The timelike pion form factor and other applications of $I = 1\ \pi\pi$ scattering

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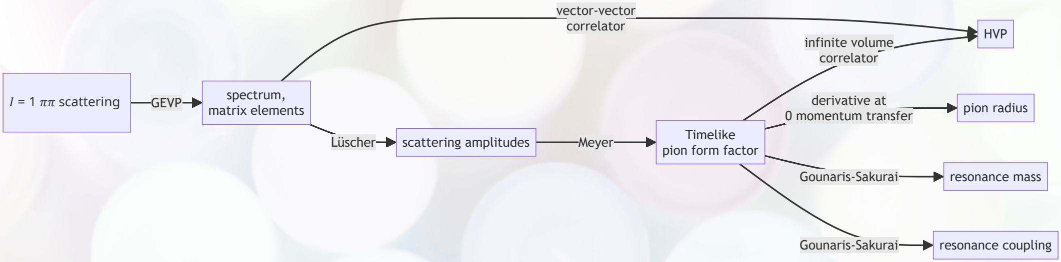
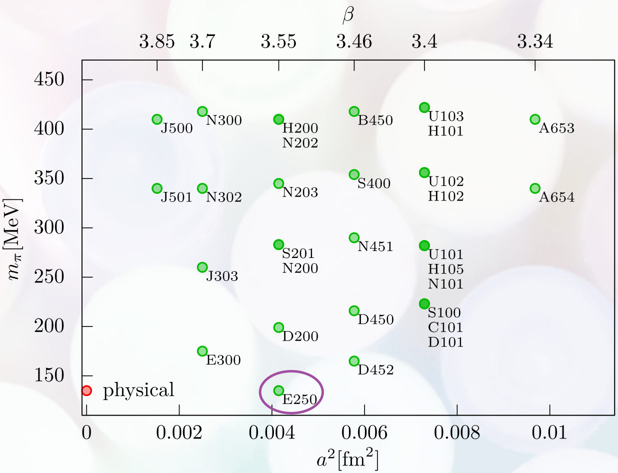
Overview



$I = 1$ $\pi\pi$ scattering
GEVP
spectrum,
matrix elements
vector-vector
correlator
HVP
Lüscher
scattering amplitudes
Meyer
Timelike
pion form factor
infinite volume
correlator
derivative at
0 momentum transfer
pion radius
Gounaris-Sakurai
resonance mass
Gounaris-Sakurai
resonance coupling

CLS ensembles/E250



[Paul et al; hep-lat/2112.07385]

E250	
m_π	132 MeV
m_K	495 MeV
a	0.0635 fm
L	6.1 fm
V/a^4	$96^3 \times 192$
$m_\pi L$	4.1

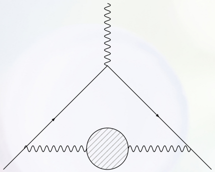
Role of QCD in muon $g-2$

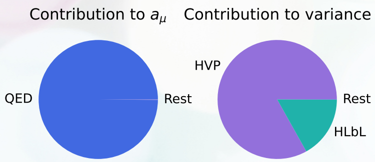
Why $a_\mu = (g-2)_\mu/2$?

- Among the most precise observables in physics
- Muon expected to play an enhanced role in physics beyond the standard model ($m_\mu^2/\Lambda_{\text{BSM}}^2 \gg m_e^2/\Lambda_{\text{BSM}}^2$)

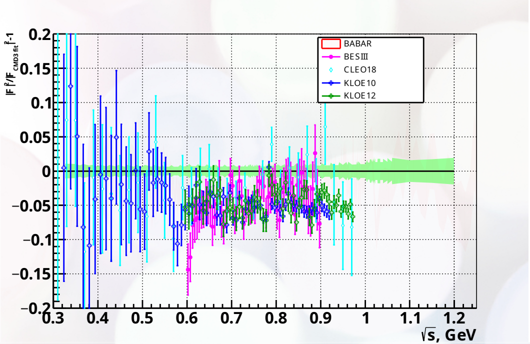
Why QCD?

