Structure-dependent electromagnetic finite-volume effects to the hadronic vacuum polarisation

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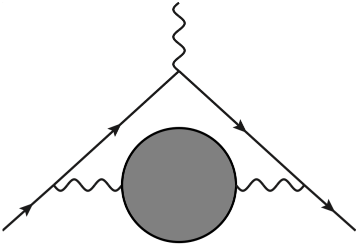
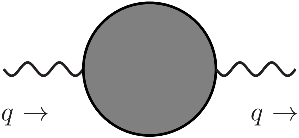
Introduction



- Hadronic vacuum polarisation (HVP) in lattice QCD+QED
- Finite-volume effects (FVEs)
- Analytically in [Bijnens, Harrison, Hermansson-Truedsson, Janowski, Jüttner, Portelli 19]
- Pointlike scalar $\mathrm{QED_L}$: Leading FVEs structure dependent and unknown
- QED long-range force: Potentially large FVEs

Hadronic vacuum polarisation



$$\Pi_{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x} \langle 0|\, T\left[J_\mu(x) J_\nu^\dagger(0)\right] |0\rangle$$

- Euclidean virtuality: $q^2 > 0$: $q = (q_0, \mathbf{0})$
- Ward identity $q_\mu \Pi_{\mu\nu} = 0$

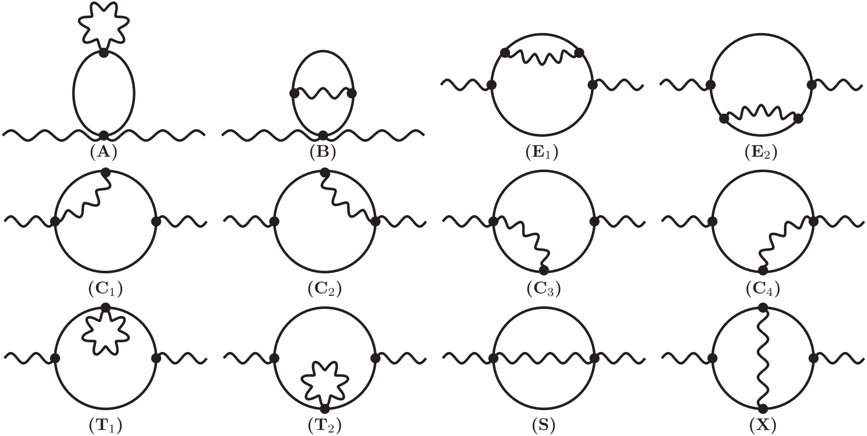
$$\Pi_{\mu\nu}(q^2) = \left(q_\mu q_\nu - q^2 \delta_{\mu\nu}\right) \Pi(q^2)$$

$$\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) = \frac{1}{3q_0^2} \sum_{j=1}^{3} \Big(\Pi_{jj}(q^2) - \Pi_{jj}(0)\Big)$$

- Finite-volume effects

$$\Delta\hat{\Pi}(q^2, L) = \hat{\Pi}(q^2, L) - \hat{\Pi}(q^2, \infty)$$

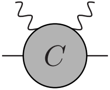
Diagrams including QED



- 7 independent diagrams: A, B, E, C, T, S, X
- 2-loop diagrams: 2 momenta in finite volume

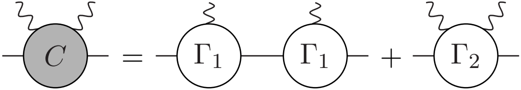
Structure dependence

- Structure dependence will enter through vertex functions
- Related to Compton tensor $C_{\mu\nu}(p, k, q)$



$$= C_{\mu\nu}(p, k, q) = e^2 \int \mathrm{d}^4x \, e^{-ik\cdot x} \, \langle \pi, \mathbf{p}| \, T\left\{J_\mu(x) J_\nu(0)\right\} |\pi, \mathbf{p}\rangle$$

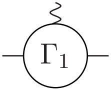
- Can insert complete set of states and decompose



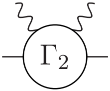
- Irreducible vertex functions $\Gamma_1 = \Gamma_\mu(p, k)$ and $\Gamma_2 = \Gamma_{\mu\nu}(p, k, q)$
- Contain structure dependence through form factors, e.g. [Fearing, Scherer 94]
- Structure for mass and leptonic decays [Di Carlo, Hansen, NHT, Portelli 22]

Structure dependence

- Structure-dependent vertex functions:



$= \Gamma_\mu(p, k)\,,$



$= \Gamma_{\mu\nu}(p, k, q)$

- We need form factor decompositions [Fearing, Scherer 94/96]:

$$\Gamma_\mu(p, k) = (2p + k)_\mu\, F(k^2) + k_\mu \frac{(p + k)^2 - p^2}{k^2}\,[1 - F(k^2)]$$

$$\Gamma_{\mu\nu}(p, k, q) = 2\delta_{\mu\nu}[1 - F(k^2) - F(q^2)] - 2\, k_\mu k_\nu\, \frac{1 - F(k^2)}{k^2} - 2\, q_\mu q_\nu\, \frac{1 - F(q^2)}{q^2} + \Gamma^{\mathrm{T}}_{\mu\nu}(p, k, q)$$

- $\Gamma^{\mathrm{T}}_{\mu\nu}(p, k, q)$ depends on 5 form-factors: $\widetilde{C}_1$, $\widetilde{D}$, $\widetilde{E}$, $\bar{F}$, $\bar{G}$
- Physically: EM polarisabilities α, $\beta + \ldots$

Diagrams

- 2-loop sum-integral differences for each diagram

$$\hat{\Pi} = 2\,\hat{\Pi}_E + 2\,\hat{\Pi}_T + \hat{\Pi}_S + \hat{\Pi}_X + 4\,\hat{\Pi}_C + \hat{\Pi}_A + \hat{\Pi}_B$$

$$\Delta\hat{\Pi}_U = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{L^3} \sum_{\boldsymbol{\ell}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{d\ell_0}{2\pi}\, \hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- Integrand $\hat{\pi}_U\left(q_0^2, k, \ell\right)$ contains propagator poles and cuts
- Can perform ℓ_0 and k_0 integrals analytically

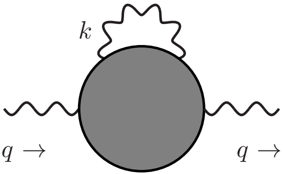
$$\hat{\rho}_U(\mathbf{k}, \boldsymbol{\ell}, q_0) = \int \frac{dk_0}{2\pi} \frac{d\ell_0}{2\pi}\, \hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- We thus have

$$\Delta\hat{\Pi} = \Delta_{\mathrm{poles}}\hat{\Pi} + \Delta_{\mathrm{cuts}}\hat{\Pi}$$

Analytical structure

- Let us go back to the HLbL tensor



$$\Delta\hat{\Pi}(q_0^2) = \left(\frac{1}{L^3} \sum_{\mathbf{k} \neq \mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{\hat{\Pi}^{\mu\mu\nu\nu}(q, q, k, -k)}{k^2}$$

- Analytical structure: Dispersion relations [Colangelo et al. 2015; Biloshytskyi et al. 23]
- Relation between 2π intermediate state and Feynman diagrams
- Can do pole part with form factors and cuts connecting to HLbL

Pole part

- We let pions be in infinite volume

$$\text{Poisson:} \quad \sum_{\boldsymbol{\ell}} = \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} + \mathcal{O}\left(e^{-m_\pi L}\right)$$

$$\Delta\hat{\Pi}_U = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3}\, \hat{\rho}_U(\mathbf{k}, \boldsymbol{\ell}, q_0)$$

- Photons give rise to FV coefficients c_j (prescription dependent)
- Pions to integrals

$$\Omega_{i,j}(z = q_0^2/m^2) = \int dx\, x^2 \frac{1}{(1+x^2)^{i/2}[z + 4(x^2+1)]^j}$$

$$\bar{\Omega}_{i,j}(z = q_0^2/m^2) = \int dx\, x^2 \frac{A_1\left(\frac{x}{\sqrt{1+x^2}}\right)}{(1+x^2)^{i/2}[z + 4(x^2+1)]^j}$$

$$A_1(x) = \frac{\operatorname{arctanh}(x)}{x}$$

Earlier work

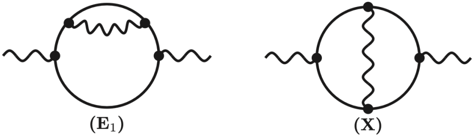
- Pointlike case: [Bijnens, Harrison, Hermansson-Truedsson, Janowski, Jüttner, Portelli 19]

$$\Delta\hat{\Pi}_U(q^2) = \frac{c_1}{(mL)^2}\,\mathcal{C}_{U,2}^{\mathrm{point}} + \frac{c_0}{(mL)^3}\,\mathcal{C}_{U,3}^{\mathrm{point}}$$

$$\Delta\hat{\Pi}(q^2) = \frac{c_0}{(mL)^3}\left(-\frac{16}{3}\Omega_{0,3} - \frac{5}{3}\Omega_{2,2} + \frac{40}{9}\Omega_{2,3} - \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} + \frac{8}{9}\Omega_{4,3}\right)$$

