Analysis of g-2 long distance two-pion correlators for reconstruction of light vector correlators

Joe Mckeon

(RBC/UKQCD collaboration)

Lattice 2024, Liverpool, 29/07/24

The RBC & UKQCD collaborations

Boston University
Nobuyuki Matsumoto

BNL and BNL/RBRC
Peter Boyle
Taku Izubuchi
Christopher Kelly
Shigemi Ohta (KEK)
Amarji Soni
Masaaki Tomii
Xin-Yu Tuo
Shuhei Yamamoto

University of Cambridge
Nelson Lachini

CERN
Matteo Di Carlo
Felix Erben
Andreas Jüttner (Southampton)
Tobias Tsang

Columbia University
Norman Christ
Sarah Fields
Ceran Hu
Yikai Huo
Joseph Karpie (JLab)
Erik Lundstrum
Bob Mawhinney
Bigeng Wang (Kentucky)

University of Connecticut
Tom Blum
Jonas Hildebrand
Luchang Jin
Vaishakhi Moningi
Anton Shcherbakov
Douglas Stewart
Joshua Swaim

DESY Zeuthen
Raoul Hodgson

Edinburgh University
Luigi Del Debbio
Vera Gülpers
Maxwell T. Hansen
Nils Hermansson-Truedsson
Ryan Hill
Antonin Portelli
Azusa Yamaguchi

Johannes Gutenberg University of Mainz
Alessandro Barone

Liverpool Hope/Uni. of Liverpool
Nicolas Garron

LLNL
Aaron Meyer

Autonomous University of Madrid
Nikolai Husung

University of Milano Bicocca
Mattia Bruno

Nara Women's University
Hiroshi Ohki

Peking University
Xu Feng
Tian Lin

University of Regensburg
Andreas Hackl
Daniel Knüttel
Christoph Lehner
Sebastian Spiegel

RIKEN CCS
Yasumichi Aoki

University of Siegen
Matthew Black
Anastasia Boushmelev
Oliver Witzel

University of Southampton
Bipasha Chakraborty
Ahmed Elgaziari
Jonathan Flynn
Joe McKeon
Rajnandini Mukherjee
Callum Radley-Scott
Chris Sachrajda

Stony Brook University
Fangcheng He
Sergey Syritsyn (RBRC)

Starting point for computing $a_\mu^{HVP,LO}$

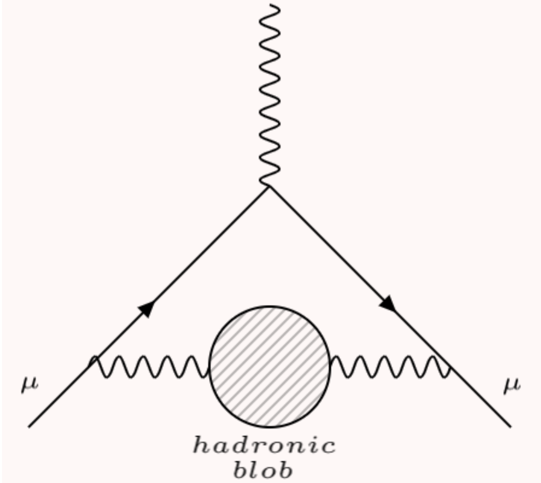
$\widetilde{K}$: QED kernel function[1].

$G(t)$: Vector correlator representing the hadronic blob (indices over quark flavours).

J_k^{em}: Electromagnetic current.

$a_\mu^{HVP,LO} = a_\mu^{SD} + a_\mu^{W} + a_\mu^{LD}$ [2,3]

Goal is to compute a_μ^{LD} with higher precision



$$a_\mu^{HVP,LO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t) \tilde{K}(t; m_\mu)$$

$$G(t)\delta_{kl} = -\int d^3x \, \langle J_k^{em}(x) J_l^{em}(0) \rangle$$

SD = Short-distance
W = Standard window
LD = Long-distance

[1] Bernecker and Meyer, EPJ A47 (2011) 148
[2] Blum et al., PRL 121, 022003 (2018)
[3] Lehner, EPJ 175 (2018) 01024