- Majority of contribution to a_μ stems from QED
- Majority of error budget for a_μ stems from uncertainty in HVP contribution
- Two approaches: (1) dispersive, data-driven and (2) lattice





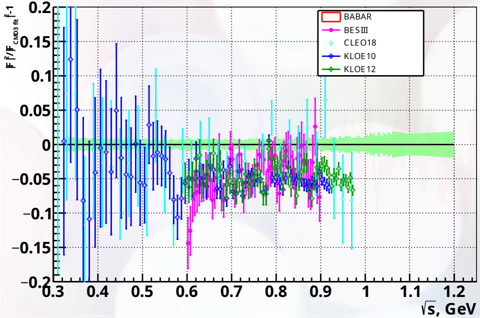
Discrepancies within the dispersive results



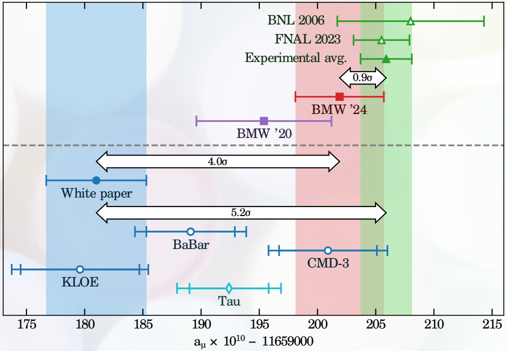
[CMD-3; hep-ex/2302.08834]

- Requires experimental input $\Longrightarrow$ not a “pure” theory calculation
- “Significant” discrepancies in cross sections

Discrepancies within & between the dispersive & lattice results



[CMD-3; hep-ex/2302.08834]



[BMW; hep-lat/2407.10913]

- Requires experimental input $\implies$ not a "pure" theory calculation
- "Significant" discrepancies in cross sections $\implies$ significant discrepancies in a_μ

Lattice approach to HVP via the time-momentum representation

From Bernecker & Meyer:

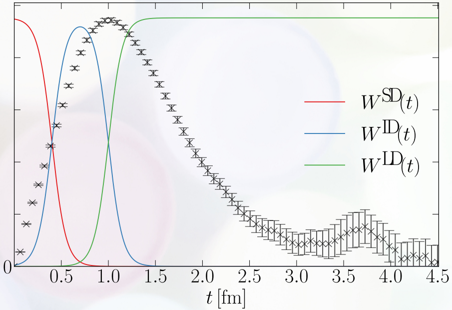
$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty G(t)\tilde{K}(t)dt$$

for kernel $\tilde{K}(t)$ and electromagnetic current correlator

$$G(t) = -\frac{1}{3}\sum_k \int d^3x \, \langle J_k^{\rm EM}(t,x) J_k^{\rm EM}(0)\rangle$$

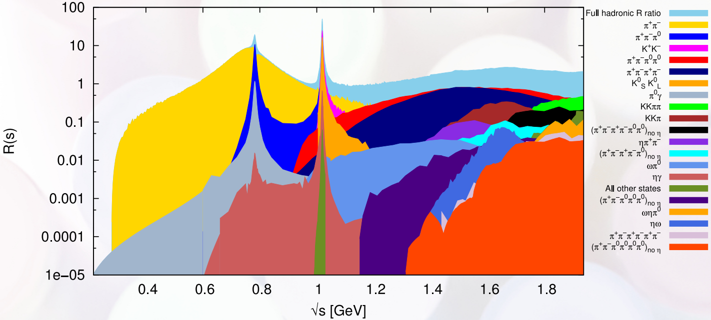
Distinguish between different "windows":

- Short distance: Lattice artifacts dominate
- Intermediate distance: precise, good for comparison
- Long distance: signal-to-noise ratio deteriorates



Plot of windows [Kuberski]

Reducing the error in the LD window through spectroscopy



[Keshavarzi et al; hep-ph/1802.02995]

