k}'_{\mathrm{cm}}| - 2M_N^2 + M_\pi^2 - i\epsilon}{2\omega_N(\boldsymbol{k}_{\mathrm{cm}})\omega_N(\boldsymbol{k}'_{\mathrm{cm}}) - 2|\boldsymbol{k}_{\mathrm{cm}}||\boldsymbol{k}'_{\mathrm{cm}}| - 2M_N^2 + M_\pi^2 - i\epsilon}\right)$$

$$\omega_N(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + M_N^2}$$

Adapted quantisation condition

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used to constrain $\overline{\mathscr{K}}^{\text{os}}(P)$ from the finite-volume spectrum

- separates the "physical ingredients": $\overline{\mathscr{K}}^{\text{os}}(P)$ contains all short-range physics
  $\mathscr{T}$ is the single π-exchange
- extended index space $\boldsymbol{k}_{\text{cm}}\ell m, \boldsymbol{k}'_{\text{cm}}\ell' m'$ (reminiscent of three-particle formalisms)
- inclusion of spin straightforward: index space expanded to include spin state labels

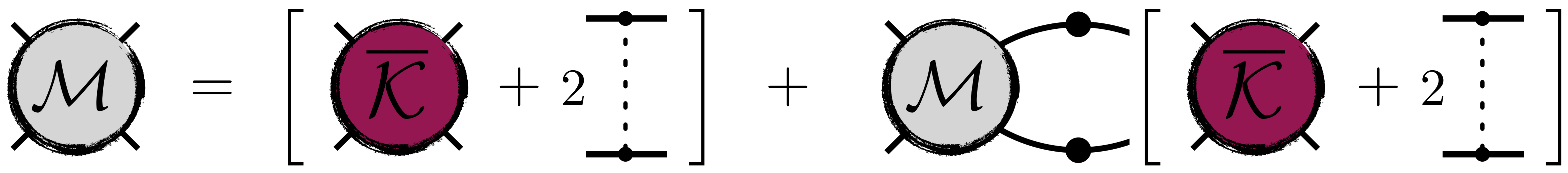
Extracting the amplitude

An extra step is needed to connect K-bar to the amplitude:

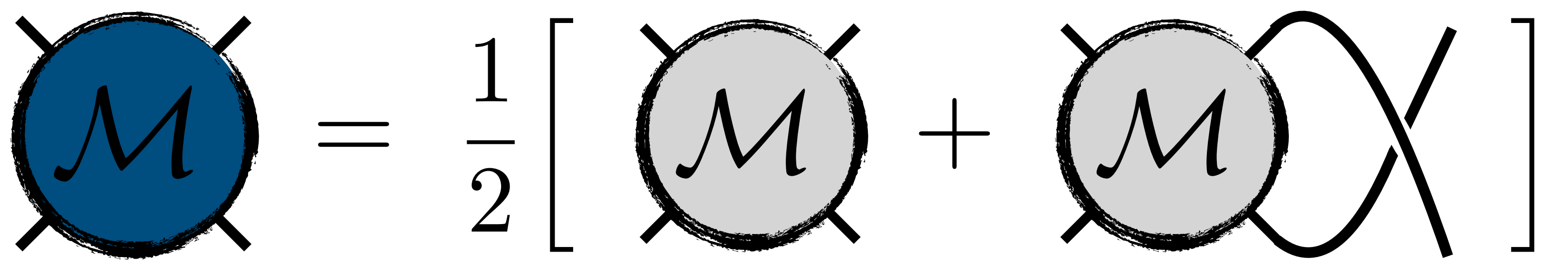
… we need to solve integral equations:

$$\mathcal{M}^{\text{aux}}(P,p,p') = \mathcal{K}^{\mathcal{T}}(P,p,p') - \frac{1}{2}\int \frac{d^3\boldsymbol{k}_{\text{cm}}}{(2\pi)^3} \frac{\mathcal{M}^{\text{aux}}(P,p,k)\,\mathcal{K}^{\mathcal{T}}(P,k,p')\,e^{\alpha((k_{\text{cm}}^{\text{os}})^2-\boldsymbol{k}_{\text{cm}}^2)}}{4\omega_N(\boldsymbol{k}_{\text{cm}})\,[(k_{\text{cm}}^{\text{os}})^2-\boldsymbol{k}_{\text{cm}}^2+i\epsilon]}$$

$$\mathcal{K}^{\mathcal{T}}(P,p,p') = \overline{\mathcal{K}}^{\text{os}}(P,p,p') + 2g^2\mathcal{T}(P,p,p')$$



solve for auxiliary amplitude



symmetrize to get amplitude

Quantisation condition in the ℓm basis

We can obtain the condition in an alternative form, purely in the ℓm basis:

$$\det_{\ell m}\left[\overline{\mathcal{K}}^{\text{os}}(P)^{-1} + F^{\mathcal{T}}(P,L)\right] = 0$$

[ABR, Hansen '23] arXiv 2311.18793

finite-volume function:

$$F^{\mathcal{T}}(P,L) = \xi S(P,L) \frac{1}{1 + 2g^2 \mathcal{T}(P) S(P,L)} \xi^{\dagger}$$

$\boldsymbol{k}_{\text{cm}}, \boldsymbol{k}'_{\text{cm}}$ indices still present internally

- more similar in form to standard Lüscher QC
- can use usual irrep projection technology
- finite-volume function is trickier to evaluate

Quantisation condition in the ℓm basis

We can obtain the condition in an alternative form, purely in the ℓm basis:

$$\det_{\ell m}\left[\overline{\mathcal{K}}^{\text{os}}(P)^{-1} + F^{\mathcal{T}}(P,L)\right] = 0$$

… can be rewritten in many ways, e.g.

$$\det_{\ell m}\left[\widetilde{\mathcal{K}}_0(P)^{-1} + \widetilde{F}(P,L) + \Delta F^{\mathcal{T}}(P,L)\right] = 0$$

