Inclusive Semileptonic Decay of the D_s meson

Lattice2024

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Extended Twisted Mass Collaboration

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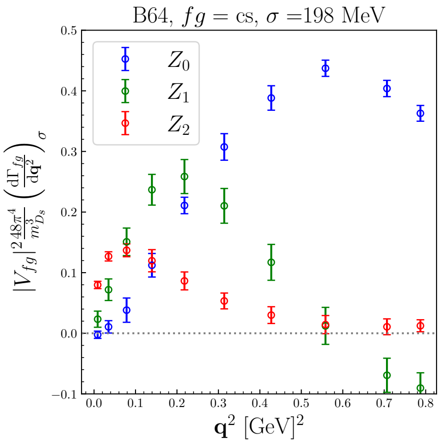
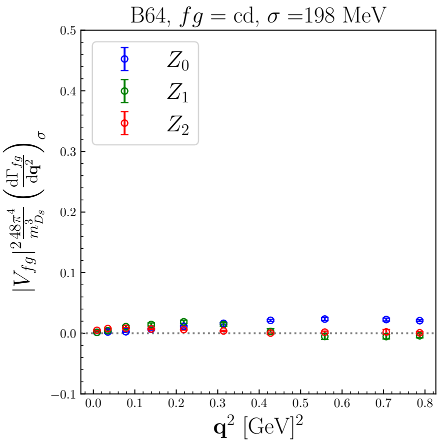
University of Roma Tre

- Giuseppe Gagliardi
- Vittorio Lubicz
- Aurora Melis
- Francesco Sanfilippo
- Silvano Simula

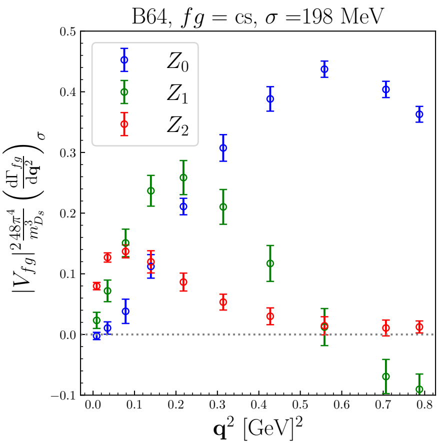
Results previous talk: differential decay rate from lattice observables

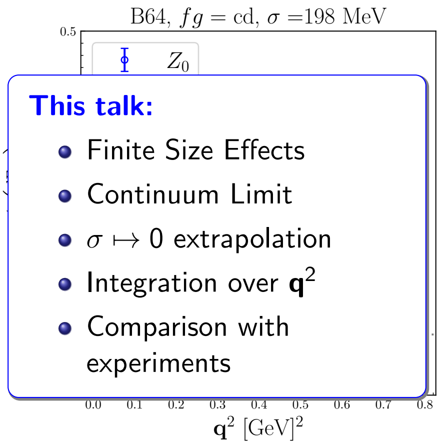




Γ_{cd} is Cabibbo suppressed

Results previous talk: differential decay rate from lattice observables





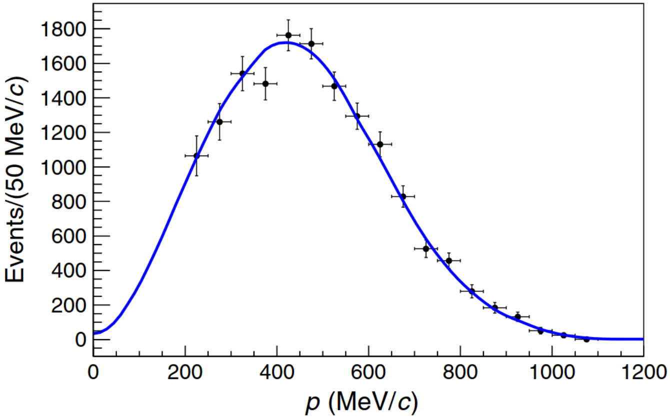
This talk:

- Finite Size Effects
- Continuum Limit
- $\sigma \mapsto 0$ extrapolation
- Integration over $\mathbf{q}^2$
- Comparison with experiments

Introduction

- Semileptonic Inclusive Decay of the D_s
- inclusive decay is an inverse problem
- ill-posed-problem, solved with HLT method [HLT19]
- preliminary investigation [Gam+22; GH20]
- theoretical details, applications in talk by Alessandro De Santis
- inclusive and exclusive do not agree in B-decays

Experimental results



BESIII
$D_s^+ \to X e^+ \nu_e$
fit to determine
low momenta

BESIII [Abl+21] $\Gamma = (8.27 \pm 0.17 \pm 0.12 \pm 0.06) \times 10^{-14}\,\mathrm{GeV}$

CLEO-C [Asn+10] $\Gamma = (8.56 \pm 0.51 \pm 0.20) \times 10^{-14}\,\mathrm{GeV}$

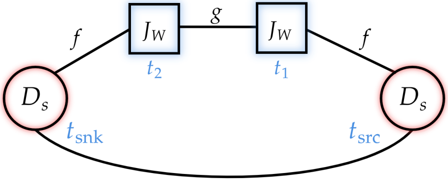
Γ_{fg} from lattice QCD

We need the hadronic tensor which is the **spectral density** of the correlation function

$$M_{fg}^{\mu\nu}(t, \mathbf{q}^2) = \int_0^\infty \mathrm{d}q_0 \, H_{fg}^{\mu\nu}(q_0, \mathbf{q}^2) e^{-q_0 t}$$

that in practice is obtained by

$$M_{fg}^{\mu\nu}(t_2 - t_1, \mathbf{q}^2) = \lim_{\substack{t_{\mathrm{snk}} \mapsto +\infty \\ t_{\mathrm{src}} \mapsto -\infty}} \frac{C_{4\mathrm{pt}}^{\mu\nu}(t_{\mathrm{snk}}, t_2, t_1, t_{\mathrm{src}}; \mathbf{q})}{C_{2\mathrm{pt}}(t_{\mathrm{snk}} - t_2) C_{2\mathrm{pt}}(t_1 - t_{\mathrm{src}})}$$



- $t = t_2 - t_1 = a, 2a, \cdots$ **Euclidean time**
- $t_2 - t_{\mathrm{snk}},\ t_{\mathrm{src}} - t_1 \gg 0$ checked

Γ_{fg} from lattice QCD

$$24\pi^3 \frac{\mathrm{d}\Gamma_{fg}}{\mathrm{d}\mathbf{q}^2} = \sum_{n=0}^{2} |\mathbf{q}|^{3-n} \int_{q_0^{\mathrm{min}}}^{q_0^{\mathrm{max}}} \mathrm{d}q_0 (q_0^{\mathrm{max}} - q_0)^n Z_n$$

- Z_0, Z_1, Z_2 can be expresses as linear combinations of $H_{fg}^{\mu\nu}$
- allowed q^2 range depends on flavour combination fg
- σ: smearing parameter

- A: systematical error in HLT
- B: statistical error in HLT
- combined with λ

Γ_{fg} from lattice QCD

$$24\pi^3 \frac{\mathrm{d}\Gamma_{fg}}{\mathrm{d}\mathbf{q}^2} = \lim_{\sigma \mapsto 0} \sum_{n=0}^{2} |\mathbf{q}|^{3-n} \int_{q_0^{\min}}^{\infty} \mathrm{d}q_0 (q_0^{\max} - q_0)^n \theta_\sigma (q_0^{\max} - q_0) Z_n$$

- Z_0, Z_1, Z_2 can be expresses as linear combinations of $H_{fg}^{\mu\nu}$
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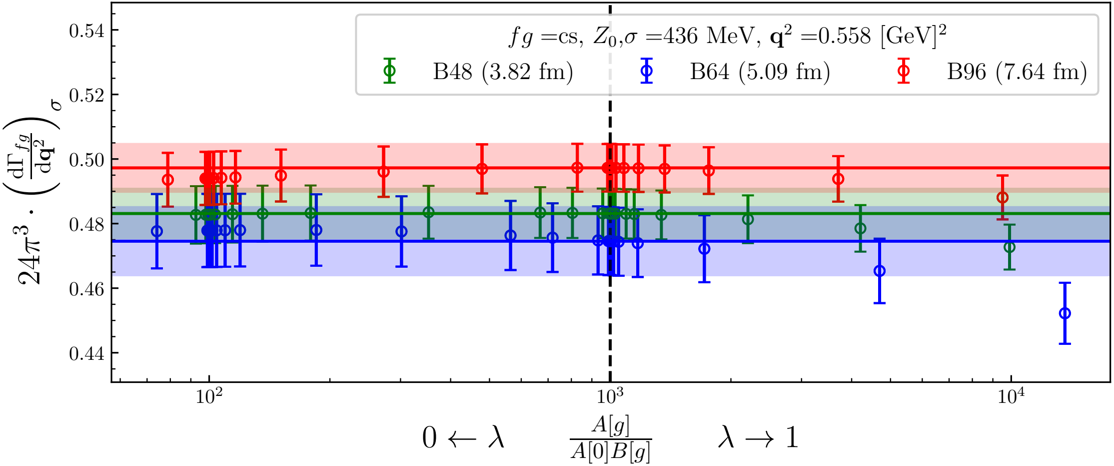
Configurations

name	L [fm]	a [fm]	M_π [MeV]
B48	3.82	0.080	≈ 135
B64	5.10	0.080	≈ 135
B96	7.64	0.080	≈ 135
C80	5.46	0.068	≈ 135
D96	5.46	0.057	≈ 135
E112	5.48	0.049	≈ 135

- ETMC-configurations
- $\mathcal{O}(a)$ and clover improved
- $N_f = 2 + 1 + 1$
- ten momenta per ensemble
- three decay channels
- two smearing kernels
- $\mathcal{O}(10)$ values of σ