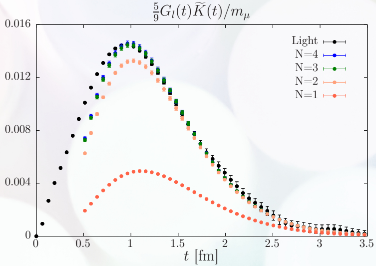
- $\pi\pi$ channel dominates at low-energy

$$\left.\begin{aligned} \langle [\pi\pi]^\dagger(t)\,[\pi\pi](0)\rangle &= Z^{(0)}_{\pi\pi} Z^{(0)\dagger}_{\pi\pi} e^{-E^{(0)}t} + \dots \\ \langle J^\dagger(t)\,[\pi\pi](0)\rangle &= Z^{(0)}_{J} Z^{(0)\dagger}_{\pi\pi} e^{-E^{(0)}t} + \dots \end{aligned}\right\} \implies \langle J^\dagger(t) J(0)\rangle = |Z^{(0)}_J|^2 e^{-E^{(0)}t} + \dots$$

Setup for the spectral reconstruction of the current correlator

Must estimate many states:

$$\langle J(t)J^\dagger(0)\rangle = \sum_n |Z_J^{(n)}|^2 e^{-E^{(n)}t}$$



[Gérardin et al; hep-lat/1904.03120]

- Rather than use a single interpolator for $\pi\pi$, use many

$$(\pi\pi)(\mathbf{p}_1, \mathbf{p}_2, t) = \pi^+(\mathbf{p}_1, t)\pi^0(\mathbf{p}_2, t) - \pi^0(\mathbf{p}_1, t)\pi^+(\mathbf{p}_2, t)$$

- 9 different operators with $\mathbf{p}^2 = 0$

$$G(t) \to \mathbf{G}(t) = \begin{bmatrix} \overbrace{G_{J\to J}(t)}^{1\times 1} & \overbrace{G_{J\to\pi\pi}(t)}^{1\times 9} \\ \underbrace{G_{\pi\pi\to J}(t)}_{9\times 1} & \underbrace{G_{\pi\pi\to\pi\pi}(t)}_{9\times 9} \end{bmatrix}$$

- Solve the correlation matrix for the overlaps and energies
- Use variational method on the 9×9 matrix

Solving the generalized eigenvalue problem

Approaches to the generalized eigenvalue problem (GEVP)

$$C(t)v_n(t,t_0) = \lambda_n(t,t_0)C(t_0)v_n(t,t_0)$$

Principal correlator
Compute eigenvalues λ on each time slice for fixed pivot t_0, then fit

$$\lambda^{(n)} = e^{-E^{(n)}(t-t_0)} + \cdots$$

Rotated correlator
Compute eigenvectors exactly once then reuse for all times, then fit

$$C_{\mathrm{rot}}^{(n)} = \mathbf{v}^{(\mathbf{n})\dagger} C \mathbf{v}^{(\mathbf{n})}$$

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Compute eigenvectors exactly once then reuse for all times, then fit

$$C_{\text{rot}}^{(n)} = \mathbf{v}^{(\mathbf{n})\dagger} C \mathbf{v}^{(\mathbf{n})}$$

"Sliding pivot" eff mass
Compute eigenvalues on each time slice for pivot $t_0 = \lceil t/2 \rceil$, then fit

$$E_{\text{eff}}^{(n)} = E^{(n)} + \dots$$

Corrections to the "sliding pivot" effective masses

Define $\delta E^{(n)} = \min_{m \neq n} |E^{(m)} - E^{(n)}|$ and $\Delta E^{(n)} = E_N - E_n$.

What is the correction $\epsilon_E^{(n)}(t, t_0)$?

Compute: $$E_{\text{eff}}^{(n)}(t) = \log\left(\frac{\lambda^{(n)}(t-1, t_0)}{\lambda^{(n)}(t, t_0)}\right)$$

Fit: $$E_{\text{eff}}^{(n)}(t) = E^{(n)} + \epsilon_E^{(n)}(t, t_0)$$

Worse case scenario (for fixed t_0) [Lüscher & Wolff (1990)]:

$$\epsilon_E^{(n)}(t, t_0) = \mathcal{O}\left(e^{-\delta E^{(n)} t}\right)$$

Next-to-leading-order (for general t_0) [Blossier; hep-lat/0902.1265]:

$$\epsilon_E^{(n)}(t, t_0) = \mathcal{O}\left(e^{-\Delta E^{(n)} t}\right) + \mathcal{O}\left(e^{-2(\Delta E^{(n)} - \delta E^{(n)}) t_0} e^{-\delta E^{(n)} t}\right)$$

Corrections to the "sliding pivot" effective masses

Define $\delta E^{(n)} = \min_{m\neq n} |E^{(m)} - E^{(n)}|$ and $\Delta E^{(n)} = E_N - E_n$.
Optimal correction when $t_0 \geq t/2$.