- $\tilde{\mathscr{K}}_0(P)$ becomes standard K-matrix as $g \to 0$ (when we turn off the π exchanges)
- $\tilde{F}(P,L)$ is the standard Lüscher finite-volume function (up to kinematic factor)

$$\Delta F^{\mathcal{T}}(P,L) = F^{\mathcal{T}}(P,L) - \xi S(P,L)\xi^{\dagger}$$

$$\widetilde{\mathcal{K}}_0(P)^{-1} = \overline{\mathcal{K}}^{\text{os}}(P)^{-1} + \widetilde{I}(P) \qquad \widetilde{I}_{\ell m,\ell' m'}(P) = \frac{1}{2}\,\text{pv}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}_{\text{cm}})\, Y^{*}_{\ell' m'}(\hat{\boldsymbol{k}}_{\text{cm}})\, |\boldsymbol{k}_{\text{cm}}|^{\ell+\ell'}\, H(\boldsymbol{k}_{\text{cm}})}{4\omega_N(\boldsymbol{k})\left[(k^{\text{os}}_{\text{cm}})^2 - (\boldsymbol{k}_{\text{cm}})^2\right]}$$

Exchanges in momentum-only basis

$$F^{\mathcal{T}}(P,L) = \xi S(P,L)\frac{1}{1+2g^2\mathcal{T}(P)S(P,L)}\xi^\dagger$$

can shuffle factors between internal building blocks and re-sum over angular momentum

$$F^{\mathcal{T}}_{\ell m,\ell' m'}(P,L) = \mathsf{Y}_{\ell m}\,\mathsf{S}(P,L)\frac{1}{1+2g^2\mathsf{T}(P,L)\,\mathsf{S}(P,L)}\mathsf{Y}^\dagger_{\ell' m'}$$

matrix in $\boldsymbol{k}_{\text{cm}}, \boldsymbol{k}'_{\text{cm}}$ space

exchange recovers exact pole form

$$(\mathsf{Y}_{\ell m})_{\boldsymbol{k}_{\text{cm}}} = \sqrt{4\pi}\,|\boldsymbol{k}_{\text{cm}}|^\ell\, Y_{\ell m}(\hat{\boldsymbol{k}}_{\text{cm}})$$

$$\mathsf{T}_{\boldsymbol{k}_{\text{cm}}\boldsymbol{k}'_{\text{cm}}}(P,L) = -\frac{1}{\left(\omega_N(\boldsymbol{k}_{\text{cm}})-\omega_N(\boldsymbol{k}'_{\text{cm}})\right)^2-(\boldsymbol{k}_{\text{cm}}-\boldsymbol{k}'_{\text{cm}})^2-M_\pi^2+i\epsilon}$$

similarities with plane-wave basis approach suggested in [Meng, Epelbaum '21], but **QC is still in ℓm basis**

→ may improve convergence of the finite-volume function to work in this basis (numerical tests needed)

Simplification of the integral equations

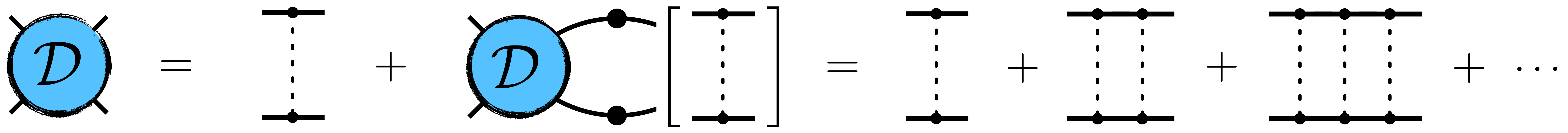
- We can show that there is a direct algebraic relation between $\overline{\mathscr{K}}^{\text{os}}$ and the infinite-volume amplitude:

$$\mathcal{M}(P) = \mathcal{D}(P) - \mathcal{D}^{\text{L}}_{\ell m}(P) \left[\frac{1}{\overline{\mathcal{K}}^{\text{os}}(P)^{-1} + \mathcal{D}^{\text{C}}(P)} \right]_{\ell m, \ell' m'} \mathcal{D}^{\text{R}}_{\ell' m'}(P)$$

(sum over repeated indices)

where $\mathscr{D}(P)$ **solves the auxiliary integral equation**:

$$\mathcal{D}(P,p,p') = 2g^2 \mathcal{T}(P,p,p') - \frac{1}{2} \int \frac{d^3\boldsymbol{k}_{\text{cm}}}{(2\pi)^3} \frac{\mathcal{D}(P,p,k)\, 2g^2 \mathcal{T}(P,k,p')\, e^{\alpha((k^{\text{os}}_{\text{cm}})^2 - \boldsymbol{k}^2_{\text{cm}})}}{4\omega_N(\boldsymbol{k}_{\text{cm}})\, [(k^{\text{os}}_{\text{cm}})^2 - \boldsymbol{k}^2_{\text{cm}} + i\epsilon]}$$

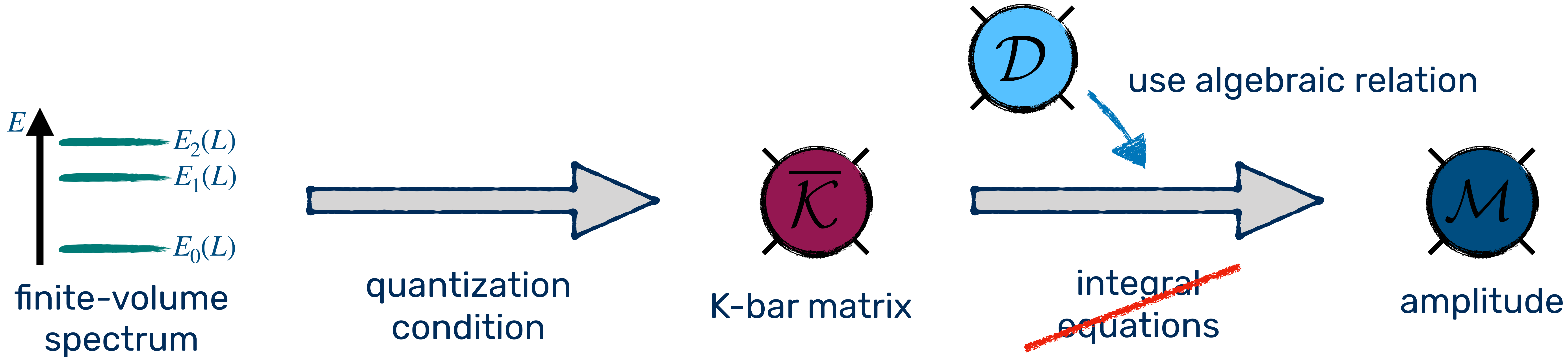


with on-shell external momenta, and the objects $\mathscr{D}^{\text{L}}$, $\mathscr{D}^{\text{R}}$, $\mathscr{D}^{\text{C}}$ are obtained from it