Finite-Volume-Effects

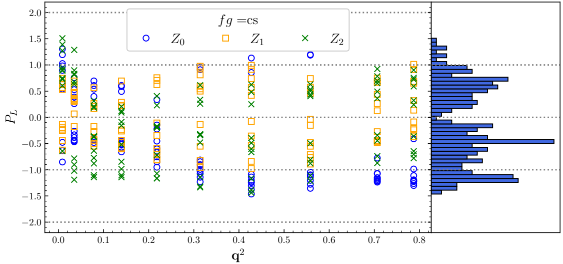
Flat volume dependence, HLT result stable



Finite-Volume-Effects

Quantify systematic effects of finite volume:

$$P_L(\sigma, q^2) = \frac{x(\sigma, q^2, L) - x\left(\sigma, q^2, \frac{3L}{2}\right)}{\sqrt{\Delta^2_{\text{stat}}(\sigma, q^2, L) + \Delta^2_{\text{stat}}\left(\sigma, q^2, \frac{3L}{2}\right)}}$$

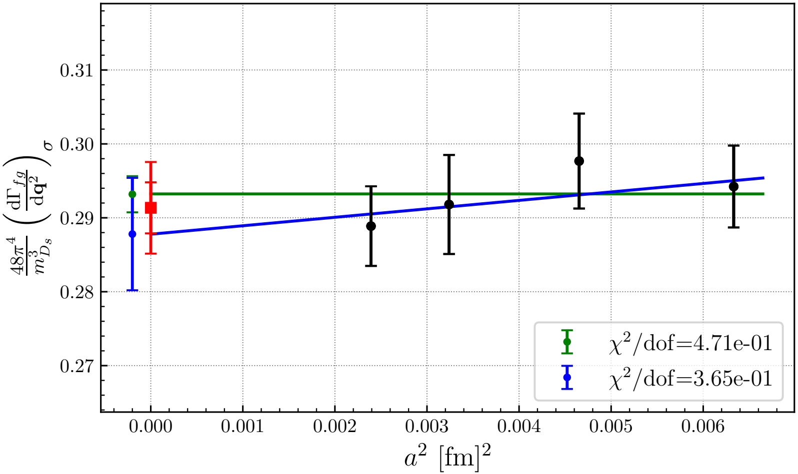


Calculate systematic error:

$$\Delta_{\text{sys}}(\sigma, q^2) = \left| x(L) - x\left(\frac{3L}{2}\right) \right| \cdot \text{erf}\left(\frac{P_L(\sigma, q^2)}{\sqrt{2}}\right)$$

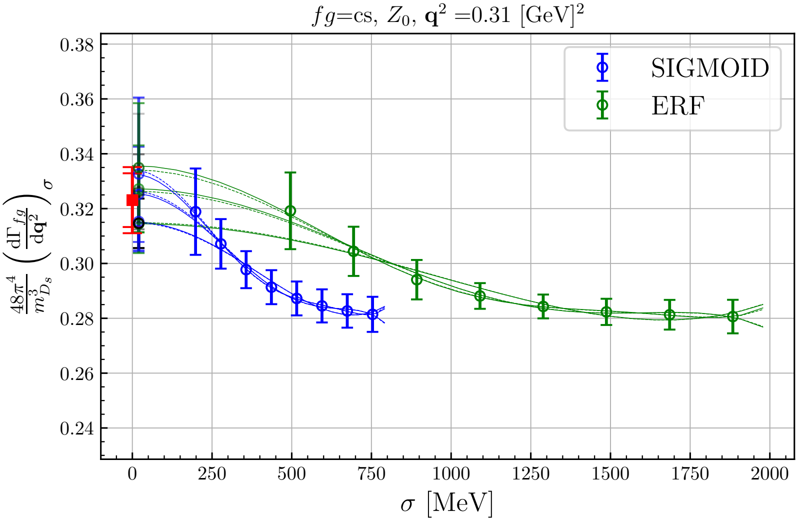
Order 1: **continuum limit**; smearing limit

fg=cs, Z_0, $\mathbf{q}^2$=0.314 [GeV]^2, σ = 436 [MeV],



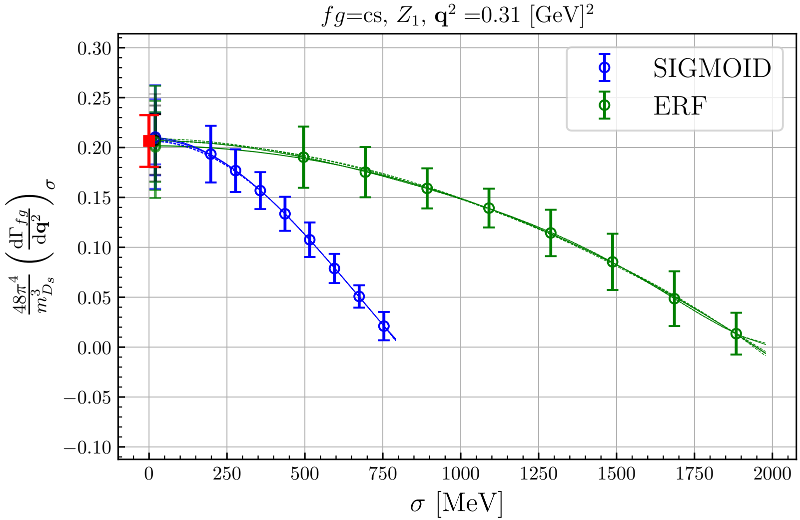
- AIC-combination of linear and constant fit
- flat limit
- small effect

Order 1: continuum limit; **smearing limit**



- combination of two kernels
- good agreement between kernels
- smooth extrapolations for all contributions
- even powers of σ

Order 1: continuum limit; **smearing limit**



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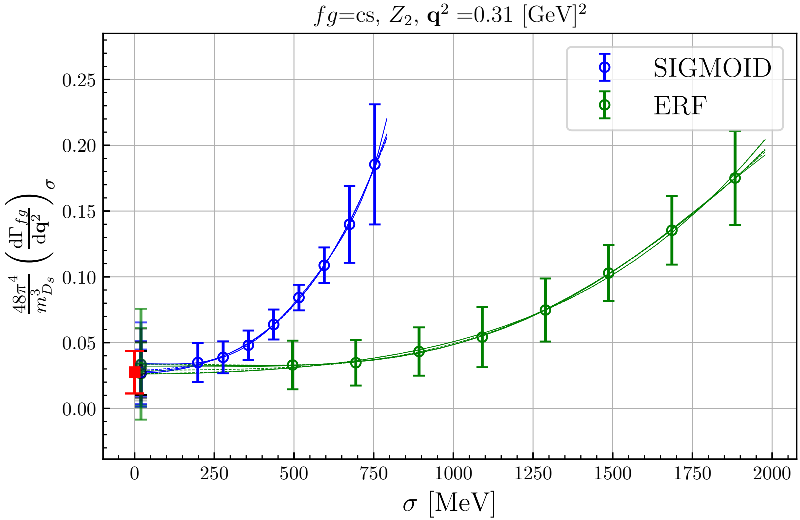
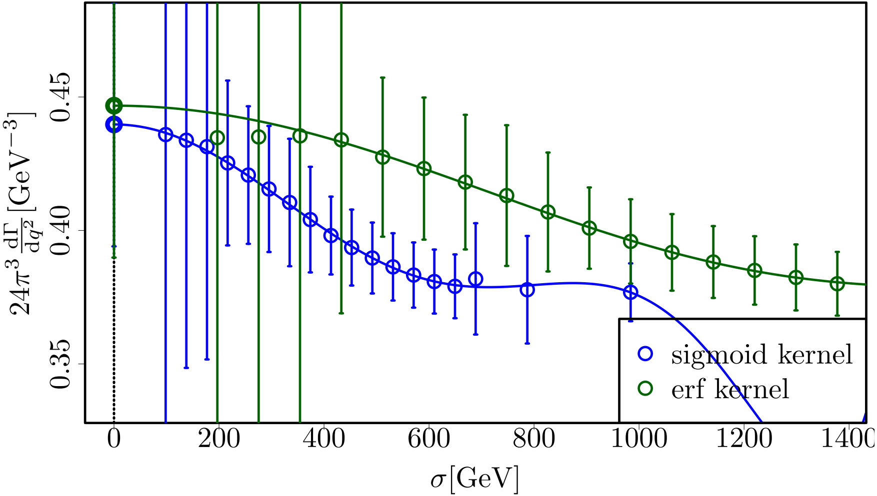


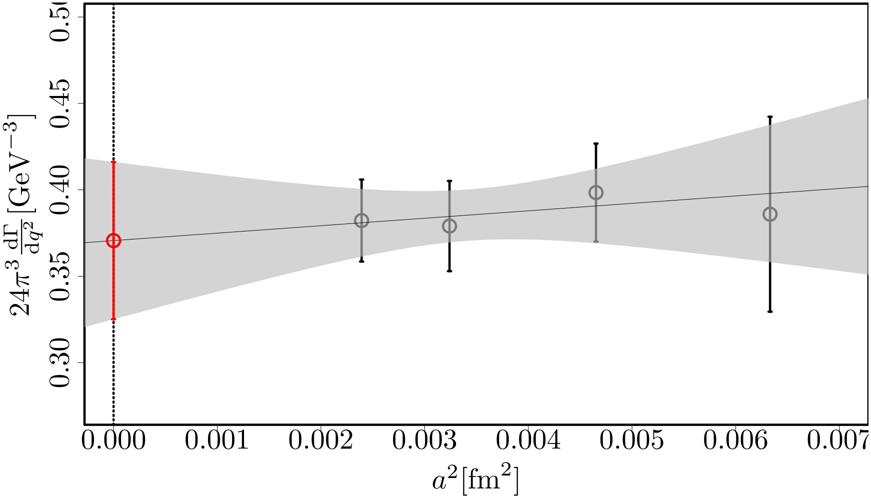
- combination of two kernels
- good agreement between kernels
- smooth extrapolations for all contributions
- even powers of σ