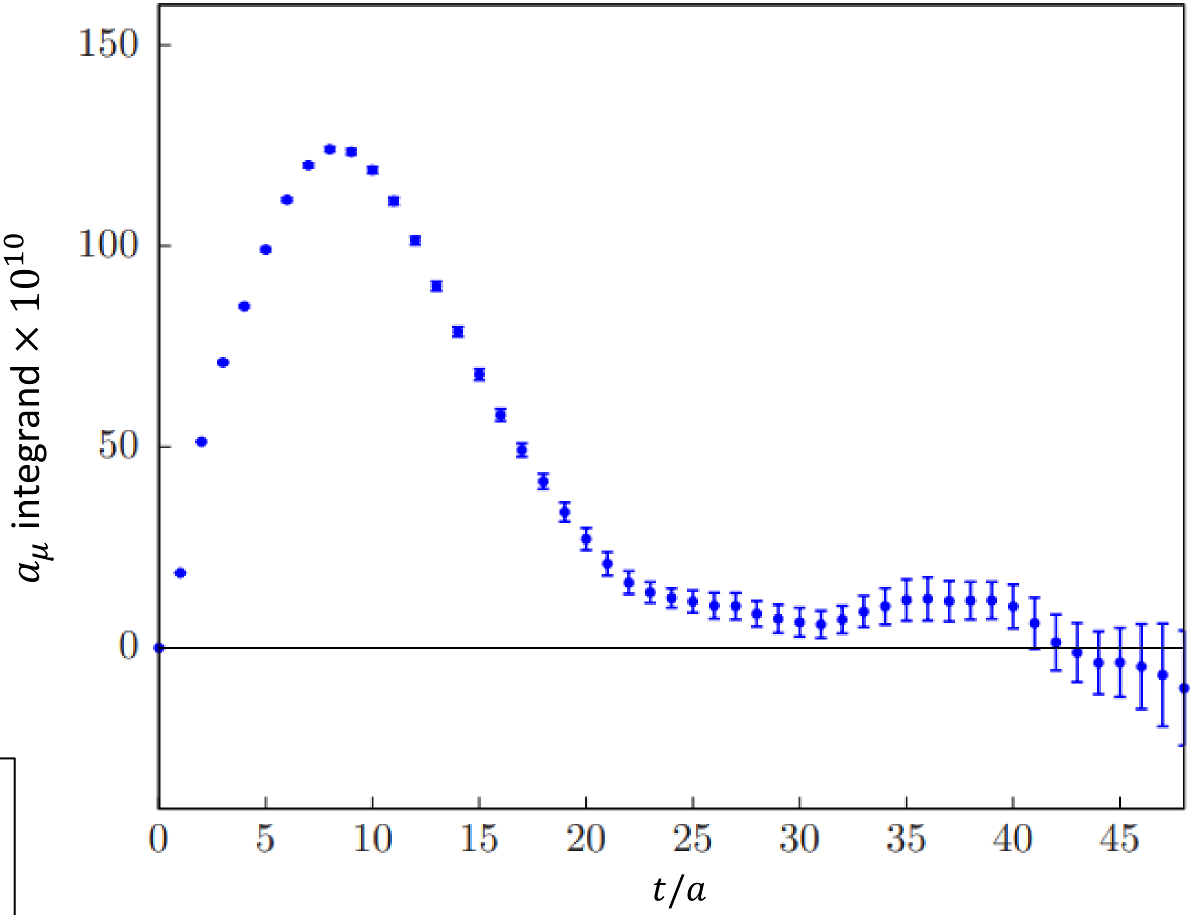
HVP image from www.bnl.gov/newsroom/news.php?a=217530

Large correlator noise at LD

- $a_\mu^{HVP,LO}$, and its error, are dominated by light connected vector correlators.
- At LD light connected vector correlators are too noisy by lattice construction
- Need to reduce this noise by reconstructing $\pi\pi$-states at LD to achieve more precise $a_\mu^{HVP,LO}$.

Aim of this talk: to give details of reconstruction of light connected vector correlator from $\pi\pi$-states at LD

(RBC/UKQCD $48^3 \times 96$, $a^{-1} \approx 1.73$GeV)



Dealing with the LD regime

- Reconstruct $G(t)$ from individual $\pi\pi$ states[1,2]

$$G(t) = \frac{10}{9} \sum_{n=0}^{n_{max}} |A_n|^2 e^{-E_n t}$$ [3,4]

- Consider a large variational basis of $\pi\pi$ operators – extract as many energy levels as precisely as possible.
- Extraction possible from the optimal linear combination of interpolating $\pi\pi$ operators (see next slide).
- Achieved via solving the Generalised Eigenvalue Problem (GEVP).
- Will also yield $|A_n|$s.

[1] Dudek et al., PRD77: 034501, 2008
[2] Bruno et al., arXiv:1910.11745
[3] Della Morte et al., JHEP 1710 (2017) 020
[4] Della Morte et al., arXiv:1710.10072

Finding E_ns

Consider a diagonal correlator, $\boldsymbol{c}(t)$, containing linear combinations of lattice generated $\pi\pi$ operators, $\mathcal{O}_i^{2\pi}$:

$$\mathbf{c}(t) = \langle 0|\Omega(t)\Omega^\dagger(0)|0\rangle \,,\ \text{where}\ \Omega = \Sigma_i v_i^* \mathcal{O}_i^{2\pi}$$

$$\mathbf{c}(t) = \sum_n W_n e^{-E_n t}, \quad W_n = \left|\langle n|\Omega^\dagger|0\rangle\right|^2 \geq 0 \ \ \forall n$$

Local minima occur when only one coefficient is **non**-zero, i.e.:

$$\mathbf{c}(t)|_{\text{local min p}} = W_p e^{-E_p t}$$

For arbitrary integer p. Hence

Fitting local minima will yield state energies.

Solving GEVP to extract energies

$$C_{ij}(t) = \langle 0|\mathcal{O}_i^{2\pi}(t)\mathcal{O}_j^{2\pi}(0)|0\rangle$$

$$\mathbf{c}(t) = \langle 0|\Omega(t)\Omega^\dagger(0)|0\rangle = \sum_{i,j} v_i^* C_{ij}(t) v_j$$

Solving GEVP to extract energies

$$C_{ij}(t) = \langle 0|\mathcal{O}_i^{2\pi}(t)\mathcal{O}_j^{2\pi}(0)|0\rangle$$

$$\mathbf{c}(t) = \langle 0|\Omega(t)\Omega^\dagger(0)|0\rangle = \sum_{i,j} v_i^* C_{ij}(t) v_j$$

Normalization condition (N) enforced by a Lagrange multiplier prevents trivial solution ($v_i = 0\ \forall i$).