- Suppression: Leading order is $1/L^3$
- Physics: Neutral current and photon far away sees no charge
- We showed cancellation independent of structure: HLbL [Colangelo et al. 15]
- Finite-volume scaling depends on QED formulation
- $\mathrm{QED_r}/\mathrm{QED_C}$: Starts at order $1/(mL)^4$
- Exponentially suppressed FVEs [Asmussen et al. 16; Biloshytskyi et al. 23]

Preliminary results



$$\Delta_{\rm poles}\hat{\Pi}_{EX}(q_0^2) = \frac{c_1}{24\pi z\,(mL)^2}\Big\{4\Big[4z^2\,\Omega_{1,3} - 7z^2\,\Omega_{3,3} + 3z^2\,\Omega_{5,3} - 96z\,\Omega_{1,3} + 40z\,\Omega_{3,3} + 8(7z-66)\,\Omega_{-1,3} + 288\Omega_{-3,3} + 240\,\Omega_{1,3}\Big] F\left(q_0^2\right)^2 - 3\Big[6z^3\,\Omega_{3,3} - 11z^3\,\Omega_{5,3} + 5z^3\,\Omega_{7,3} + 72z^2\,\Omega_{1,3} - 132z^2\,\Omega_{3,3} + 60z^2\,\Omega_{5,3} - 528z\,\Omega_{1,3} + 240z\,\Omega_{3,3} + 32(9z-22)\,\Omega_{-1,3} + 384\,\Omega_{-3,3} + 320\,\Omega_{1,3}\Big]\Big\} + \frac{c_0\,\mathcal{C}}{(mL)^3}$$

- Form factors
- $1/L^3$ depends on charge radius $\langle r_\pi^2\rangle$ as well, and $\bar{\Omega}_{i,j}$
- Pointlike limit gives back old result

Conclusions and outlook

- $\mathrm{QED_L}$: HVP starts at order $c_0/(mL)^3$
- Proven in [Bijnens, Harrision, Hermansson-Truedsson, Janowski, Jüttner, Portelli 19]
- New steps towards evaluating structure-dependent $c_0/(mL)^3$
- Poles + cuts: can be cancellations
- Only then will we know how big the leading FVE is
- Numerical validation from e.g. lattice perturbation theory

Backup slides

Other QED prescriptions

- $\mathrm{QED_L}$: Starts at $1/L^3$ with structure dependence

$$\Delta\hat{\Pi}(q^2) \overset{\mathrm{QED_L}}{\propto} \frac{c_0}{(m_\pi L)^3}\,, \qquad c_0 = -1$$

- $\mathrm{QED_C}$:

$$\Delta\hat{\Pi}(q^2) \overset{\mathrm{QED_C}}{\propto} \frac{c_0^\star}{(m_\pi L)^3}\,, \qquad c_0^\star = 0$$

- So $\mathrm{QED_C}$ expected to have smaller FVEs than $\mathrm{QED_L}$ here
- $\mathrm{QED_r}$ [Di Carlo, Hansen, NHT, Portelli In prep.]: Designed to remove c_0

$$\Delta\hat{\Pi}(q^2) \overset{\mathrm{QED_r}}{\propto} \frac{\bar{c}_0}{(m_\pi L)^3}\,, \qquad \bar{c}_0 = 0$$

- Leading scaling the same in $\mathrm{QED_r}$ as $\mathrm{QED_C}$
- Approach of [Biloshytskyi et al. 23]: Exponentially suppressed

Numerical validation

Lattice scalar QED simulations:

- Eight different volumes with L/a between 16 and 64;
  Time extent $T/a = 128$
- $z = q^2/m^2 = 0.964$, $am = 0.2$

Lattice perturbation theory (LPT): $k_\mu,\ \ell_\mu \in (-\pi/a, \pi/a)$

- Cuba VEGAS Monte Carlo integration for both FV and IV pions

$$\left(\frac{1}{L^3} \sum_{\mathbf{k} \neq \mathbf{0}} \frac{1}{L^3} \sum_{\boldsymbol{\ell}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{d\ell_0}{2\pi}$$

- We do this for several lattice spacings and then make a continuum extrapolation

Numerical validation