→ similar to ladder equation in 3-particle formalisms, can profit from the progress in solving this type of integral equation e.g. [Romero-Lopez et al. '19], [Jackura et al. '21], [Dawid et al. '23]

Simplification of the integral equations

This suggests a more direct approach for finding the amplitude, (partially) bypassing the integral equations



Summary

- we have presented relativistic quantisation conditions that apply on the nearest left-hand cut, extending the validity of the standard Lüscher condition

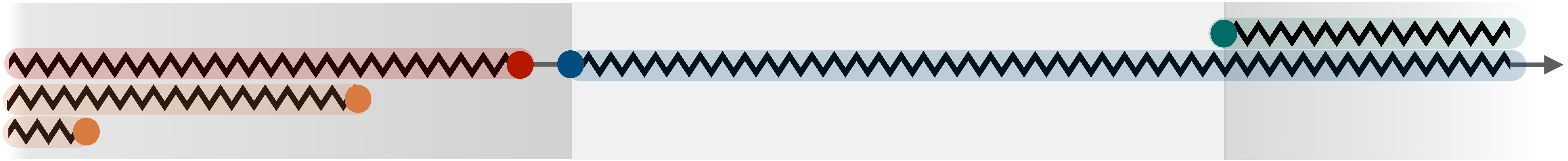
- QCs were introduced in both a mixed momentum and angular-momentum basis and angular-momentum-only basis

- the formalism applies in moving frames and has been extended to nonidentical particles and arbitrary spins

- different strategies have been proposed to extract amplitudes from intermediate K-matrices, including integral equations and a direct algebraic relation

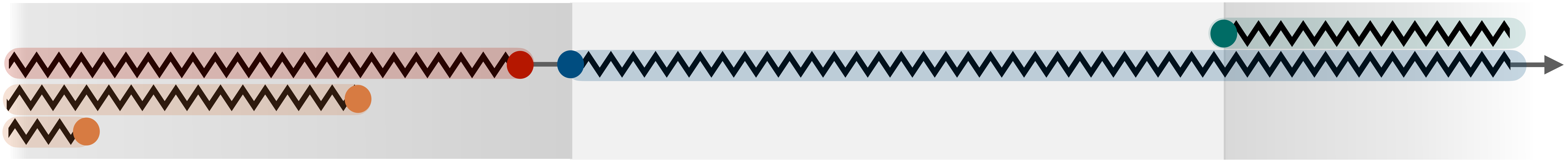
$$\det_{\boldsymbol{k}_{\mathsf{cm}}\ell m}\left[S(P,L)^{-1}+\xi^\dagger\overline{\mathcal{K}}^{\mathsf{os}}(P)\xi+2g^2\mathcal{T}\right]=0$$

$$\det_{\ell m}\left[\overline{\mathcal{K}}^{\mathsf{os}}(P)^{-1}+F^{\mathcal{T}}(P,L)\right]=0$$



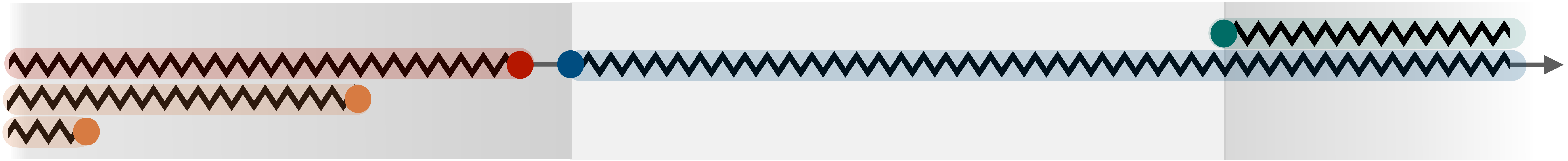
Outlook

- numerical testing of the formalism and consistency check with three-particle RFT formalism is underway
- extension to multichannel systems
- formal clarification of connections to three-particle formalisms (e.g. this method as a limiting case?)
- comparison with alternative approaches



Thank you for your attention!

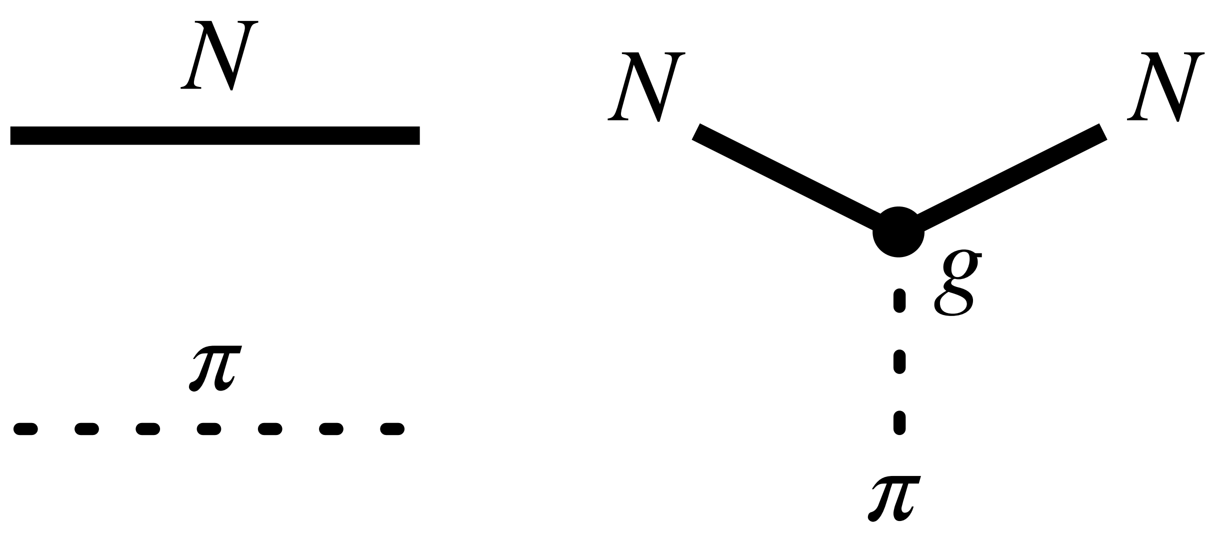
Any questions?