$$N = \sum_{i,j} v_i^* C_{ij}(t_0) v_j$$

t_0 should be chosen large enough to avoid contamination from higher states.[1]

[1] Dudek et al., PRD77: 034501, 2008

Solving GEVP to extract energies

$$C_{ij}(t) = \langle 0|\mathcal{O}_i^{2\pi}(t)\mathcal{O}_j^{2\pi}(0)|0\rangle$$

$$\mathbf{c}(t) = \langle 0|\Omega(t)\Omega^\dagger(0)|0\rangle = \sum_{i,j} v_i^* C_{ij}(t) v_j$$

Normalization condition (N) enforced by a Lagrange multiplier prevents trivial solution ($v_i = 0\ \forall i$).

$$N = \sum_{i,j} v_i^* C_{ij}(t_0) v_j$$

$$\mathbf{c}(t) = \sum_{i,j} v_i^* \left[C_{ij}(t) - \lambda C_{ij}(t_0)\right] v_j + \lambda N$$

t_0 should be chosen large enough to avoid contamination from higher states.[1]

$$\frac{\partial \mathbf{c}(t)}{\partial v_i^*} = 0 \implies C(t)v = \lambda C(t_0)v$$

Generalized Eigenvalue Problem

Will see that fitting λ yield energies.

[1] Dudek et al., PRD77: 034501, 2008

Reordering of Spectra in GEVP

General

Exploit

$$v_n^\dagger C(t_0) v_m = \delta_{nm}$$

→

$$\begin{pmatrix} v_1(t)C(t_0)v_1(t+1) & \dots & v_1(t)C(t_0)v_n(t+1) \\ \vdots & \ddots & \vdots \\ v_n(t)C(t_0)v_1(t+1) & \dots & v_n(t)C(t_0)v_n(t+1) \end{pmatrix}$$

Occasionally, eigenvalues are not ordered correctly after solving the GEVP.

Reordering of Spectra in GEVP

Exploit

$$v_n^\dagger C(t_0) v_m = \delta_{nm}$$

→

General

$$\begin{pmatrix} v_1(t)C(t_0)v_1(t+1) & \dots & v_1(t)C(t_0)v_n(t+1) \\ \vdots & \ddots & \vdots \\ v_n(t)C(t_0)v_1(t+1) & \dots & v_n(t)C(t_0)v_n(t+1) \end{pmatrix}$$

Occasionally, eigenvalues are not ordered correctly after solving the GEVP.

Expectation

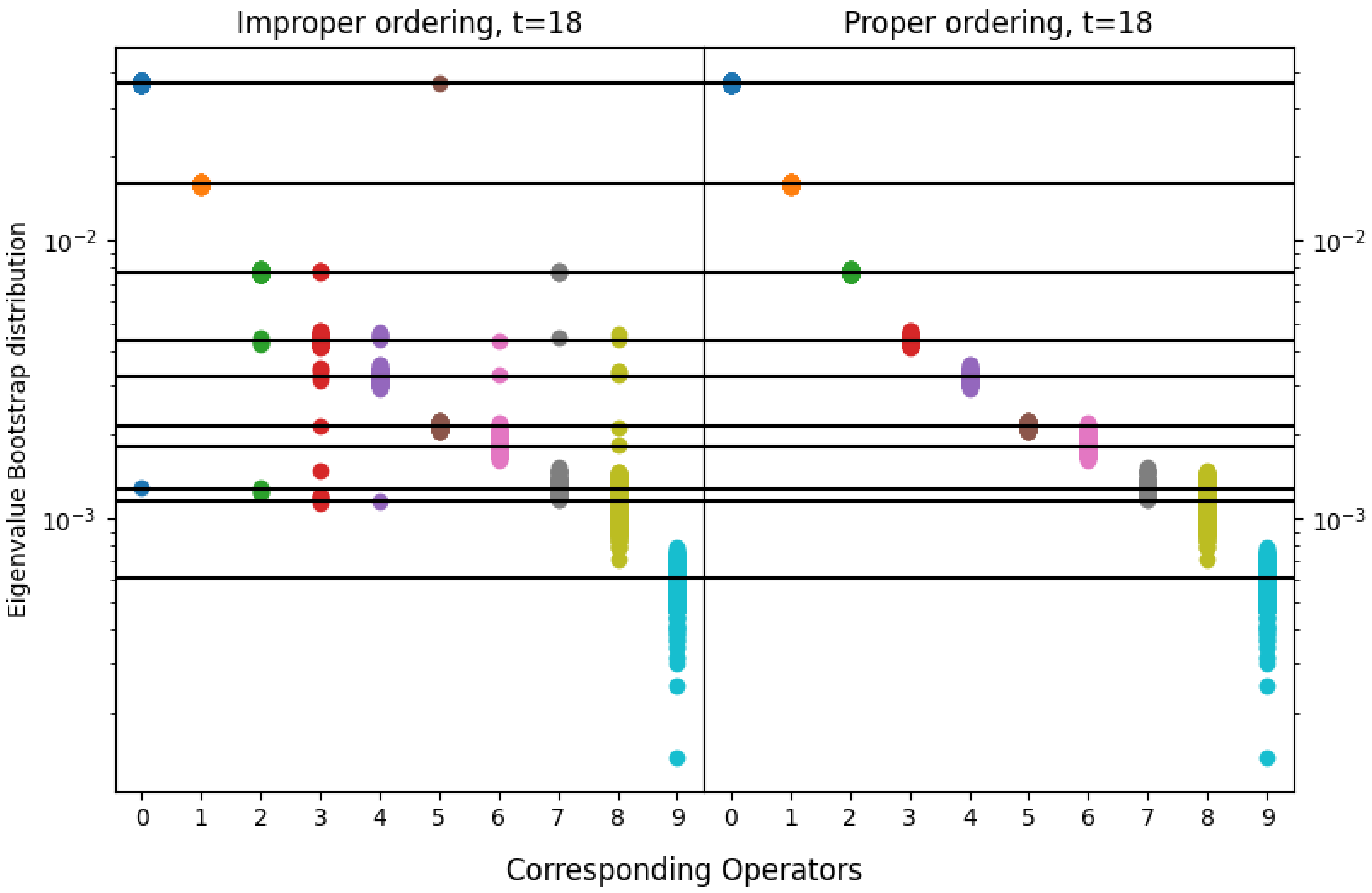
$$I_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Reality