Order 2: **smearing limit**; continuum limit



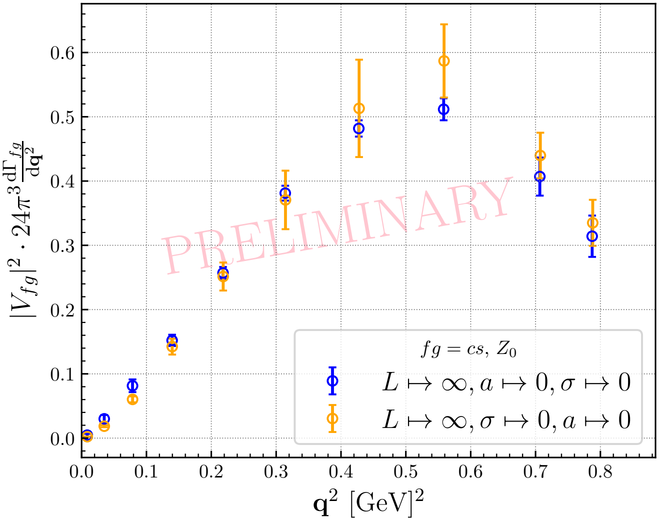
- B64, Z_0
- $q^2 = 0.31\text{GeV}^2$
- good agreement between kernels
- even powers of σ

Order 2: smearing limit; **continuum limit**



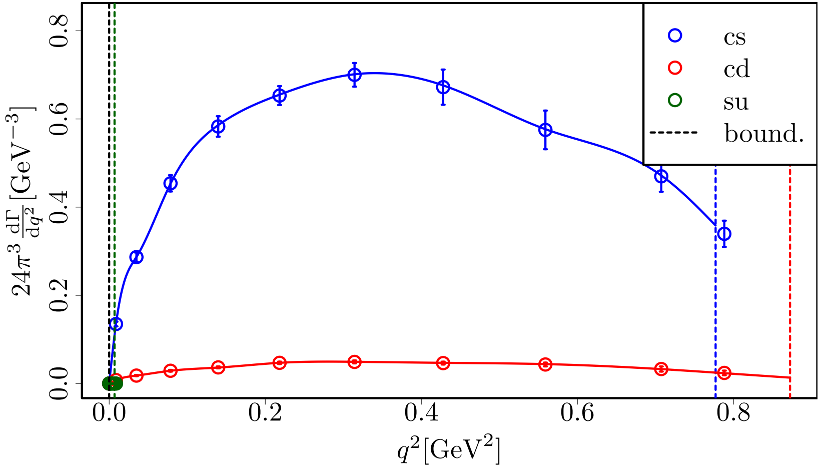
- Z_0, $q^2 = 0.31 \mathrm{GeV}^2$

Comparing order of limits



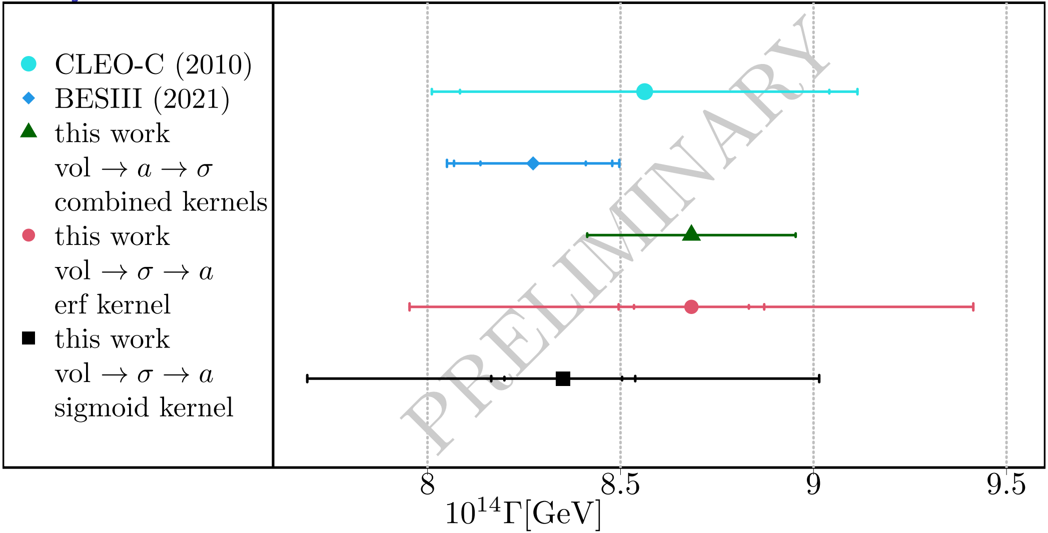
$\Rightarrow$ The two different orders are compatible

Calculation total decay rates



- B64, stat. error
- $Z_0 + Z_1 + Z_2$
- interpolation with cubic splines
- piecewise integration
- different momenta regions for different decay channels

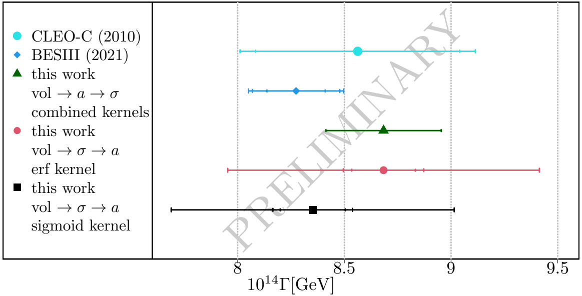
Summary


CLEO-C (2010)
BESIII (2021)
this work
vol → a → σ
combined kernels
this work
vol → σ → a
erf kernel
this work
vol → σ → a
sigmoid kernel
PRELIMINARY
8
8.5
9
9.5
10^14 Γ[GeV]

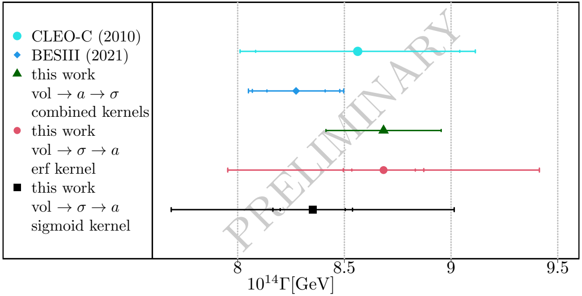
Summary

Summary

- HLT method well suited
- systematics under control
- good agreement with experimental results

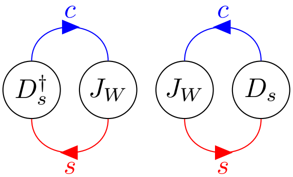


Summary



Outlook

- ✓ Quark Mass Dependence
- ✓ Contribution from $fg = su$
- ✓ Leptonic Moments
- ✓ Disconnected Diagrams
- ✓ Exclusive Contributions
- ! next step: B-decay



Summary

Outlook

- ✓ Quark Mass Dependence
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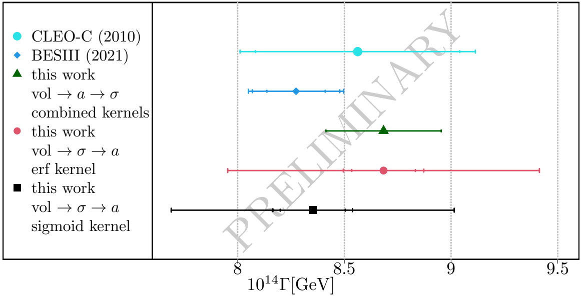
Thank you for your attention!



Definition of Z_n

$$Z_0 \equiv Y_2 + Y_3 - 2Y_4 \qquad Z_1 \equiv 2(Y_3 - 2Y_1 - Y_4) \qquad Z_2 \equiv Y_3 - 2Y_1$$

Form factors decomposition of the hadronic tensor

$$m_{D_s}^3 H^{\mu\nu}(p, p_x) = g^{\mu\nu} m_{D_s}^2 h_1 + p^\mu p^\nu h_2 + (p - p_X)^\mu (p - p_X)^\nu h_3$$
$$+[p^\mu (p - p_X)^\nu + (p - p_X)^\mu p^\nu] h_4 - i\varepsilon^{\mu\nu\alpha\beta} p_\alpha (p - p_X)_\beta h_5$$

$$Y_1 = -m_{D_s} \sum_{ij} \hat{n}^i \hat{n}^j H^{ij} = h_1$$

$$Y_2 = m_{D_s} H^{00} = h_1 + h_2 + \left(1 - \frac{q_0}{m_{D_s}}\right)^2 h_3 + 2\left(1 - \frac{q_0}{m_{D_s}}\right) h_4$$

$$Y_3 = m_{D_s} \sum_{ij} \hat{q}^i \hat{q}^j H^{ij} = -h_1 m_{D_s}^2 + |\mathbf{q}|^2 h_3$$

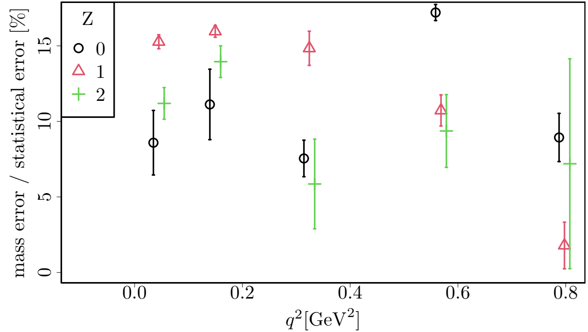
$$Y_4 = -m_{D_s} \sum_{i} \hat{q}^i H^{0i} = \left(1 - \frac{q_0}{m_{D_s}}\right) |\mathbf{q}| h_3 + |\mathbf{q}| h_4$$

$$Y_5 = \frac{i m_{D_s}}{2} \sum_{ijk} \varepsilon^{ijk} \hat{q}^k H^{ij} = |\mathbf{q}| h_5$$