Introductory details

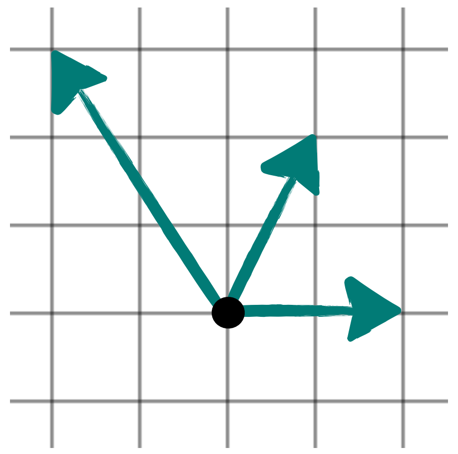
Theoretical setup:

- generic low-energy EFT with "nucleons" N and lighter "pions" π (masses M_N and M_π)
- N and π with arbitrary spins
- generic interactions, including $N\overline{N}\pi$ vertex with coupling g

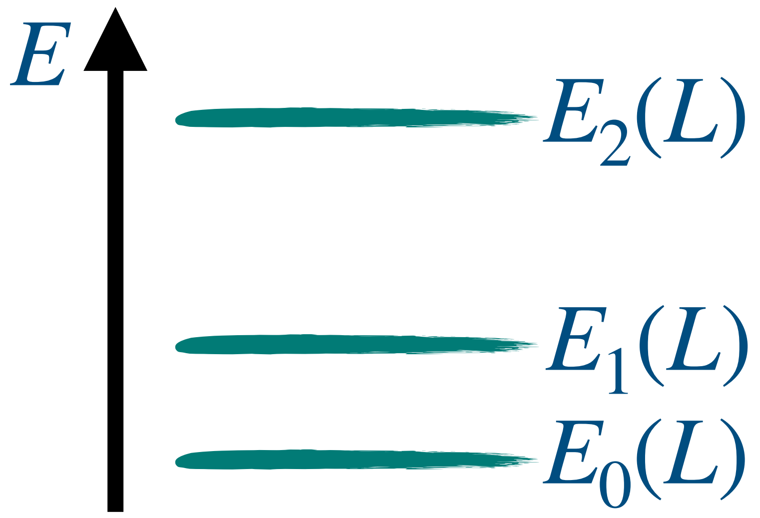


Finite volume setup:

- periodic cubic spatial volume of side L, finite time extent T
- L large enough to neglect $\mathcal{O}\left(e^{-M_\pi L}\right)$ effects