$$\begin{pmatrix} \sim 1 & \sim 0 & \sim 0 & \sim 0 & \sim 0 & \sim 0 \\ \sim 0 & \sim 1 & \sim 0 & \sim 0 & \sim 0 & \sim 0 \\ \sim 0 & \sim 0 & \sim 1 & \sim 0 & \sim 0 & \sim 0 \\ \sim 0 & \sim 0 & \sim 0 & \sim 0 & \sim 1 & \sim 0 \\ \sim 0 & \sim 0 & \sim 0 & \sim 1 & \sim 0 & \sim 0 \\ \sim 0 & \sim 0 & \sim 0 & \sim 0 & \sim 0 & \sim 1 \end{pmatrix}$$

Tells us two eigenvectors have swapped order.

Comparison Plot (Ensemble Ca)



2pi.g5.0.0.1
2pi.g5.0.0.2
2pi.g5.0.1.1
2pi.g5.0.1.2
2pi.g5.0.1.2.v2
2pi.g5.0.2.2
2pi.g5.1.1.1
2pi.g5.1.1.2
2pi.g5.1.1.2.v2
svec.gi

Parameter slide

ID	a^{-1}/GeV	N_f	$L^3 \times T$	m_π/MeV	m_K/MeV	N_{conf}
48I	1.7312(28)	2+1	$48^3 \times 96$	139.32(30)	499.44(88)	27
Ca	1.7312(28)	2+1	$64^3 \times 128$	139.32(30)	499.44(88)	25
64I	2.3549(49)	2+1	$64^3 \times 128$	138.98(43)	507.5(1.5)	31
96I	2.6920(67)	2+1	$96^3 \times 192$	131.29(66)	484.5(2.3)	18

Parameters from Blum et al., arXiv:2301.08696

The $\pi\pi$ data

For 48I/64I we use:

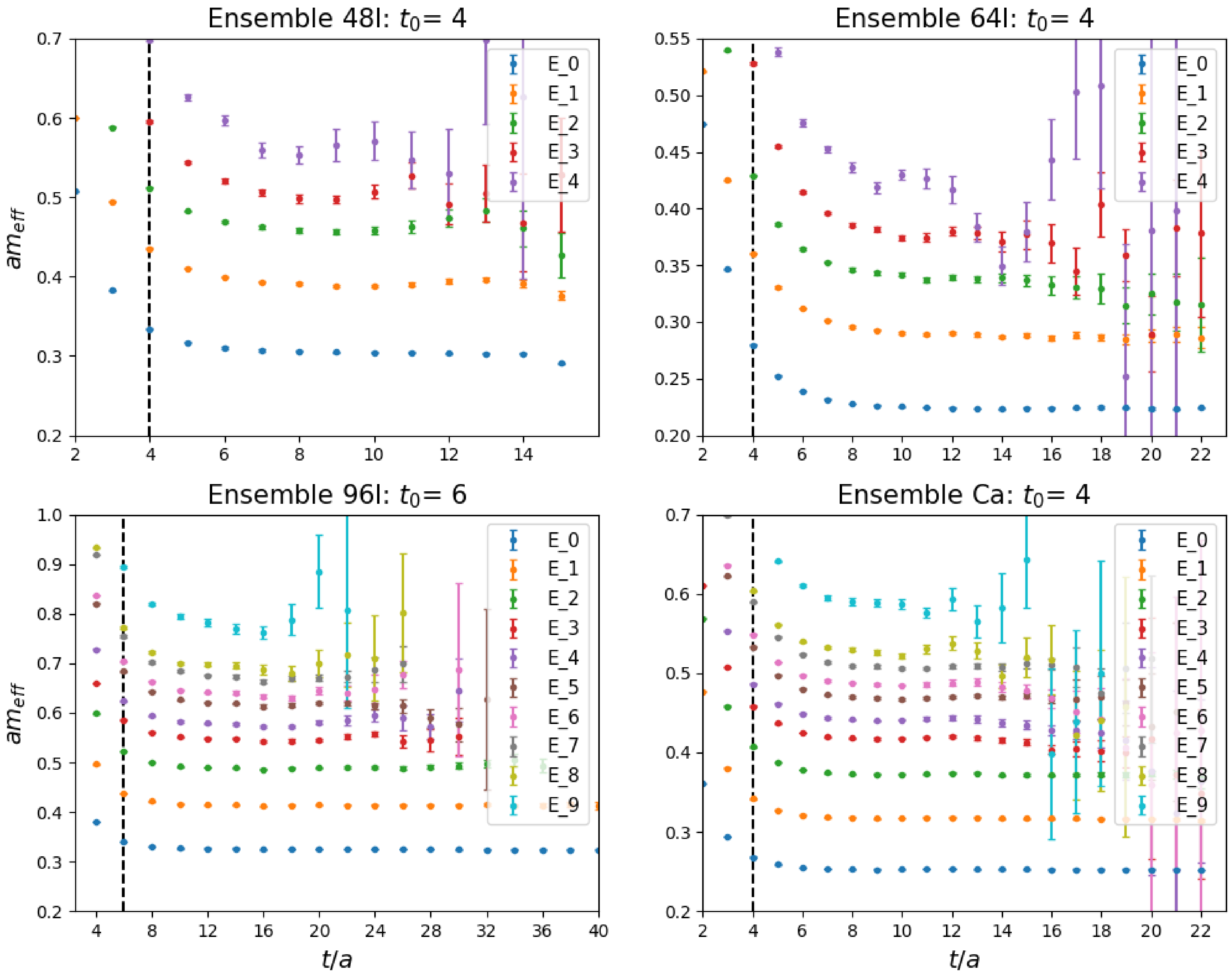
- 2pi.g5.0.0.1
- 2pi.g5.0.0.2
- 2pi.g5.0.1.1
- 2pi.g5.1.1.1
- svec.gi