Compute: $$E_{\text{eff}}^{(n)}(t) = \log\left(\frac{\lambda^{(n)}(t-1, \lceil t/2 \rceil)}{\lambda^{(n)}(t, \lceil t/2 \rceil)}\right)$$

Fit: $$E_{\text{eff}}^{(n)}(t) = E^{(n)} + \mathcal{O}\left(e^{-\Delta E^{(n)} t}\right)$$

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Corrections to the “sliding pivot” effective masses & matrix elements

Define $\delta E^{(n)} = \min_{m \neq n} |E^{(m)} - E^{(n)}|$ and $\Delta E^{(n)} = E_N - E_n$.
Optimal correction when $t_0 \geq t/2$.

Compute: $$E_{\text{eff}}^{(n)}(t) = \log\left(\frac{\lambda^{(n)}(t-1, \lceil t/2 \rceil)}{\lambda^{(n)}(t, \lceil t/2 \rceil)}\right)$$

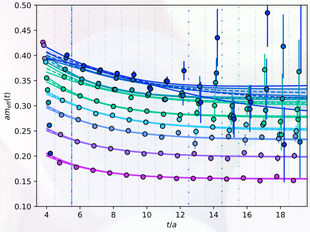
Fit: $$E_{\text{eff}}^{(n)}(t) = E^{(n)} + \mathcal{O}\left(e^{-\Delta E^{(n)} t}\right)$$

Effective matrix elements for optimized operator $[\Pi\Pi]^{(n)}(t; t_0) \equiv ([\pi\pi](t), v^{(n)}(t, t_0))$:

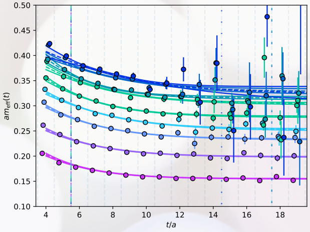
Compute: $$\tilde{Z}_J^{(n)}(t) = \frac{\langle J(t)\, [\Pi\Pi]^{(n)\,\dagger}(0; \lceil t/2 \rceil)\rangle}{\sqrt{\langle [\Pi\Pi]^{(n)}(t; \lceil t/2 \rceil)\, [\Pi\Pi]^{(n)\,\dagger}(0; \lceil t/2 \rceil)\rangle}} \left(\frac{\lambda^{(n)}(\lceil t/2 \rceil + 1, \lceil t/2 \rceil)}{\lambda^{(n)}(\lceil t/2 \rceil + 2, \lceil t/2 \rceil)}\right)^{t/2}$$

Fit: $$\tilde{Z}_J^{(n)}(t) = Z_J^{(n)} + \mathcal{O}\left(e^{-\Delta E^{(n)} \lceil t/2 \rceil}\right)$$ *Depends on t_0, not t!*

Fits to the effective masses ($p^2 = 0,\ T_1^+$)



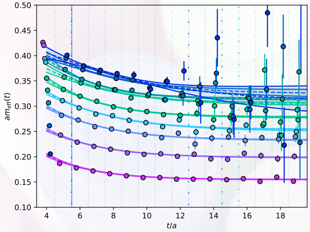
Fit to principal correlators



Fit to rotated correlators

- Principal & rotated correlators fits: include generic "garbage exponential" term, independent of each level

Fits to the effective masses ($p^2 = 0,\ T_1^+$)