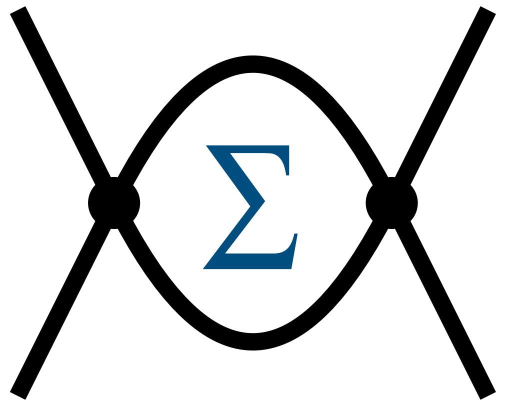


$\boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^3$



discretized momenta

discretized spectrum

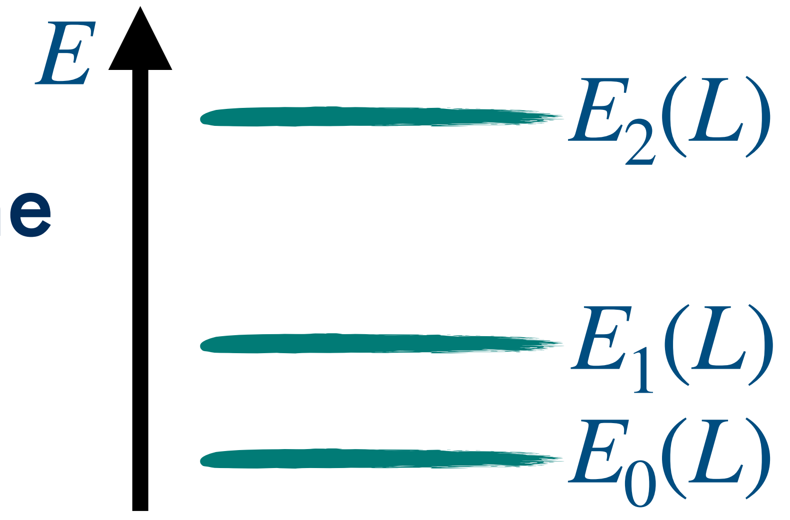


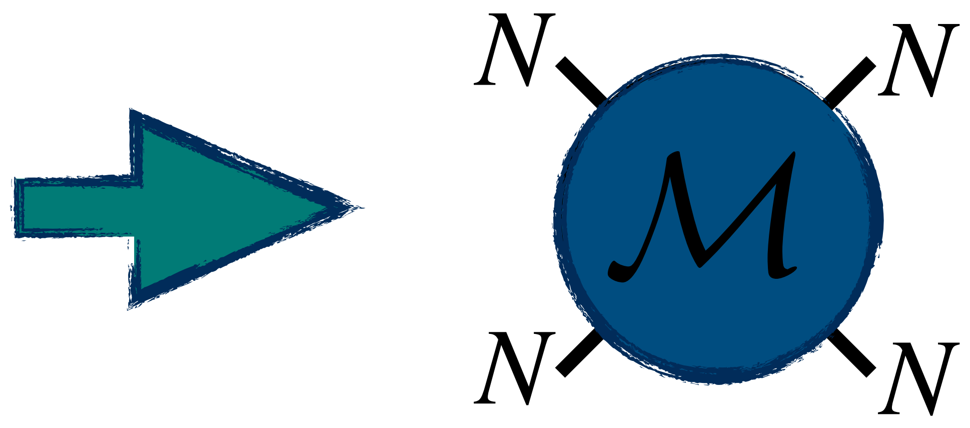
spatial loop momenta summed

Finite-volume scattering formalism

What we want from a **scattering formalism:**

finite-volume spectrum

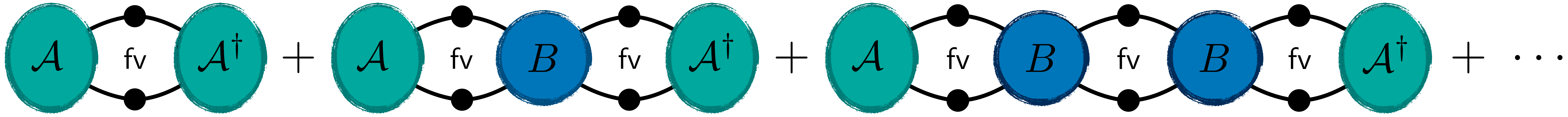




infinite-volume scattering amplitude

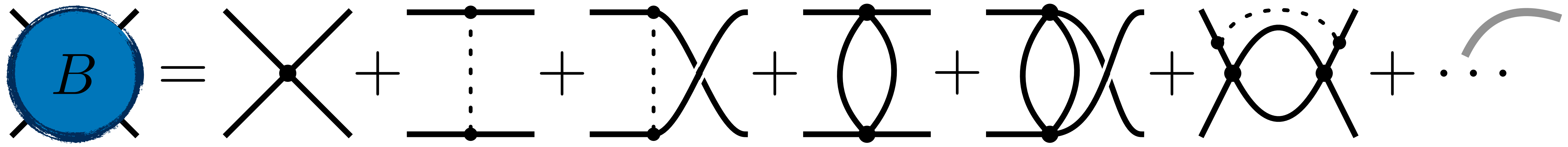
Consider **finite-volume correlator** – has poles at the finite-volume energy levels

$C_L(P) =$



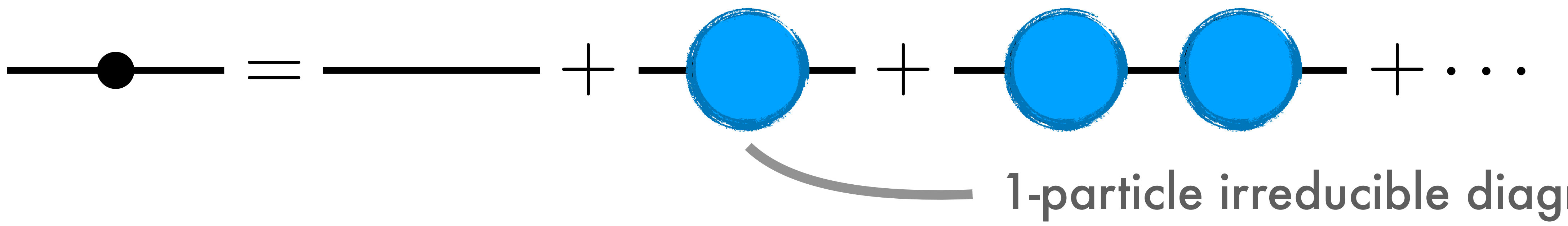
[Kim, Sachrajda, Sharpe 2005]

Bethe-Salpeter kernel



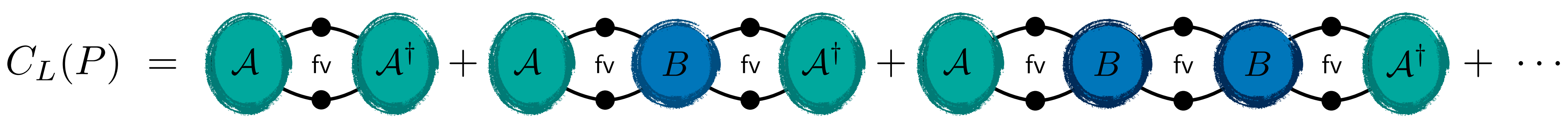
all other diagrams which are 2-particle irreducible in the s-channel

dressed N propagator

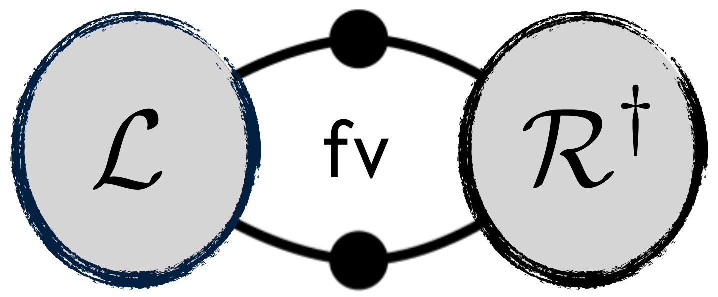


1-particle irreducible diagrams

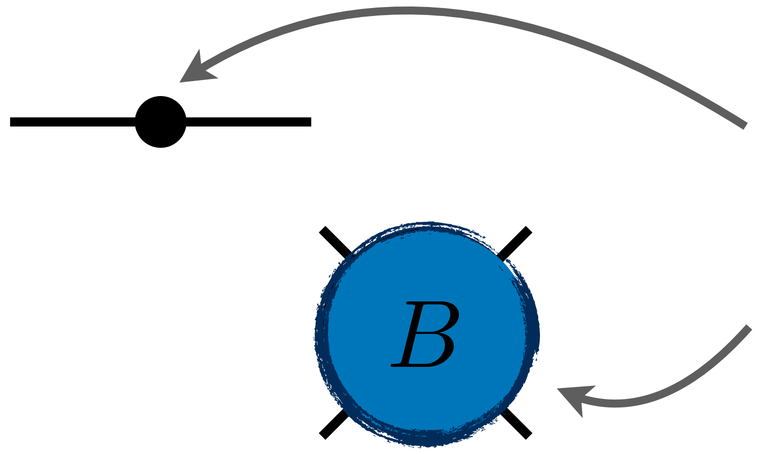
The standard derivation



Analysis of finite-volume loops in elastic regime $(2M_N)^2 < s < (2M_N + M_\pi)^2$:

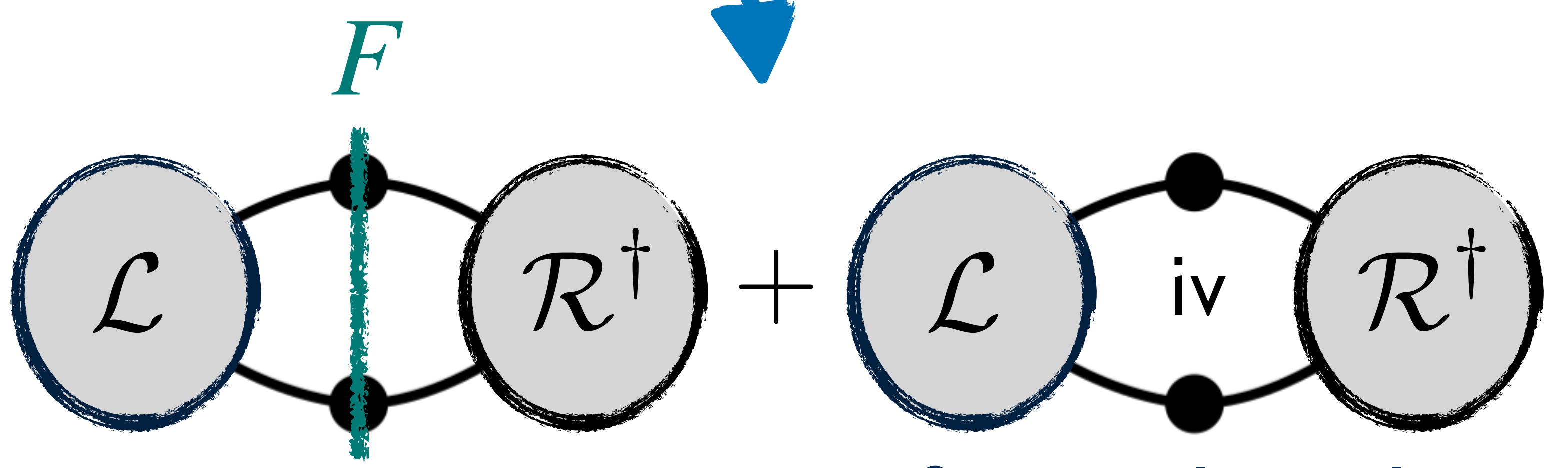


2-particle loops lead to $\mathcal{O}\left(L^{-n}\right)$ effects



other **fv** loops can be replaced by **iv** loops up to $\mathcal{O}\left(e^{-M_\pi L}\right)$ corrections

[Lüscher 1986]
[Kim, Sachrajda, Sharpe 2005]



infinite-volume loop

F cut $\sum_k - \int_k$

- intermediate two-particle (NN) state dominates
- left and right functions set to on-shell kinematics

infinite-volume correlator

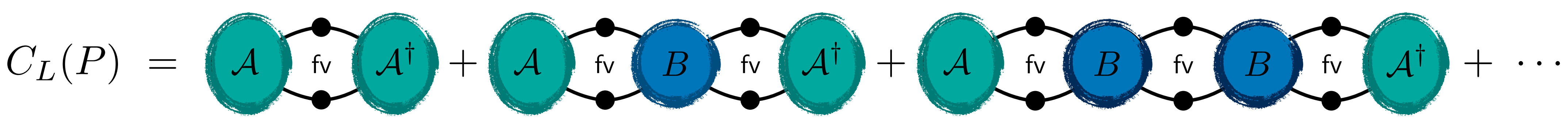
$$C_L(P) = C_\infty(P) + A(P) \frac{i}{F(P,L)^{-1} + \mathcal{K}(P)} A(P)^\dagger$$

matrix of known functions — K-matrix — operator "overlaps"

$$\det\left[F(P,L)^{-1} + \mathcal{K}(P)\right] = 0 \quad \text{at fv energy levels}$$

Lüscher quantization condition

The standard derivation



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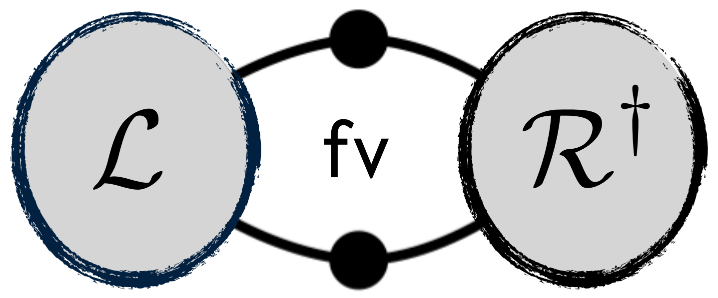
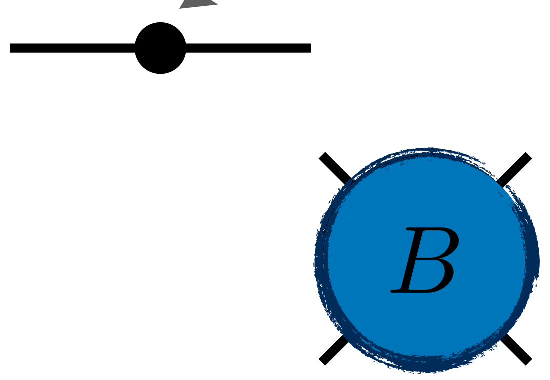
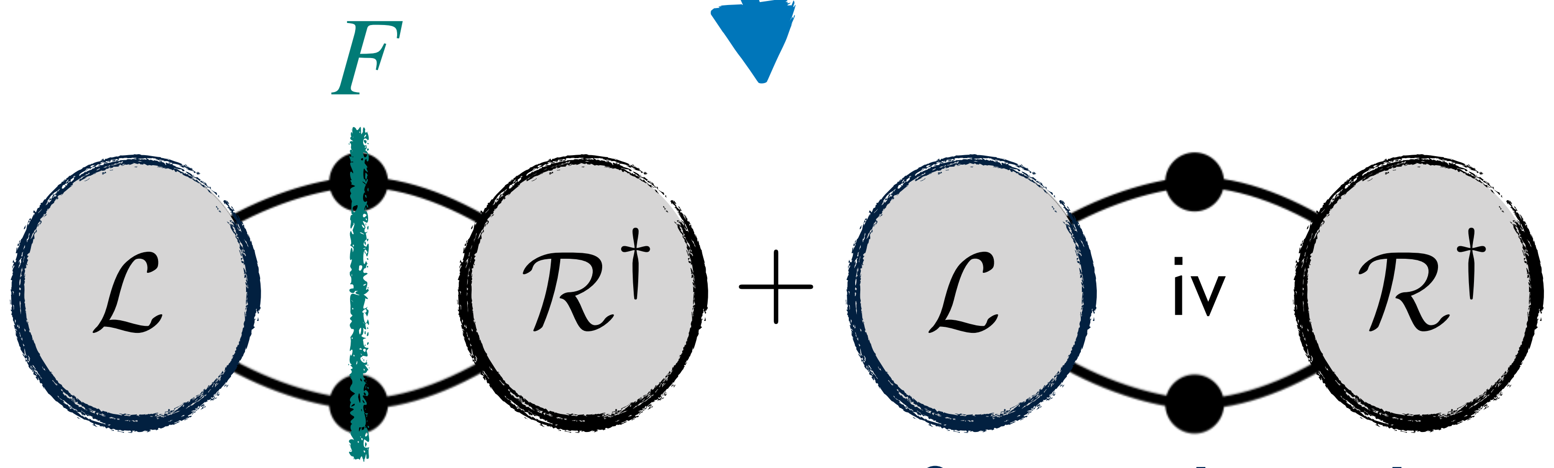