Data is generated using distillation methods

For 96I/Ca we use:

- 2pi.g5.0.0.1
- 2pi.g5.0.0.2
- 2pi.g5.0.1.1
- 2pi.g5.0.1.2
- 2pi.g5.0.1.2.v2
- 2pi.g5.0.2.2
- 2pi.g5.1.1.1
- 2pi.g5.1.1.2
- 2pi.g5.1.1.2.v2
- svec.gi

Spectra Plots



Extracting energies from the eigenvalues

$$C(t)v = \lambda C(t_0)v$$

Can be shown that[1]:

$$\lambda_n(t) \overset{t\to\infty}{=} c_n e^{-tE_n}\left[1 + \mathcal{O}\left(e^{-t\Delta E_n}\right)\right]$$

We use below function due to finite lattice volume

$$\lambda_n^{fit}(t) = (1 - A_n)e^{-E_n(t-t_0)} + A_n e^{-E'_n(t-t_0)}$$

where $$E'_n = E_n + \Delta E_n$$

[1] M. Luscher and U. Wolff, NP B339 222 (1990)

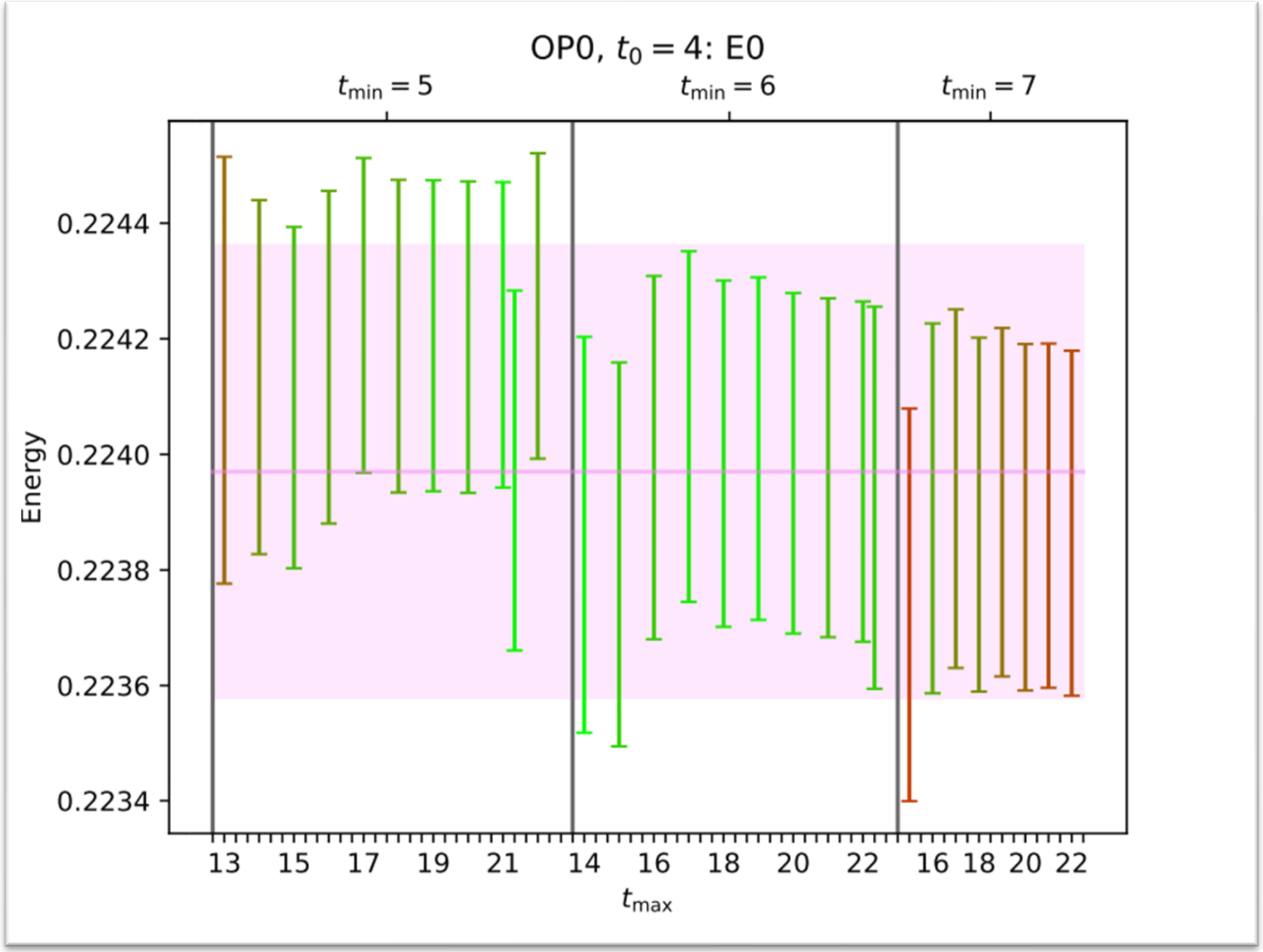
Fitting of GEVP spectra (Ensemble 64)