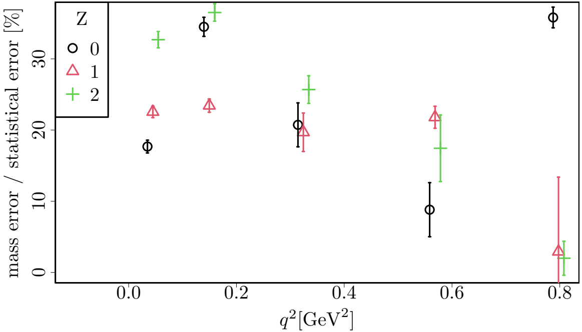
$$\hat{n}^2 = 1$$
$$\hat{n} \cdot \mathbf{q} = 0$$
$$\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$$

Contribution of different strange and charm quark mass

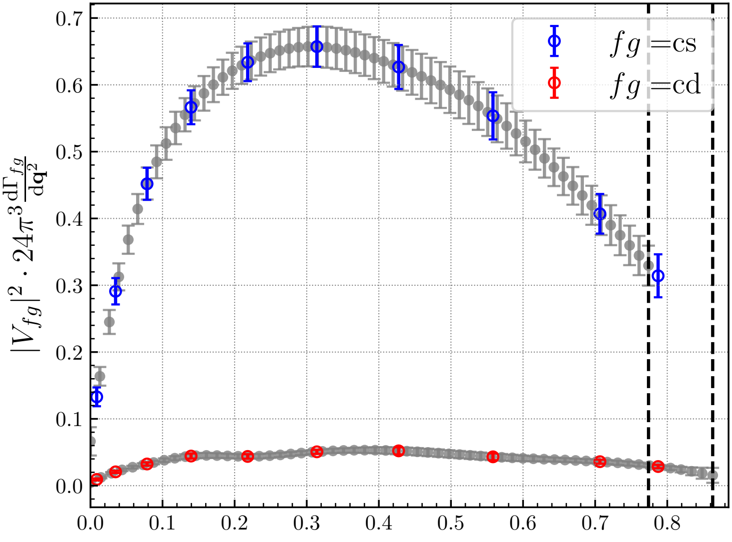
cd channel



cs channel



Total decay rate



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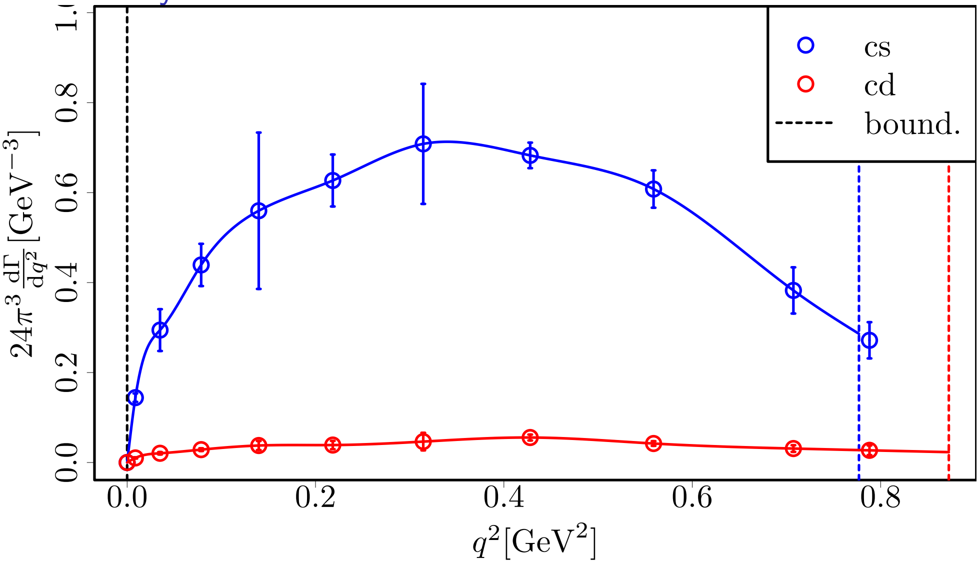
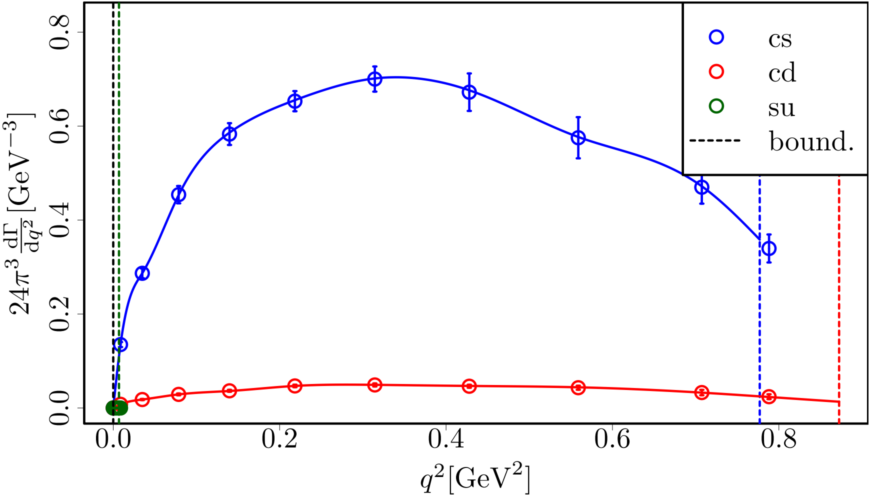
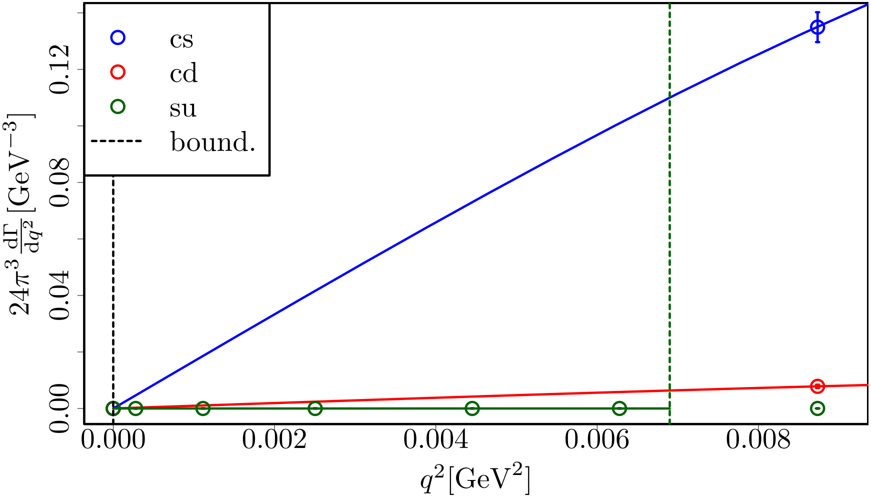


Contribution $fg = su$



B64, statistical error, $Z_0 + Z_1 + Z_2$

Contribution $fg = su$



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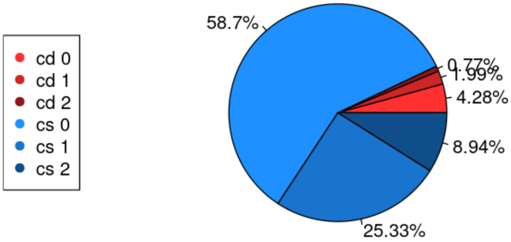
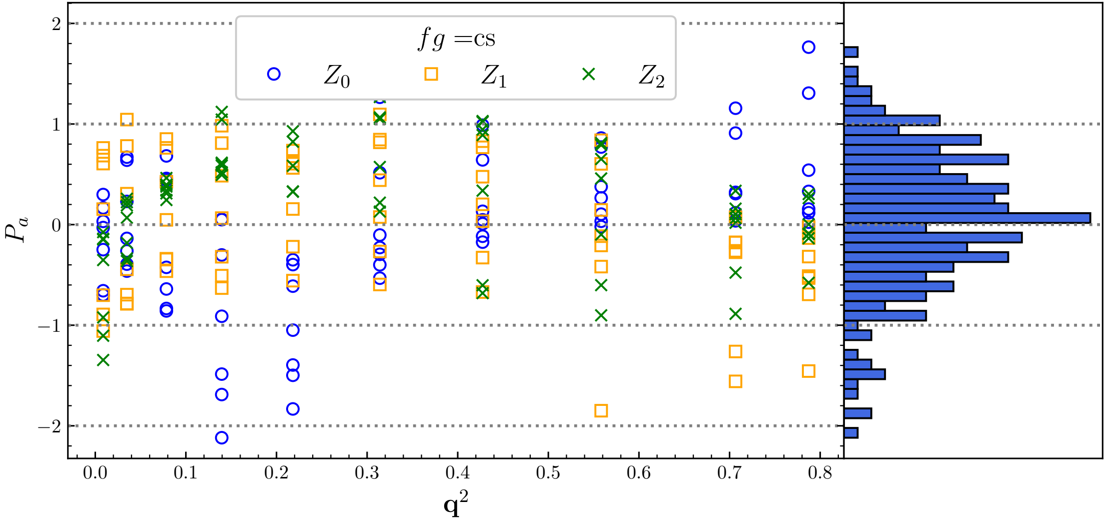