2-particle loops lead to $\mathcal{O}\left(L^{-n}\right)$ effects



other **fv** loops can be replaced by **iv** loops up to $\mathcal{O}\left(e^{-M_\pi L}\right)$ corrections

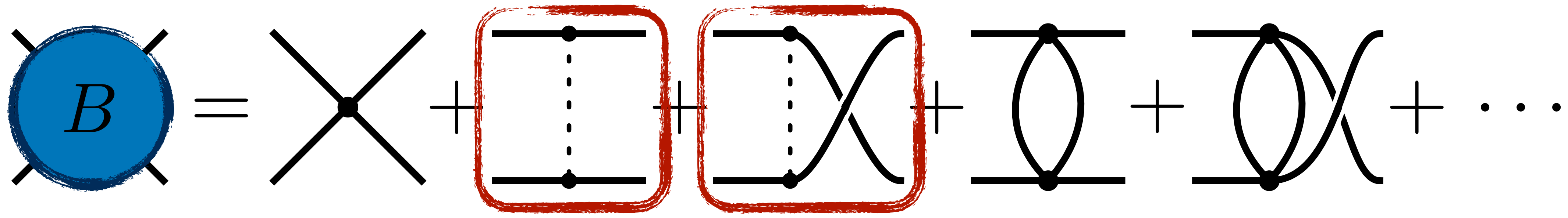
[Lüscher 1986]
[Kim, Sachrajda, Sharpe 2005]



infinite-volume loop

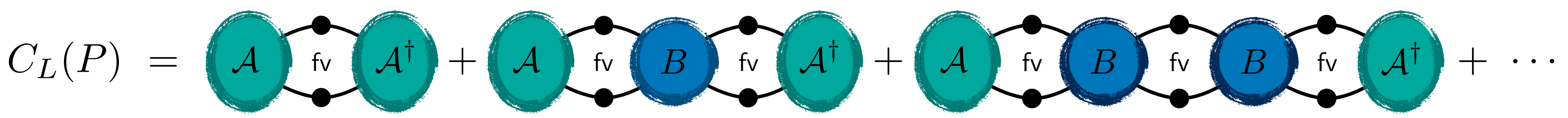
F cut $\sum_k - \int_k$

- intermediate two-particle (NN) state dominates
- left and right functions set to on-shell kinematics

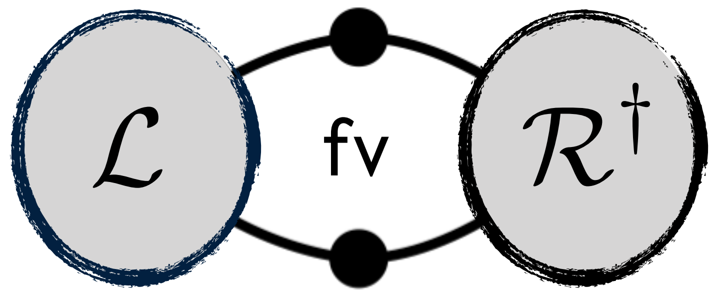


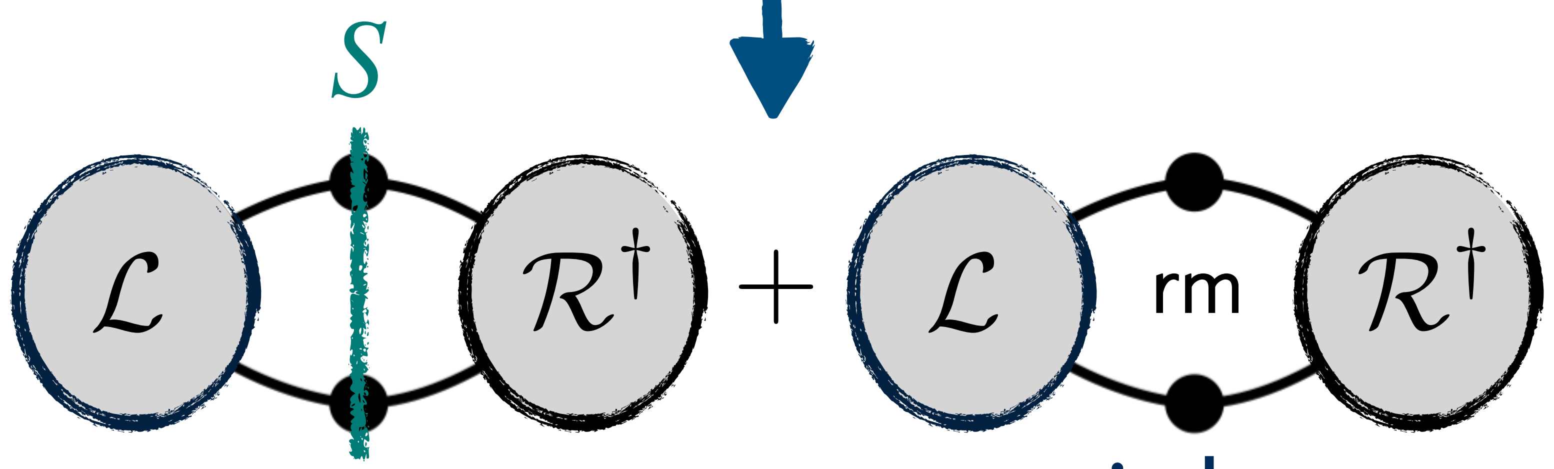
- placing BS kernels on shell introduces singularities and left-hand cuts below threshold – not present in the correlator
- cut near threshold arises from the π exchanges shown
- invalidates next steps in derivation