Fit form:

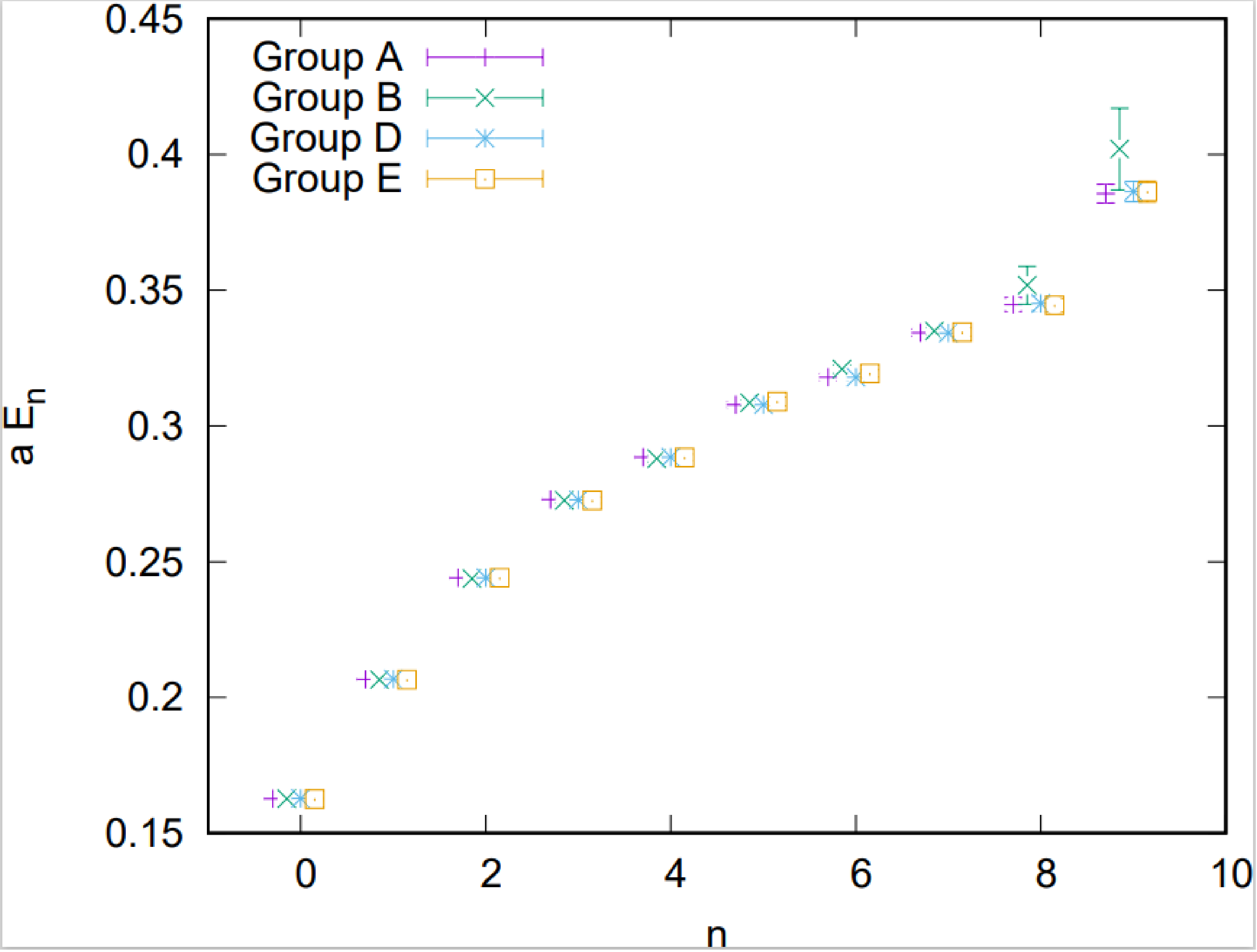
$$\lambda_n^{fit}(t) = (1 - A_n)e^{-E_n(t-t_0)} + A_n e^{-E'_n(t-t_0)}$$

Fit criteria:

- $N_{dof} \geq 4$ (relaxed for higher operators)
- t_{min} and t_{max} chosen to avoid noise and excited state contamination.
- $0.05 < p_{val} < 0.95$



Comparison between groups (Ensemble 96I)