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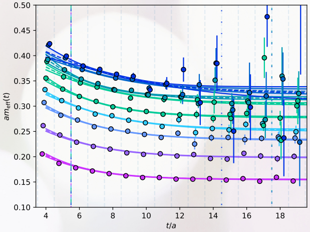
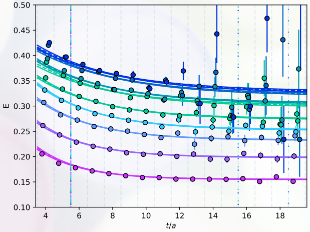


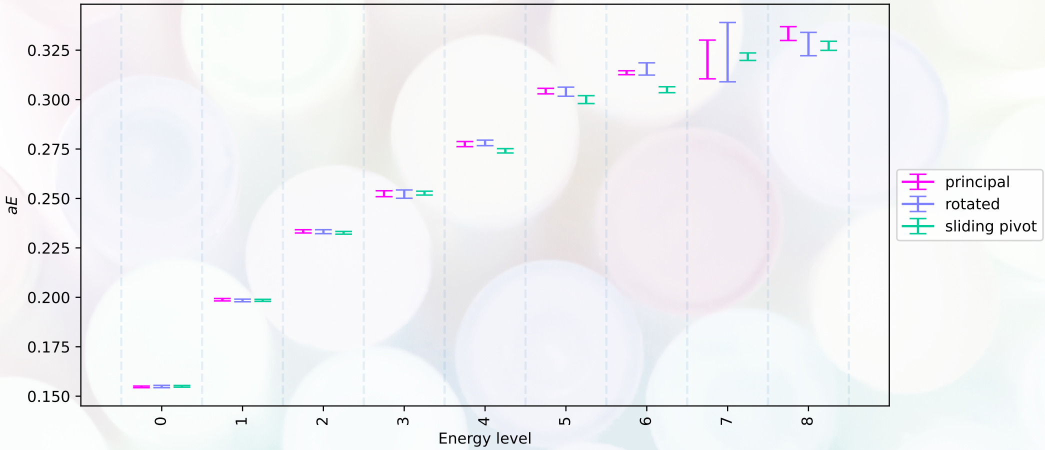
Fit to rotated correlators



Fit to "sliding pivot" masses

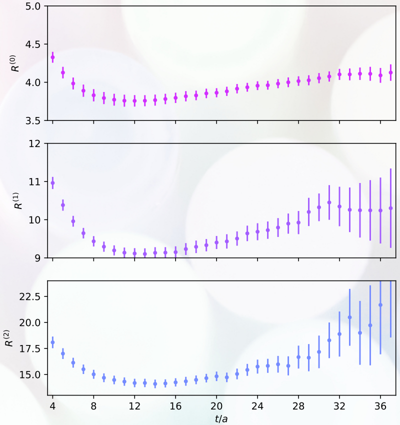
- Principal & rotated correlators fits: include generic "garbage exponential" term, independent of each level
- "Sliding pivot" fits: include shared $\mathcal{O}\left(e^{-\Delta E^{(n)} t}\right)$ correction

Direct comparison of spectrum ($p^2 = 0,\ T_1^+$)



- Comparison after model averaging over choice $t_{\text{min}}/a = [4, \ldots, 9]$

Comparison of matrix elements



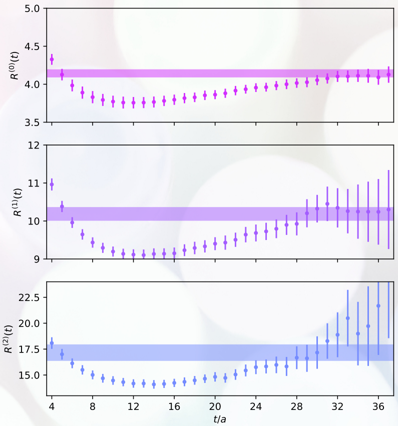
Effective overlaps using [hep-lat 1808.05007]

Previous study:

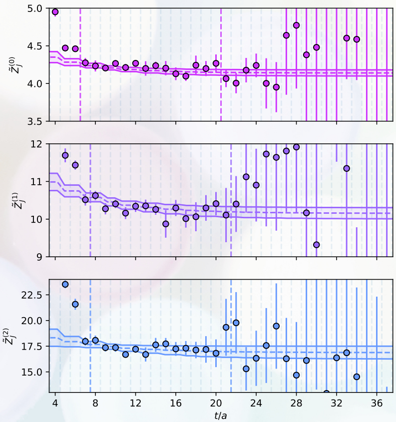
$$R^{(n)}(t) = \left| \frac{\langle J(\Pi\Pi)^{(n)\dagger}\rangle}{\sqrt{\langle(\Pi\Pi)^{(n)}(\Pi\Pi)^{(n)\dagger}\rangle \, e^{-E_{\pi\pi}^{(n)} t}}} \right| \xrightarrow{t\to\infty} \left| Z_J^{(n)} \right|$$

- Requires fitted energies as input
- Sensitive to GEVP parameters
- Only plateaus at very late times or large t_0

Comparison of matrix elements



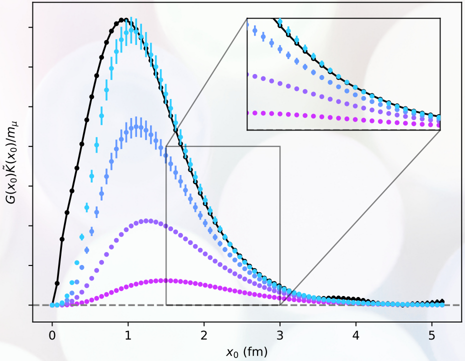
Effective overlaps using [hep-lat 1808.05007]



Effective overlaps from this talk

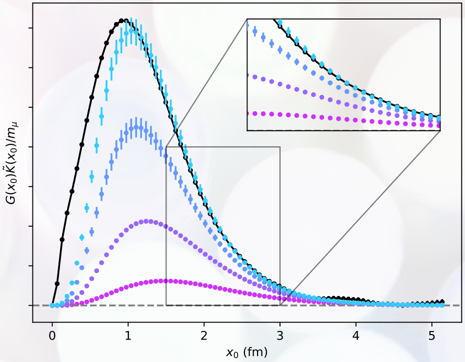
- Since corrections are known, possible to fit earlier & more precise data (right)

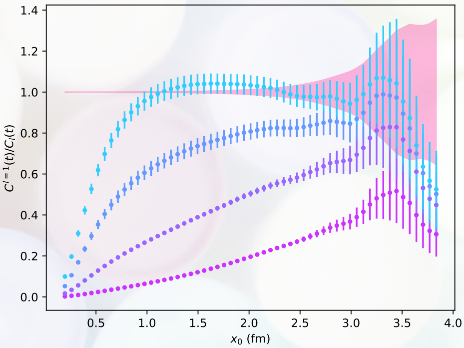
Putting it all together: reconstructing the current correlator



- Current correlator fully saturated after including 4 states

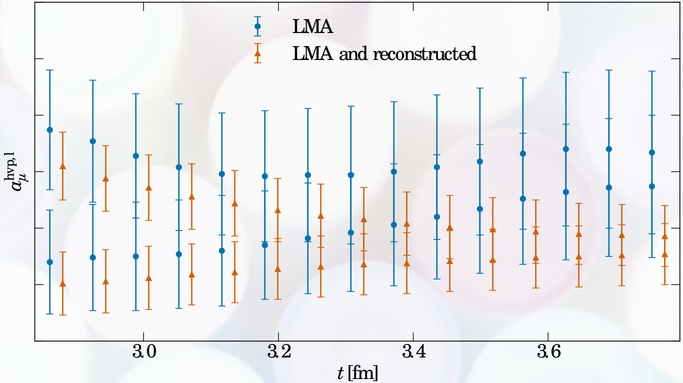
Putting it all together: reconstructing the current correlator





- Current correlator fully saturated after including 4 states
- Spectroscopy more precise than low-mode averaging (LMA) at $x_0 \approx 2.5$ fm