Proposed formalism



We propose the following instead:

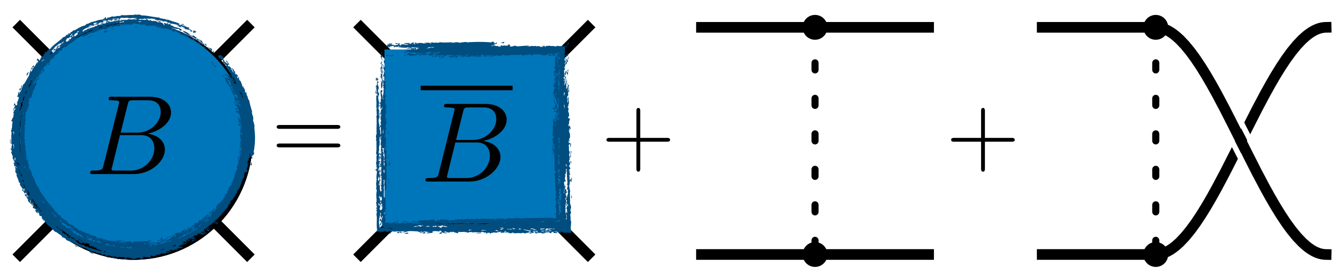




S cut:
keep the sum over momenta

remainder term

key step



split BS kernel: remove π exchanges

dangerous π exchanges are never put fully on shell, only the safe kernels $\overline{B}$

remainder — endcap vectors

$$C_L(P) = \mathcal{I}(P) + \tilde{A}(P)\xi \frac{i}{S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}(P)\xi + 2g^2 T(P)} \xi^\dagger \tilde{A}(P)^\dagger$$

$$\det\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}(P)\xi + 2g^2 T(P)\right] = 0$$

at finite-volume energy levels

modified quantization condition

- quantities in QC live in angular momentum plus discrete spatial momentum index space: $\boldsymbol{k}^\star \ell m; \boldsymbol{k}^{\star\prime} \ell' m'$ with $\boldsymbol{k}, \boldsymbol{k}' \in 2\pi\mathbb{Z}^3/L$
- determinant taken over this full space
  (similarity to 3-particle RFT formalism [Hansen, Sharpe 2014])

Quantization condition

$$\det\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}(P)\xi + 2g^2 T(P)\right] = 0$$

→ constrains the K-bar $\overline{\mathscr{K}}(P)$ (and coupling g) from the finite-volume spectrum

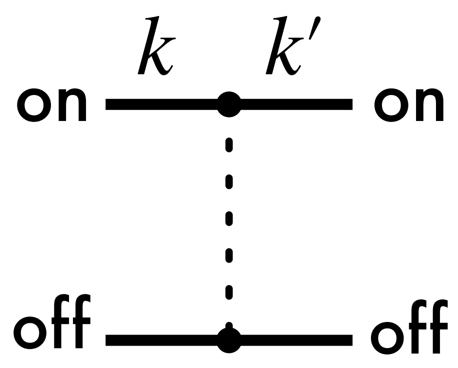
- **S-cut matrix:**

$$S_{\boldsymbol{k}^\star \ell m, \boldsymbol{k}'^\star \ell' m'}(P,L) = \frac{1}{2L^3} \frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\, Y^*_{\ell' m'}(\hat{\boldsymbol{k}}^\star)\, \delta_{\boldsymbol{k}^\star \boldsymbol{k}'^\star}\, |\boldsymbol{k}^\star|^{\ell+\ell'}\, H(\boldsymbol{k}^\star)}{4\omega_N(\boldsymbol{k}) \left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2\right]}$$

regulator function

on-shell CM momentum magnitude $(k^\star_{\text{os}})^2 = s/4 - M_N^2$

- **T matrix:** partial wave projections of partially off-shell t-channel diagram



e.g. *S-wave:*

$$\mathcal{T}^{\ell=0}_{\boldsymbol{k}^\star, \boldsymbol{k}^{\star\prime}}(P) = \frac{1}{2|\boldsymbol{k}^\star||\boldsymbol{k}'^\star|} \log\left(\frac{2\omega_N(\boldsymbol{k}^\star)\omega_N(\boldsymbol{k}'^\star) + 2|\boldsymbol{k}^\star||\boldsymbol{k}'^\star| - 2M_N^2 + M_\pi^2 - i\epsilon}{2\omega_N(\boldsymbol{k}^\star)\omega_N(\boldsymbol{k}'^\star) - 2|\boldsymbol{k}^\star||\boldsymbol{k}'^\star| - 2M_N^2 + M_\pi^2 - i\epsilon}\right)$$

$(\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + M_N^2})$

- **Trivial projectors ξ, $\xi^\dagger$:** $\xi_{\boldsymbol{k}^\star} = 1$

- **$\overline{\mathscr{K}}(P)$ matrix:** matrix in AM index space, projections of a Lorentz scalar $\overline{\mathscr{K}}(s)$

Particles with nonzero spin taken into account by incorporating spin state indices into the above quantities.