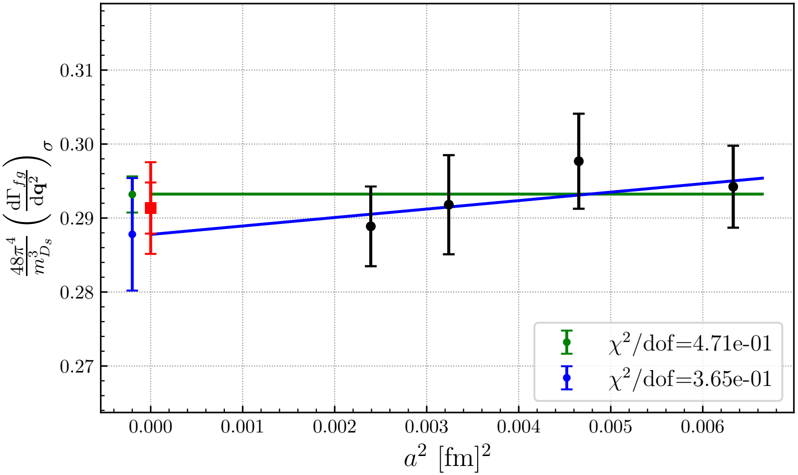
cs	cd	su
93%	7%	$< 10^{-5}\%$

systematics from Continuum Limit



Order 1: **Continuum Limit**; Smearing Limit

fg=cs, Z_0, $\mathbf{q}^2$=0.314 [GeV]^2, σ = 436 [MeV],



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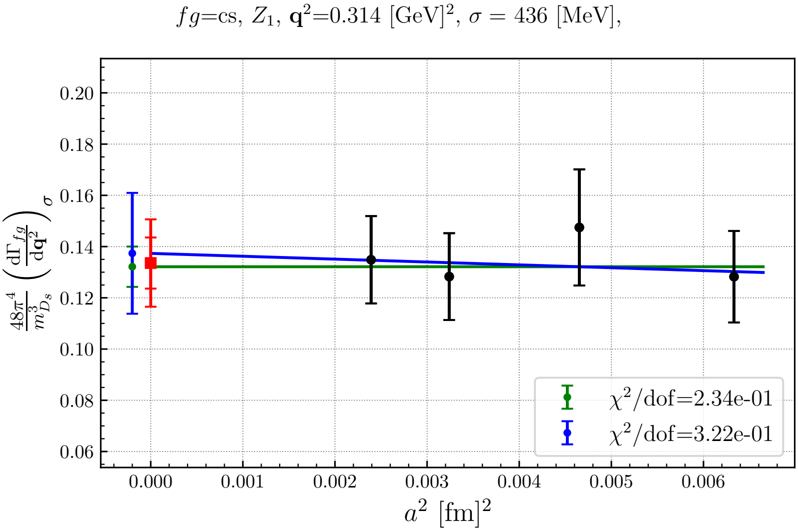

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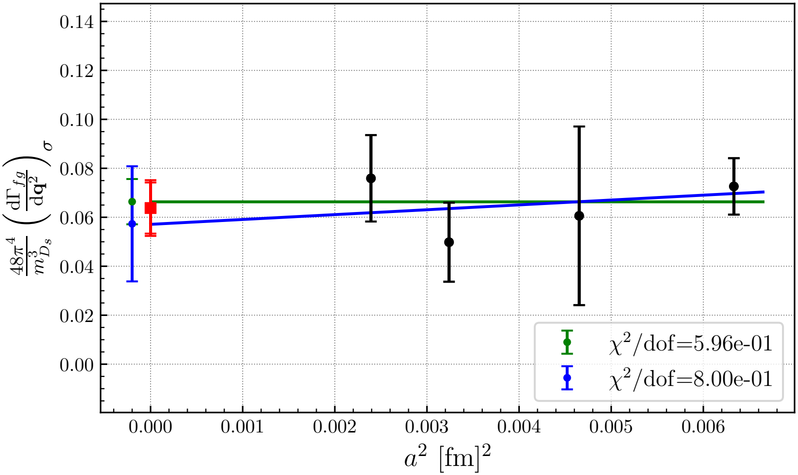

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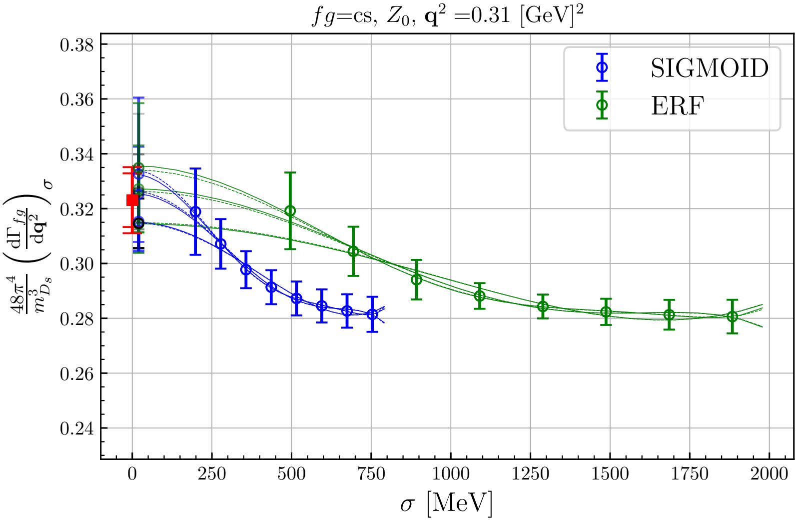

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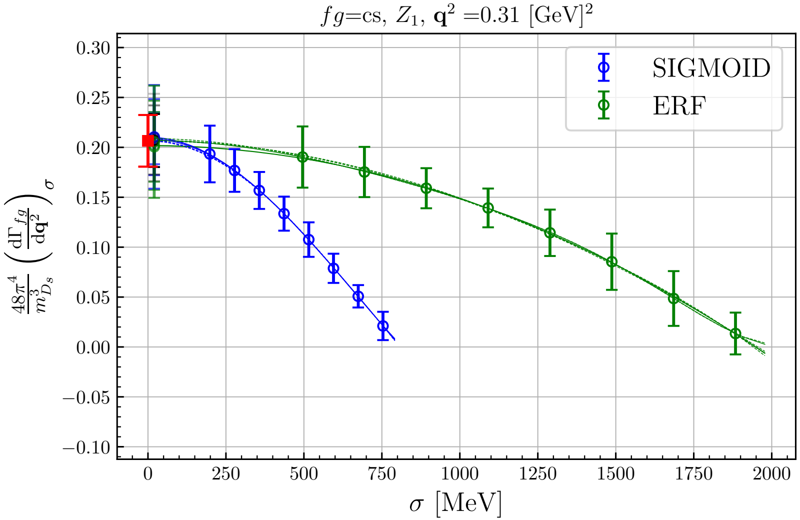

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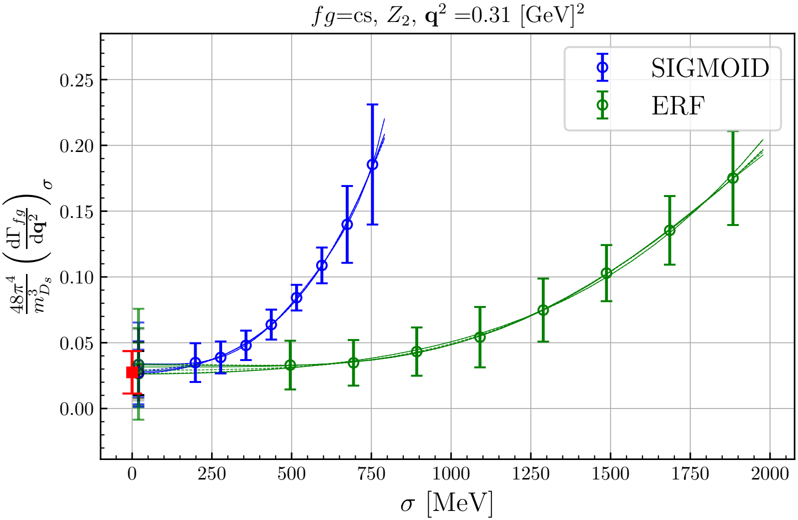
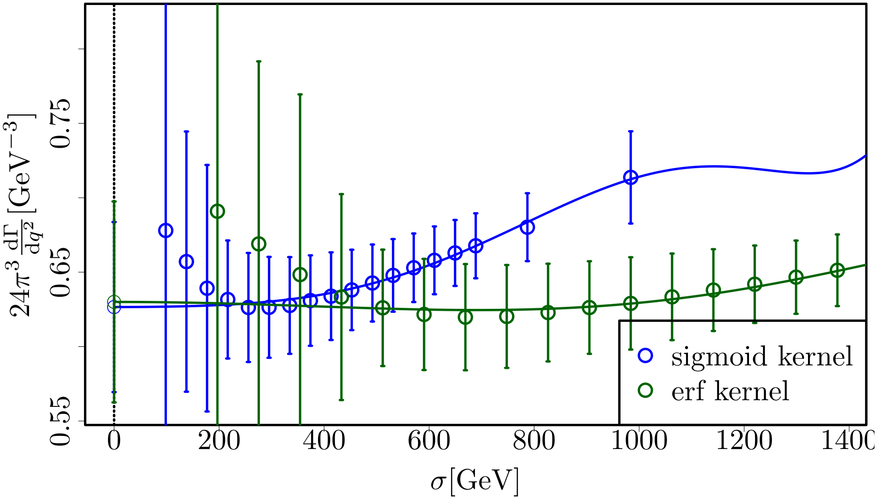
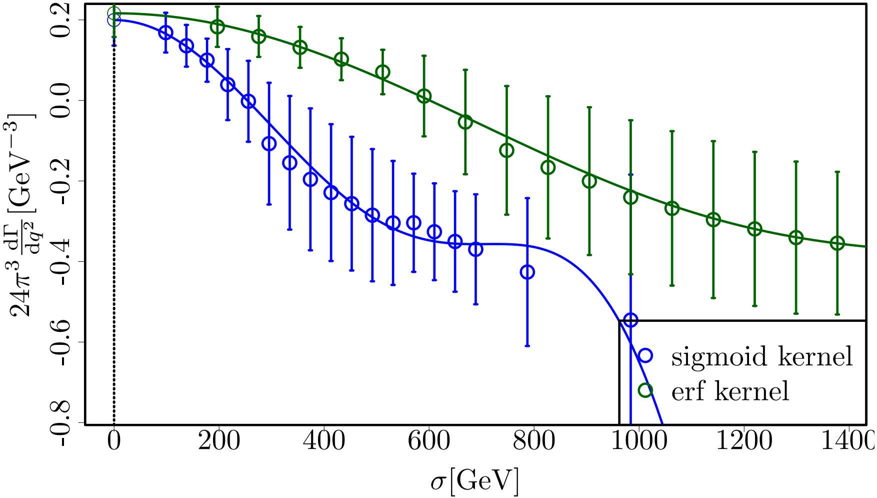