Finding A_n's

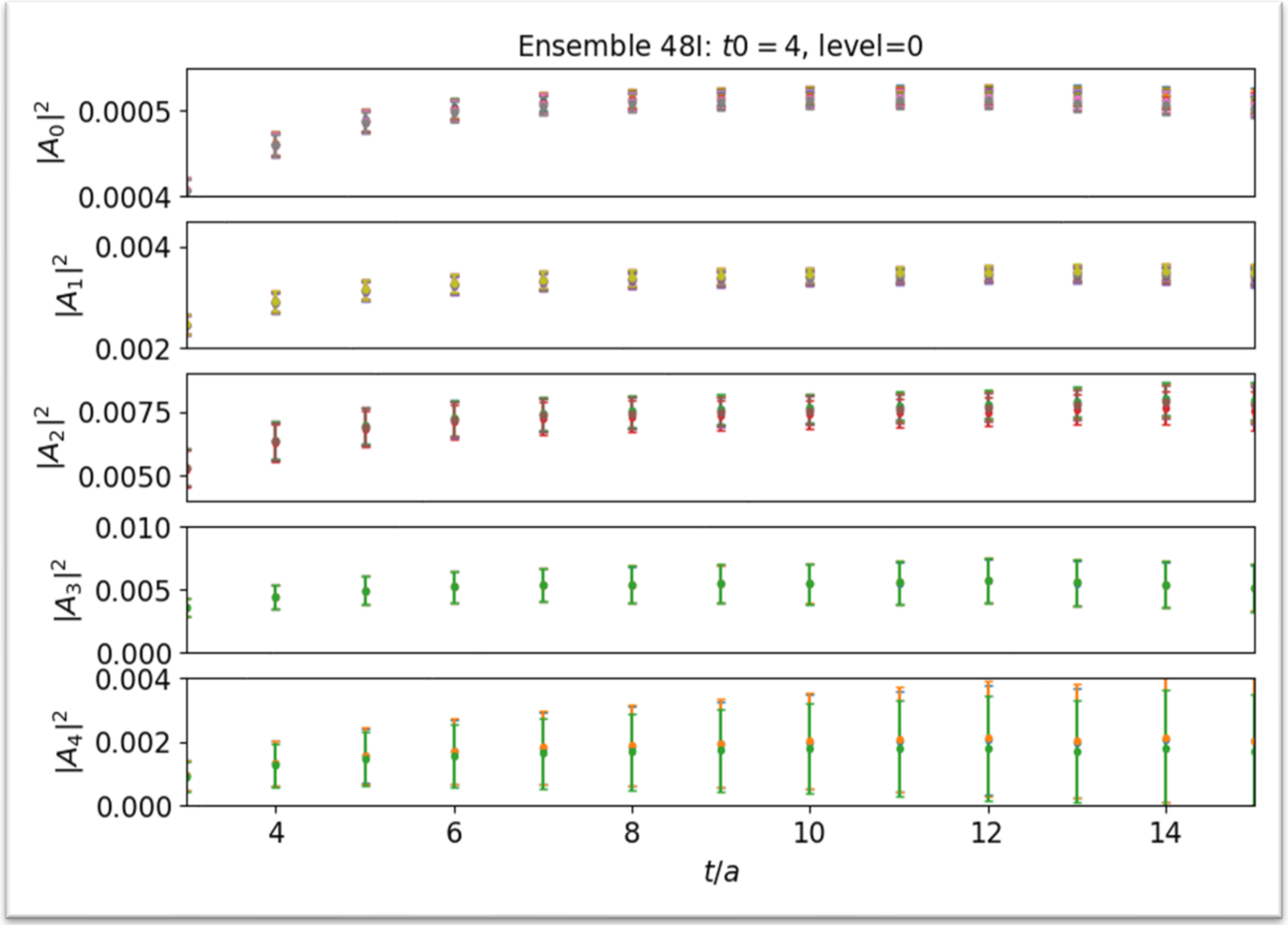
- Overlap factors can be found from below ratio[1,2].

$$D_{nn}(t) = \langle \Omega_n(t)\Omega_n^\dagger(0)\rangle = v_n^\dagger C(t) v_n$$

$$\langle J(t)\Omega^\dagger(0)\rangle = \sum_i v_{ni} \langle J(t)\mathcal{O}_i^{2\pi\dagger}(0)\rangle$$

$$R^{E_n}(t) = \frac{\sum_i v_{ni}\langle J(t)\mathcal{O}_i^{2\pi\dagger}(0)\rangle}{\sqrt{D_{nn}}e^{-E_n t/2}}$$

