The finite-volume spectrum in the presence of a long-range force

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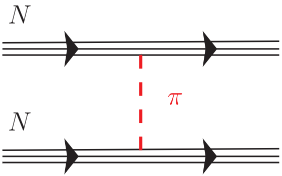
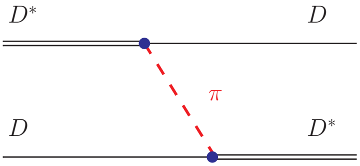
Plan

- Introduction: why long-range forces?
- Modified effective range expansion in the EFT framework
- Modified Lüscher equation
- Analysis of data in different approaches
- Conclusion, outlook

Why long-range forces?



NN scattering



$T_{cc}^+(3875)$

- Left hand cut close to threshold: the energy levels below the left-hand branch point cannot be used
- Slowly converging partial-wave expansion: expecting strong admixture of higher partial waves in the quantization condition (Meng & Epelbaum, 2021)
- Exponentially suppressed corrections still sizable

Plane-wave basis (Meng & Epelbaum, 2021)

- Describe the system in terms of the parameters of the effective Lagrangian which, by definition, encode only faraway singularities
- Work in the plane wave basis; do not resort to the partial-wave expansion
  - For the *NN* scattering, it was shown that, at the physical quark masses, the partial-wave mixing is sizable (Meng & Epelbaum, 2021)
  - A consistent fit of the DD^* scattering phases to lattice data in the left-hand cut region has been performed (Meng *et al.*, 2023)

Alternative approaches

- Splitting long- and short-range interactions (Hansen & Raposo, 2023)
  - Fit short-range part to the scattering data, get full amplitude through solving integral equations
  - Quantization condition is written down both in the plane-wave basis and the partial-wave basis
- Applying three-particle formalism to the $DD\pi$ system (Hansen, Romero-Lopez and Sharpe, 2024)
  - Two-particle quantization condition for a stable D^*
  - Plane wave basis is used
- Using Lüscher equation plus EFT with long-range force in the infinite volume above the left-hand cut (Collins *et al.*, 2024)
- HAL QCD approach (Lyu *et al.*, 2023)

Modified effective range expansion (van Haeringen & Kok, 1982)

- Lüscher equation is based on the assumption $R \sim M^{-1} \ll L$
  ...violated by a long-range force with a small M!
- Splitting of the potential

$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

- Effective-range expansion: very small radius of convergence

$$q^{2\ell+1} \cot \delta_\ell(q) = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell q^2 + O(q^4)$$

- Define modified effective-range function:

$$K_\ell^M(q^2) = M_\ell(q) + \frac{q^{2\ell+1}}{|f_\ell(q)|^2} \left(\cot(\delta_\ell(q) - \sigma_\ell(q)) - i\right)$$

Jost functions and all that

- Jost function for the long-range interaction:

$$f_\ell(q) = \frac{q^\ell e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!} \lim_{r\to 0} r^\ell f_\ell(q,r)$$

- The function $M_\ell(q)$:

$$M_\ell(q) = \frac{1}{\ell!}\left(-\frac{iq}{2}\right)^\ell \lim_{r\to 0} \frac{d^{2\ell+1}}{dr^{2\ell+1}} \frac{f_\ell(q,r)}{f_\ell(q)}$$

- Larger radius of convergence for the modified effective-range function:

$$K_\ell^M(q^2) = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2}\,\tilde{r}_\ell q^2 + O(q^4)$$

- Relation between $K_\ell^M(q^2)$ and the full phase $\delta_\ell(q)$ is *algebraic*

Requirements on the potential

- The long-range potential $V_L(r)$ is *local*
- The long-range potential must be *superregular*

$$\left| \lim_{r \to 0} r^{-2\ell} V_L(r) \right| < \infty$$

- In case of Yukawa interaction, $V_L(r) = \frac{g e^{-M_\pi r}}{r} - \sum_{i=1}^{2\ell+1} c_i \frac{g e^{-M_i r}}{r}$

$$M_i \sim M \text{ (heavy scale)}, \qquad M_\pi^k = \sum_{i=1}^{2\ell+1} c_i M_i^k \quad \text{for} \quad k = 0, \cdots, 2\ell$$

- The short-range potential is a low-energy polynomial:

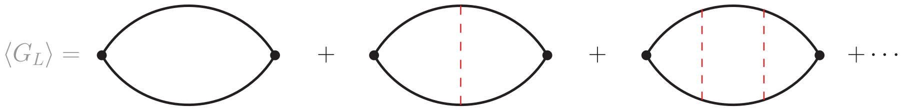
$$\langle \boldsymbol{p} | V_S | \boldsymbol{q} \rangle = C_0^{00} + 3 C_1^{00} \boldsymbol{p}\boldsymbol{q} + C_0^{10} (\boldsymbol{p}^2 + \boldsymbol{q}^2) + \cdots$$