Order 2: **Smearing Limit**; Continuum Limit



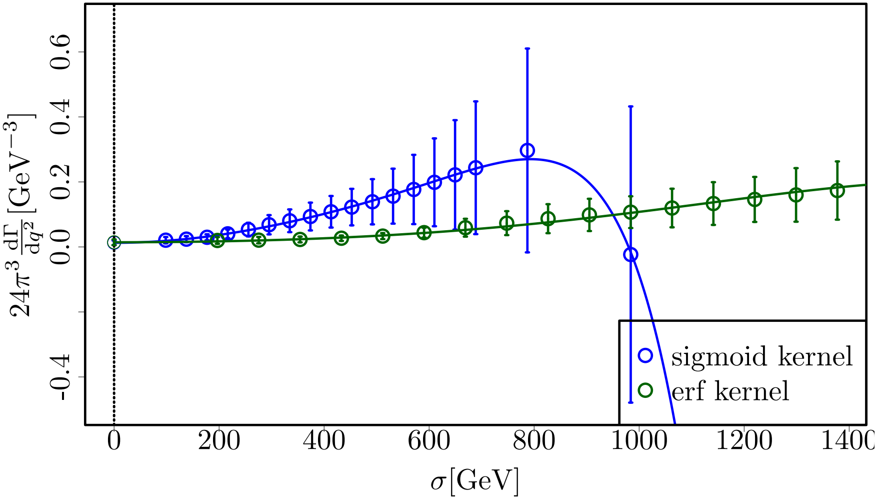
B64 total error Z_0 $q^2 = 0.56\mathrm{GeV}^2$

Order 2: **Smearing Limit**; Continuum Limit



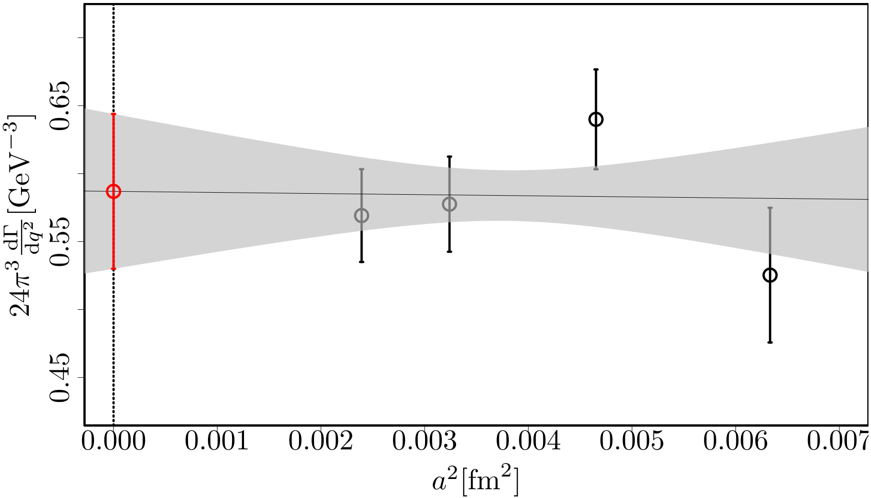
B64 total error Z_1 $q^2 = 0.56\text{GeV}^2$

Order 2: **Smearing Limit**; Continuum Limit



B64 total error Z_2 $q^2 = 0.56\text{GeV}^2$

Order 2: Smearing Limit; **Continuum Limit**



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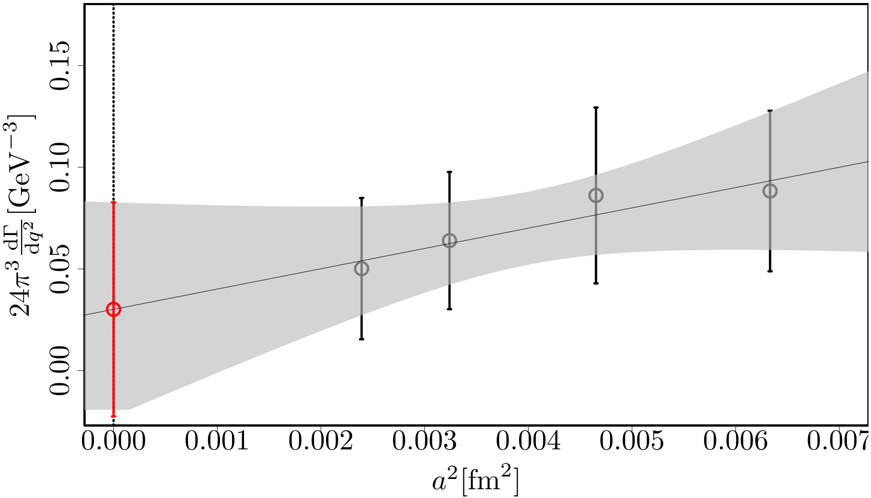

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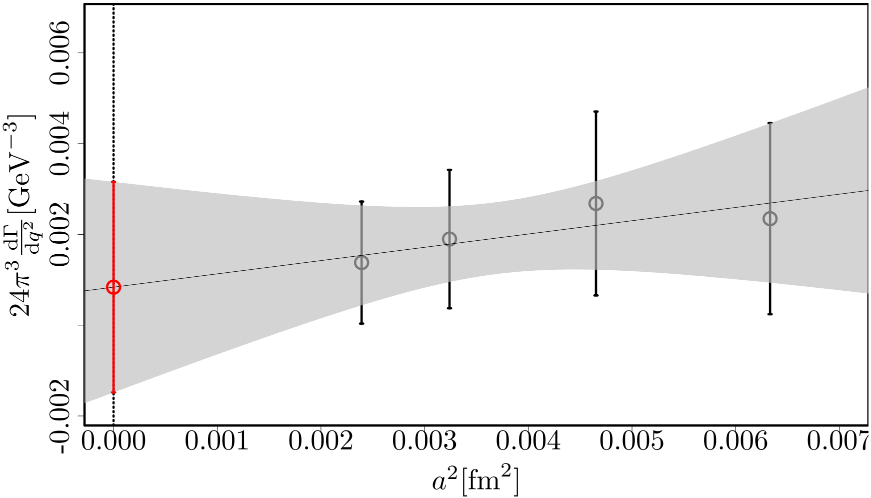

total error Z_2 $q^2 = 0.56\mathrm{GeV}^2$

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[Gam+22] Paolo Gambino et al. 'Lattice QCD study of inclusive semileptonic decays of heavy mesons'. In: *JHEP* 07 (2022), p. 083. DOI: 10.1007/JHEP07(2022)083. arXiv: 2203.11762 [hep-lat].

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July 30, 2024

Alessandro De Santis

Inclusive semi-leptonic $D_s \mapsto X\ell\nu$ decay from lattice QCD

Part 1 : theory and method

Part 2 : results, by Christiane Groß

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Marco Garofalo
Christiane Groß
Bartosz Kostrzewa
Carsten Urbach

University of Swansea
Antonio Smecca

University of Torino
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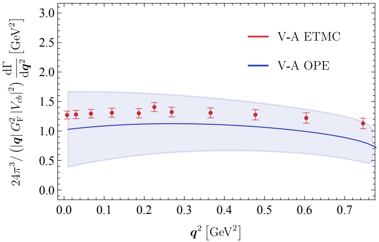
Silvano Simula

Motivations



Exploratory study of the $H \mapsto X\ell\nu$ decay done in P. Gambino et al. (2022), but

- ▷ Unphysical ensemble
- ▷ $L \mapsto \infty$ and $a \mapsto 0$ limits missing
- ▷ Comparison only with OPE

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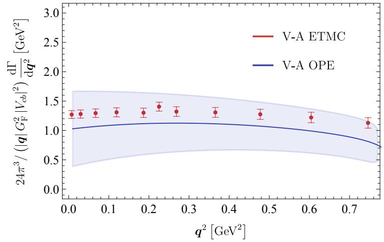

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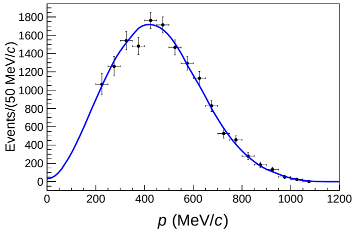
$\Gamma_{\text{semi-lep.}} = 8.27(21) \times 10^{-14}$ GeV (2.5%) BES-III

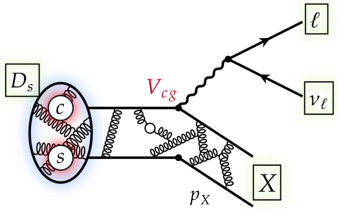


The experimental precision of $D_s \mapsto X\ell\nu$ data, achievable from the lattice, offers the opportunity to do a **complete phenomenologically relevant calculation** and at the same time to **validate the method**

Theoretical background

Inclusive semi-leptonic $D_s \mapsto X\ell\nu$ decay

[P. Gambino and S. Hashimoto (2020), S. Hashimoto (2017)]