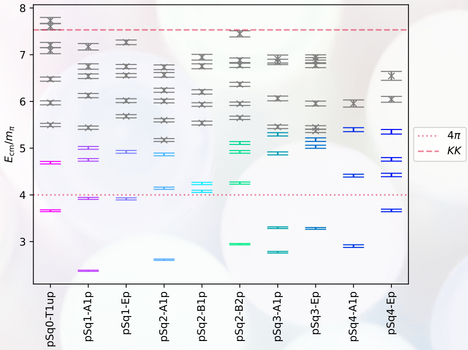
Improvement to a_μ^{hvp} via the bounding method

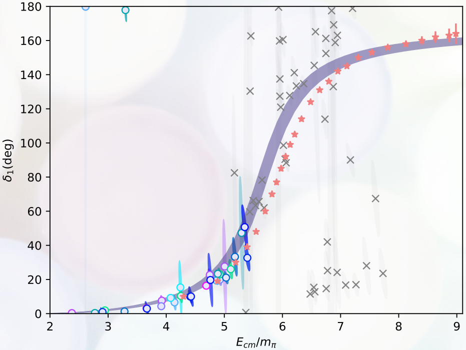


[Kuberski]

- Including spectroscopy halves the error of a_μ^{hvp} on our most important ensemble!

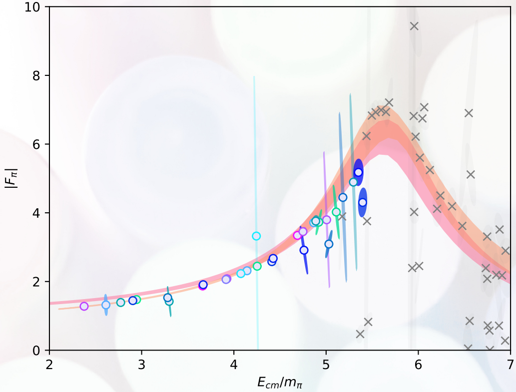
Extension to other irreps: the phase shifts [preliminary]





- Fit spectrum to a Breit-Wigner line resonance using all irreps
- For this first study, constraint to $E_{\rm cm} \lesssim 5 m_\pi$

The timelike pion form factor [preliminary]



Gounaris-Sakurai (pink) & 2-subtracted Omnès (orange) representations

Lellouch-Lüscher-Meyer:

$$\left|F_\pi^{(\Lambda,\mathbf{d})}\right|^2 = g_\Lambda(\gamma)\left(k\frac{\partial\delta_1(k)}{\partial k} + q\frac{\partial\phi^{(\Lambda,\mathbf{d})}(q)}{\partial q}\right) \times \frac{3\pi E_{\rm cm}^2}{2q_{\rm cm}^5 L^3}\left|Z_J^{(\Lambda,\mathbf{d})}\right|^2$$

Posterior [$\chi^2_\nu = 1.1$]:

$$m_\rho/m_\pi = 5.760(88)$$
$$g_{\rho\pi\pi} = 6.02(30)$$
$$\Gamma_\rho/m_\pi = 1.14(12)$$

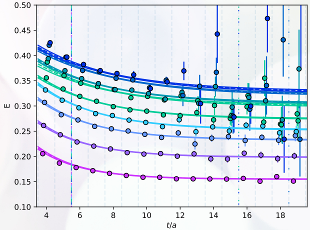
- Does not include systematics!

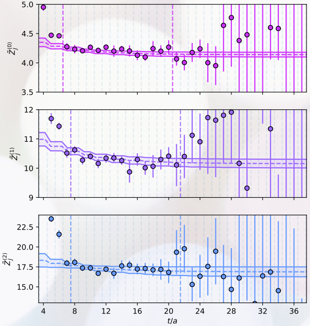
Advertisements

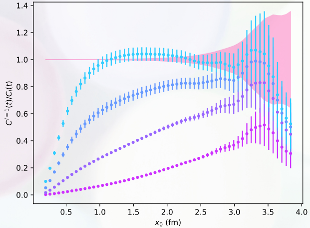
Related talks by Mainz folk:

- "$\pi^0 \to \gamma^*\gamma^*$ transition form factor and the pion pole contribution to a_μ on CLS ensembles", Jonna Koponen, Mon 11:35
- "Hadronic vacuum polarization contribution to the muon g-2 at short and long distances", Simon Kuberski, Mon 12:35
- "Machine-learning techniques as noise reduction strategies in lattice calculations of the muon $g-2$", Hartmut Wittig, Wed 11:35
- "The hadronic contribution to the running of α and the electroweak mixing angle", Alessandro Conigli, Thu 9:40
- "UV-finite QED correction to the hadronic vacuum polarization contribution to $(g-2)_\mu$", Julian Parrino, Thu 9:40
- "The isospin-violating part of the hadronic vacuum polarisation", Dominik Erb, Thu 10:00

Summary



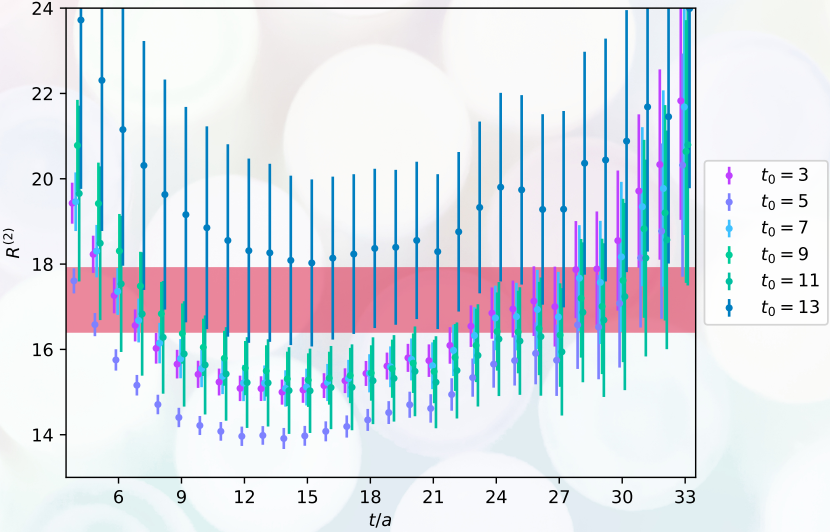




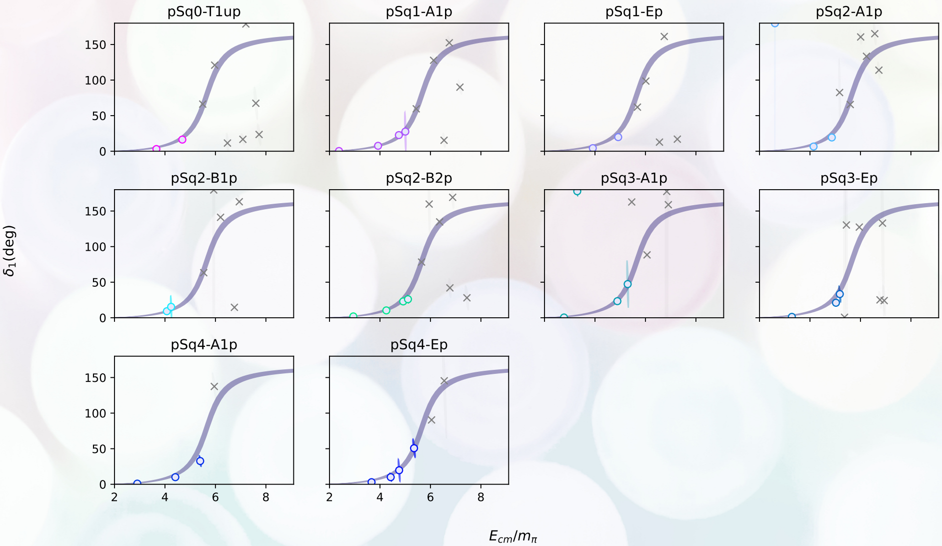
- Reminder: in the regime $t_0/2 \geq t$, the NLO corrections to effective quantities are known! $\implies$ We can use these constraints to increase precision.
- Including spectroscopy halves the error of a_μ^{hvp} on our most important ensemble

Backup

Sensitivity of old ratios to t_0



Phase shifts by irrep



Form factor by irrep