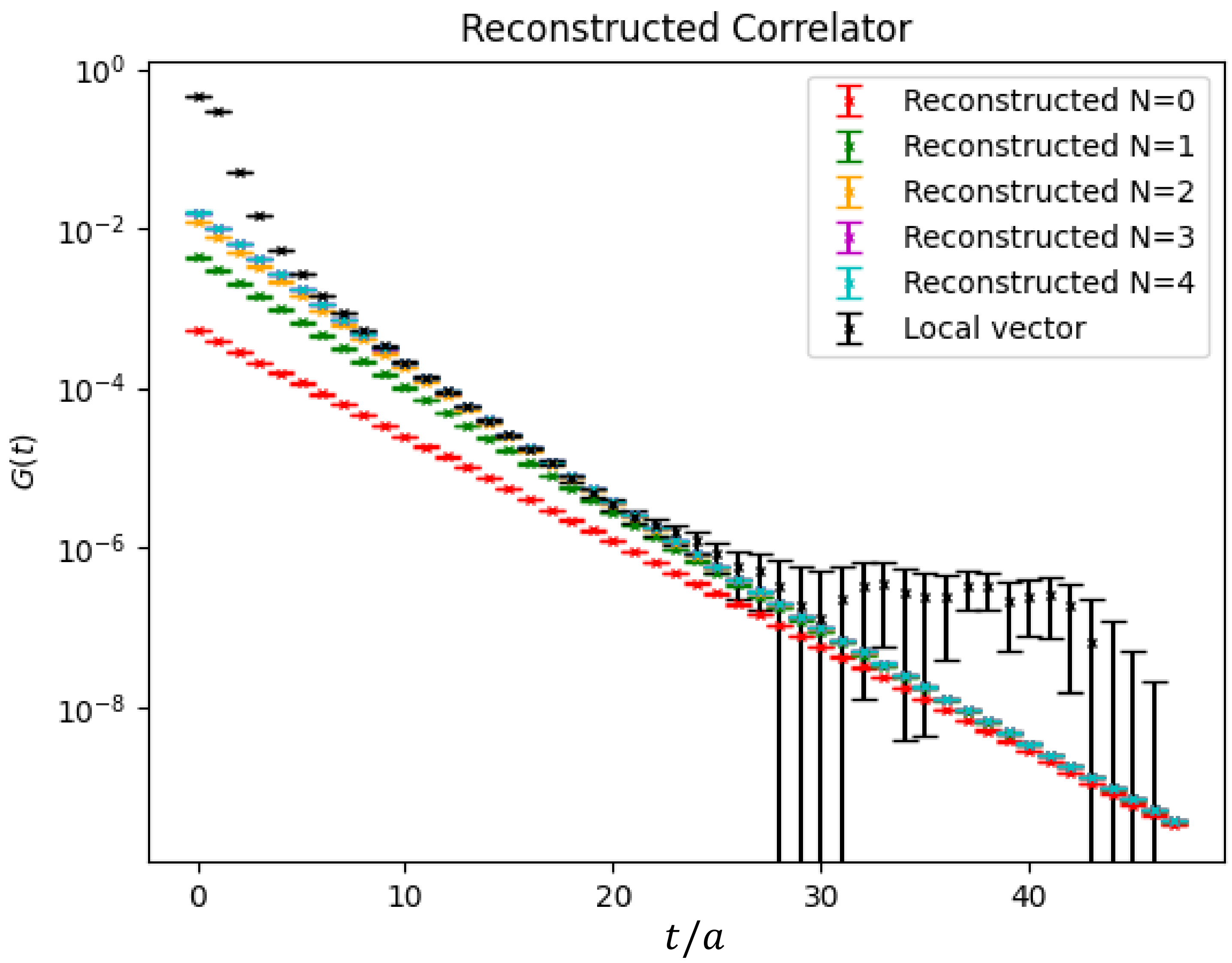
$$|R^{E_n}(t)|^2 \xrightarrow{t} |A_n|^2$$



[1] F. Erben et al., PRD 101 054504 (2020)
[2] Gérardin et al., PRD 100 014510 (2019)

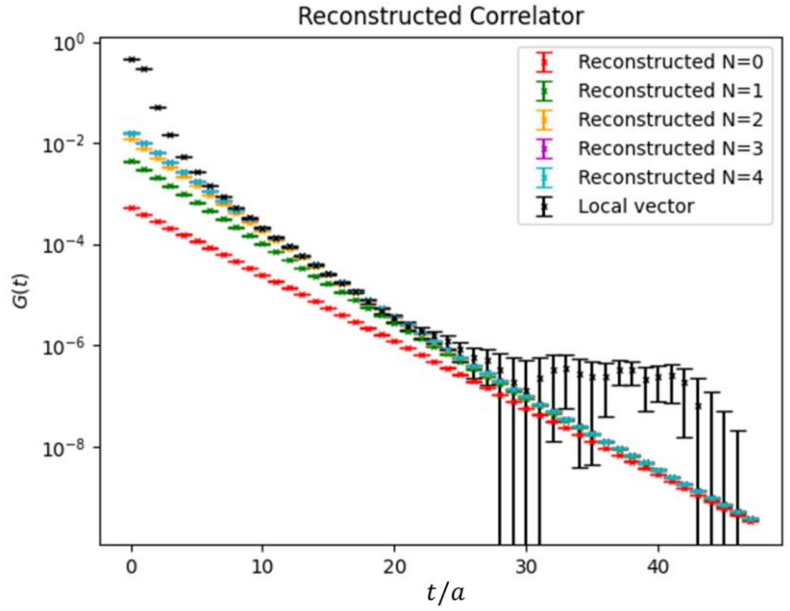
Reconstructing $G(t)$ (Ensemble 48I)

$$G(t) = \frac{10}{9} \sum_{n=0}^{n_{max}} |A_n|^2 e^{-E_n t}$$



Summary:

- a_μ^{HVP} and its error are dominated by light connected vector correlators which are too noisy at LD
- We have shown how using a large variational bases of $\pi\pi$ operators and GEVP methods we can reconstruct the $\pi\pi$-states at LD and significantly reduce the errors at LD
- This in turn significantly improved a_μ kernel at LD after implementing improved bounding method (as detailed in Christoph's talk)



Science and Technology Facilities Council

University of Southampton

Integrand Reconstruction (Ensemble 48I)

$$a_\mu^{HVP,LO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\tilde{K}(t; m_\mu)$$

Improved bounding method used (as detailed in Christoph's talk).