- Incoming D_s meson at rest, $p^2 = m_{D_s}^2$
- Outgoing X hadron, $p_X = (q_0, \mathbf{q})$
- $\hat{J}^{\mu}_{gf}(x) = i\bar{g}(x)\gamma^{\mu}(\mathbb{1} - \gamma_5)f(x)$

$$\Gamma = G_F^2\left(|V_{cd}|^2\Gamma_{cd} + |V_{cs}|^2\Gamma_{cs} + |V_{us}|^2\Gamma_{su}\right)$$

Each contribution is given by

$$\Gamma_{fg} = \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2E_\nu} \frac{\mathrm{d}^3 p_\ell}{(2\pi)^3 2E_\ell} L_{\mu\nu}(p_\ell, p_\nu) \boldsymbol{H_{fg}^{\mu\nu}(p, p_X)}$$

with $L_{\mu\nu}$ standard leptonic tensor and the **fully non-perturbative hadronic tensor**

$$\boldsymbol{H_{fg}^{\mu\nu}(p, p_X)} = \frac{(2\pi)^4}{2m_{D_s}} \langle D_s | \hat{J}^\mu_{fg}(0) \delta^4(\mathbb{P} - p_X) \hat{J}^{\nu\dagger}_{fg}(0) | D_s \rangle$$

After a lengthy (but straightforward) derivation ...

$$24\pi^3 \frac{\mathrm{d}\Gamma_{fg}}{\mathrm{d}\mathbf{q}^2} = \sum_{n=0}^{2} |\mathbf{q}|^{3-n} \int_{q_0^{\min}}^{q_0^{\max}} \mathrm{d}q_0 (q_0^{\max} - q_0)^n Z_n,$$

$Z_n =$ linear combinations of $H_{fg}^{\mu\nu}(q_0, \mathbf{q}^2)$

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To treat numerically the integral we introduce a regularized version of the θ-function

$$24\pi^3 \frac{d\Gamma_{fg}}{d\mathbf{q}^2} = \lim_{\sigma \mapsto 0} \sum_{n=0}^{2} |\mathbf{q}|^{3-n} \int_{q_0^{\min}}^{\infty} dq_0 (q_0^{\max} - q_0)^n \theta_\sigma(q_0^{\max} - q_0) Z_n$$

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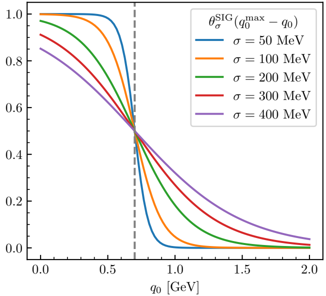
$$\lim_{\sigma \mapsto 0} \theta_\sigma(x) = \theta(x)$$

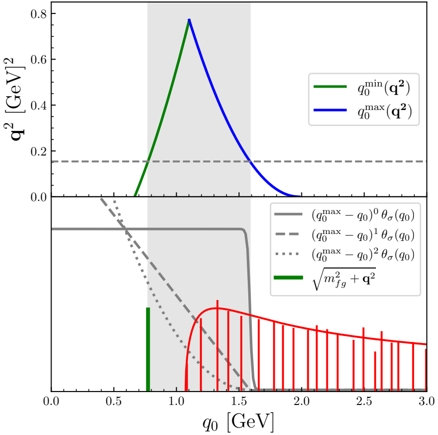


$$\theta_\sigma^{\mathrm{SIG}}(x) = \frac{1}{1 + e^{-x/\sigma}}$$

$$\theta_\sigma^{\mathrm{ERF}}(x) = \frac{1 + \mathrm{erf}(x/\sigma)}{2}$$

The final hadron phase-space

$$q_0 \in \left[\sqrt{m_{fg}^2 + \mathbf{q}^2}, m_{D_s} - |\mathbf{q}|\right] \qquad m_{fg}^2 \text{ lightest mass in the spectrum}$$



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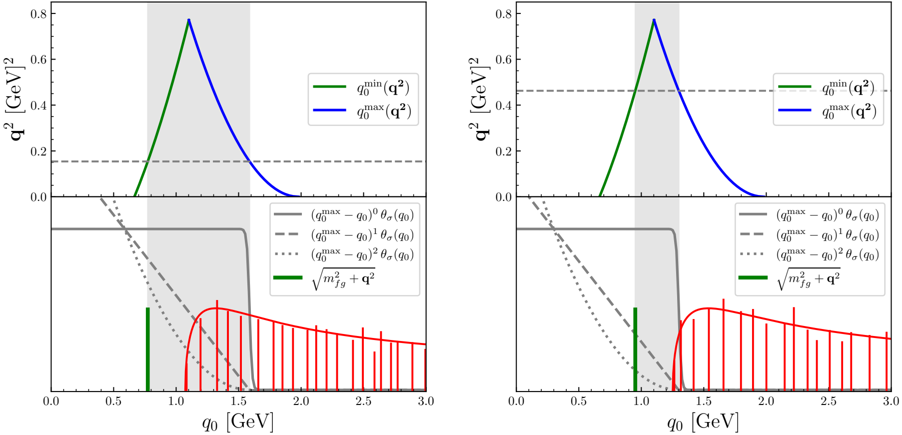

Asymptotic expansion for small σ: fit Ansatz for $\sigma \mapsto 0$ extrapolation

[A. Evangelista et. al (2023)]

$$\begin{aligned}\Delta\rho_\sigma &= \int_0^\infty \mathrm{d}q_0 x^{n} \left[\theta_\sigma(x) - \theta(x)\right] \rho(q_0) \\ x &= q_0^{\max} - q_0\end{aligned}$$

- If $\rho(q_0)$ is **regular** at $q_0^{\max}$

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  - $n = 0, 1$ $\Delta\rho_\sigma = \mathcal{O}(\sigma^2)+$ even powers $(Z_{0,1})$
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- If $\rho(q_0) = Z \cdot \boldsymbol{\delta(q_0 - q_0^{\max})} + \cdots$
  - $n = 0$ $\Delta\rho_\sigma = \frac{1}{2} Z$!?
  - $n > 0$ $\Delta\rho_\sigma = 0$

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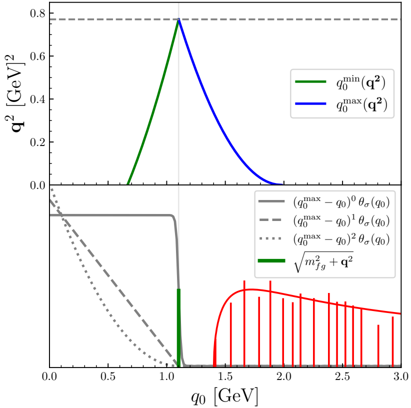

! Decay rate is not vanishing at $\mathbf{q}^2_{\max}$

? Experimental prescription may differ

Γ_{fg} from lattice QCD

We need the hadronic tensor which is the **spectral density** of the correlation function

$$M_{fg}^{\mu\nu}(t, \mathbf{q}^2) = \int_0^\infty \mathrm{d}q_0 \, H_{fg}^{\mu\nu}(q_0, \mathbf{q}^2) e^{-q_0 t}$$

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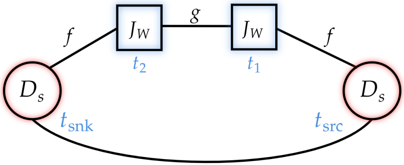
that in practice is obtained by

$$M^{\mu\nu}_{fg}(t_2 - t_1, \mathbf{q}^2) = \lim_{\substack{t_{\mathrm{snk}} \mapsto +\infty \\ t_{\mathrm{src}} \mapsto -\infty}} \frac{C^{\mu\nu}_{4\mathrm{pt}}(t_{\mathrm{snk}}, t_2, t_1, t_{\mathrm{src}}; \mathbf{q})}{C_{2\mathrm{pt}}(t_{\mathrm{snk}} - t_2) C_{2\mathrm{pt}}(t_1 - t_{\mathrm{src}})}$$



- ▷ $t = t_2 - t_1 = a, 2a, \cdots$ **Euclidean time**
- ▷ $t_2 - t_{\mathrm{snk}},\ t_{\mathrm{src}} - t_1 \gg 0$ checked

Going from

$$M^{\mu\nu}_{fg}(t,\mathbf{q}) = \int_0^\infty \mathrm{d}q_0 H^{\mu\nu}_{fg}(q_0,\mathbf{q}^2) e^{-q_0 t}$$

to

$$\int_{q_0^{\min}}^\infty \mathrm{d}q_0 (q_0^{\max} - q_0)^n \theta_\sigma(q_0^{\max} - q_0) H^{\mu\nu}_{fg}(q_0,\mathbf{q}^2)$$

implies solving a **numerically ill-conditioned** (but mathematically well-posed) **inverse Laplace transform**

- $t = a, 2a, 3a, \cdots < \infty$, scarce information
- signal-to-noise ratio of $M^{\mu\nu}_{fg}(t,\mathbf{q})$ deteriorates exponentially

One way out: HLT (R. Kellermann's talk for another approach)

One way out: HLT

(R. Kellermann's talk for another approach)

Extraction of spectral densities from lattice correlators

Martin Hansen,[1] Alessandro Lupo,[2] and Nazario Tantalo[3]

[1]*INFN Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Rome, Italy*

[2]*University of Rome Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Rome, Italy*

[3]*University of Rome Tor Vergata and INFN Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Rome, Italy*

Many applications by now

R-ratio
Phys.Rev.Lett. 130 (2023) 24, 241901
F. Margari's poster, D. Stewart's talk

Hadronic τ decay
A. Evangelista et al. (2023), Phys.Rev.Lett. 132 (2024)
G. Gagliardi's talk

Heavy $H \mapsto X\ell\nu$ inclusive decay
P. Gambino et al. (2022)

Spectroscopy at non-zero temperature
A. Smecca's talk Meson spectroscopy

Ed. Bennet et al. (2024)
N. Forzano's talk

Exclusive scattering amplitudes from lattice QCD
A. Patella & N. Tantalo (2024)
A. Patella's talk

Many others!