Scattering on two potentials: the EFT framework

$$T = T_L + (1 + T_L G_0) T_S (1 + G_0 T_L)$$
$$T_S = V_S + V_S G_L T_S$$

- The Green function with the long-range potential only: $G_L = G_0 + G_0 V_L G_L$

The loop with infinite number of long-range insertions



$$\langle \mathbf{r} | G_L(q_0^2) | \mathbf{r}' \rangle = 4\pi \sum_{\ell m} \mathscr{Y}_{\ell m}(\mathbf{r}) \tilde{G}_L^{\ell}(r, r'; q_0^2) \mathscr{Y}_{\ell m}^{*}(\mathbf{r}') , \qquad \langle G_L^{\ell}(q_0^2) \rangle = \lim_{r, r' \to 0} G_L^{\ell}(r, r'; q_0^2)$$

- Relation to the Jost functions:

$$\langle G_L^{\ell}(q_0^2) \rangle = \frac{1}{4\pi((2\ell+1)!!)^2} M_{\ell}(q_0) + \underbrace{\text{real low-energy polynomial in } q_0^2}_{\text{renormalization prescription}}$$

Modified effective range expansion: EFT framework

- Lowest order: $\langle \boldsymbol{p} | V_S | \boldsymbol{q} \rangle = C_0^{00}$

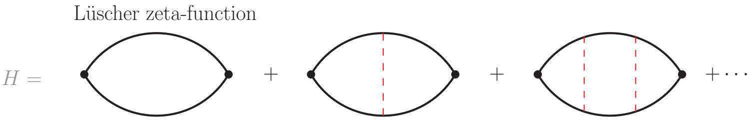
$$\underbrace{4\pi / C_0^{00}}_{=K_0^M(q_0^2)\text{ at lowest order}} = M_0(q_0) + \frac{q_0}{|f_0(q_0)|^2}\left(\cot(\delta_0(q_0) - \sigma_0(q_0)) - i\right)$$

- Higher orders:
  - The quantity $K_0^M(q_0^2)$ is a low-energy polynomial in q_0^2, expressed in terms of couplings $C_0^{00}, C_1^{00}, C_0^{10}, \ldots$
  - In the proof, the locality of $V_L(r)$ plays crucial role. The proof is not valid for a general, non-local potential

Modified Lüscher equation

$$\det \mathscr{A}_{\ell m,\ell' m'} = 0\,, \qquad \mathscr{A}_{\ell m,\ell' m'} = \delta_{\ell\ell'}\delta_{mm'} K_\ell^M(q_0^2) - H_{\ell m,\ell' m'}(q_0)$$

- Modified Lüscher zeta-function, finite volume:



$$H_{\ell m,\ell' m'}(q_0) = \frac{4\pi}{L^6} \sum_{\boldsymbol{p},\boldsymbol{q}} \mathscr{Y}^*_{\ell m}(\boldsymbol{p}) \langle \boldsymbol{p} | G_L(q_0^2) | \boldsymbol{q} \rangle \mathscr{Y}_{\ell' m'}(\boldsymbol{q})$$

- Taking into account the renormalization prescription:

$$H_{\ell m,\ell' m'}(q_0) = (H_{\ell m,\ell' m'}(q_0) - H^\infty_{\ell m,\ell' m'}(q_0)) + \frac{1}{4\pi} \delta_{\ell\ell'}\delta_{mm'} M_\ell(q_0)$$

Analysis of data

- Partial-wave mixing in a finite volume is expected to be significantly reduced
- For a single partial wave:

Energy level $\quad\rightarrow\quad$ scattering phase at a given energy

- A parameterization of phase shifts in a restricted energy interval is needed, if partial-wave mixing is included.

Solution in the plane wave basis

- Conceptually, very straightforward and transparent
- Parameterization of the infinite-volume amplitude in terms of the effective couplings is assumed in the whole energy range
- Does the EFT expansion converge?

Conclusions, outlook

- A novel quantization condition in the presence of the long-range forces has been proposed
  - Solves the left-hand cut problem
  - Reduces partial-wave mixing
  - Relates the energy level to the scattering phase(s) *at the same energy*
- Outlook:
  - Technical issues:
    - Efficient algorithm for the calculation of the modified Lüscher zeta-function
    - Moving frames
  - Long-range force in the three-body quantization condition
  - Electromagnetic interactions: is the non-perturbative resummation of the Coulomb photon exchanges needed/possible?