In general we want to extract $\rho_\sigma = \int \mathrm{d}\omega\, K_\sigma(\omega)\rho(\omega)$ from $C(t) = \int_0^\infty \mathrm{d}\omega\, e^{-\omega t}\rho(\omega)$

- A linear estimator for the solution can be written by **approximating the target smearing (Schwartz) kernel**

$$\rho_\sigma = \sum_{\tau=1}^{T} g_\tau C(a\tau)$$

$$K^{\text{approx}}_{\sigma,T} = \sum_{\tau=1}^{T} g_\tau(T) e^{-a\omega\tau}$$

- The estimator is **model independent and unbiased** in the limits $T \mapsto \infty$ and vanishing statistical errors

$$\lim_{T\mapsto\infty} K^{\text{approx}}_{\sigma,T} = K_\sigma$$

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$$\lim_{T\mapsto\infty} K_{\sigma,T}^{\text{approx}} = K_\sigma$$

For $T < \infty$ one needs to estimate the residual **systematic** uncertainty due to the kernel approximation in addition to **statistical** error

- The coefficients $\boldsymbol{g}$ are calculated by minimizing

$$W[\lambda, \boldsymbol{g}] = (1-\lambda)\frac{A[\boldsymbol{g}]}{A[\mathbf{0}]} + \lambda B[\boldsymbol{g}]$$

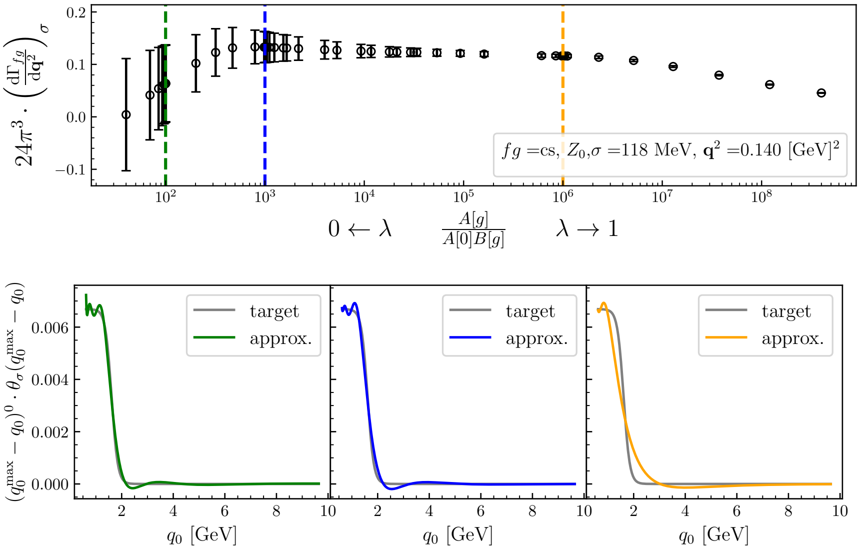
- Suppression of the **statistical error**

$$B[\boldsymbol{g}] = \boldsymbol{g}^T \cdot \hat{\mathsf{COV}}\left[C(t)\right] \cdot \boldsymbol{g} \equiv \left(\delta\rho\right)^2$$

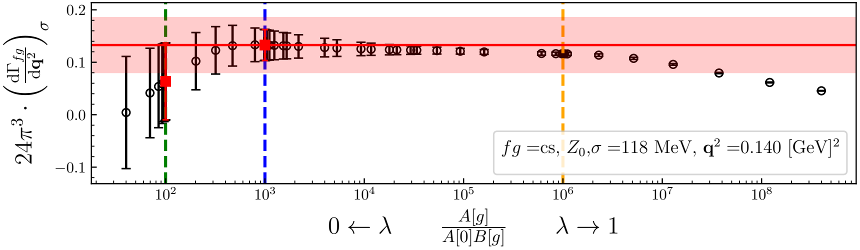
- **Accuracy of the approximated** kernel

$$A[\boldsymbol{g}] = \int_{E_0}^{\infty} \mathrm{d}\omega \left\{ \sum_{\tau=1}^{T} g_\tau e^{-a\omega\tau} - K_\sigma^{\mathrm{target}} \right\}^2 \qquad E_0 \sim 0.9 \cdot q_0^{\mathrm{min}}$$

Stability analysis to tune λ



$24\pi^3 \cdot \left(\frac{d\Gamma_{fg}}{d\mathbf{q}^2}\right)_\sigma$
fg =cs, Z_0,σ =118 MeV, $\mathbf{q}^2$ =0.140 [GeV]^2
$0 \leftarrow \lambda$
$\frac{A[g]}{A[0]B[g]}$
$\lambda \rightarrow 1$
$(q_0^{\max} - q_0)^0 \cdot \theta_\sigma(q_0^{\max} - q_0)$
target
approx.
q_0 [GeV]



$\rho_\star$: $\dfrac{A[\boldsymbol{g}]}{A[\mathbf{0}]B[\boldsymbol{g}]} = 10^3$ plateaux

$\rho_{\star\star}$: $\dfrac{A[\boldsymbol{g}]}{A[\mathbf{0}]B[\boldsymbol{g}]} = 10^2$ systematic

pull variable to assess systematic over statical error

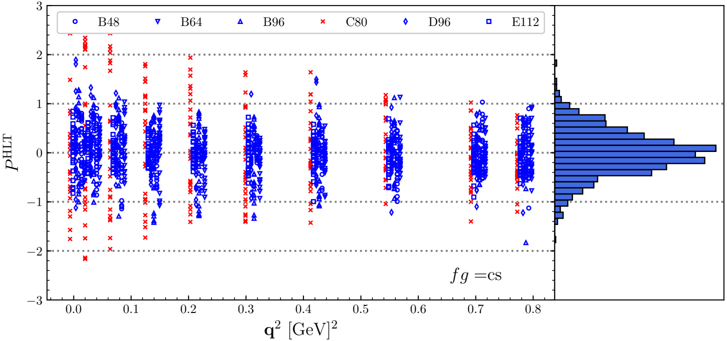
$$P^{\mathrm{HLT}} = \frac{\rho_\star - \rho_{\star\star}}{\sqrt{\delta\rho_\star{}^2 + \delta\rho_{\star\star}{}^2}}$$

$$\Delta^{\mathrm{syst}} = |\rho_\star - \rho_{\star\star}| \,\mathrm{erf}\left(\frac{P^{\mathrm{HLT}}}{\sqrt{2}}\right)$$

$\star$(Z_1 and Z_2 in backup)

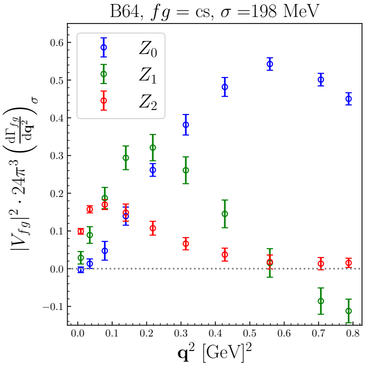
$\mathcal{O}(3000)$ stability analysis in one plot

Distribution of the pull variable P^{HLT} across all the stability analysis

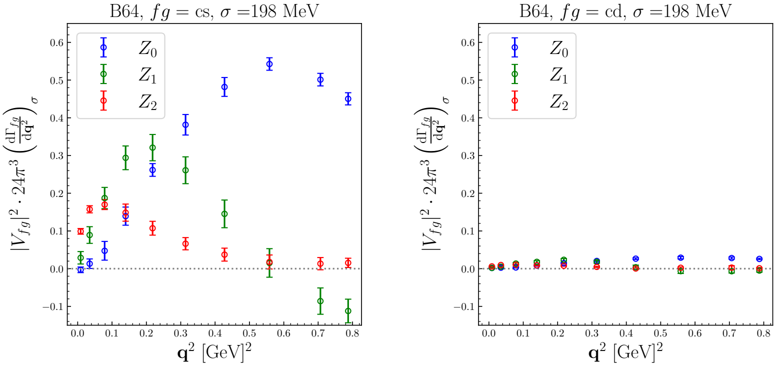


C80 requires more statistics

Results ad fixed ensemble and σ



Results ad fixed ensemble and σ



Γ_{cd} is Cabibbo suppressed

Conclusions

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Netx-to-do list

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- Continuum Limit
- $\sigma \mapsto 0$ extrapolation
- Integration over $\mathbf{q}^2$
- Comparison with experiments

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Christiane Groß's talk right after me

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Christiane Groß's talk right after me

Thank you for the attention and don't run away!!!

Backup

Definition of Z_n

$$Z_0 \equiv Y_2 + Y_3 - 2Y_4 \qquad Z_1 \equiv 2(Y_3 - 2Y_1 - Y_4) \qquad Z_2 \equiv Y_3 - 2Y_1$$

Form factors decomposition of the hadronic tensor

$$\begin{aligned} m_{D_s}^3 H^{\mu\nu}(p,p_x) = & \, g^{\mu\nu} m_{D_s}^2 h_1 + p^\mu p^\nu h_2 + (p-p_X)^\mu (p-p_X)^\nu h_3 \\ & + [p^\mu (p-p_X)^\nu + (p-p_X)^\mu p^\nu] h_4 - i\varepsilon^{\mu\nu\alpha\beta} p_\alpha (p-p_X)_\beta h_5 \end{aligned}$$

$$Y_1 = -m_{D_s} \sum_{ij} \hat{n}^i \hat{n}^j H^{ij} = h_1$$

$$Y_2 = m_{D_s} H^{00} = h_1 + h_2 + \left(1 - \frac{q_0}{m_{D_s}}\right)^2 h_3 + 2\left(1 - \frac{q_0}{m_{D_s}}\right) h_4$$

$$Y_3 = m_{D_s} \sum_{ij} \hat{q}^i \hat{q}^j H^{ij} = -h_1 m_{D_s}^2 + |\mathbf{q}|^2 h_3$$

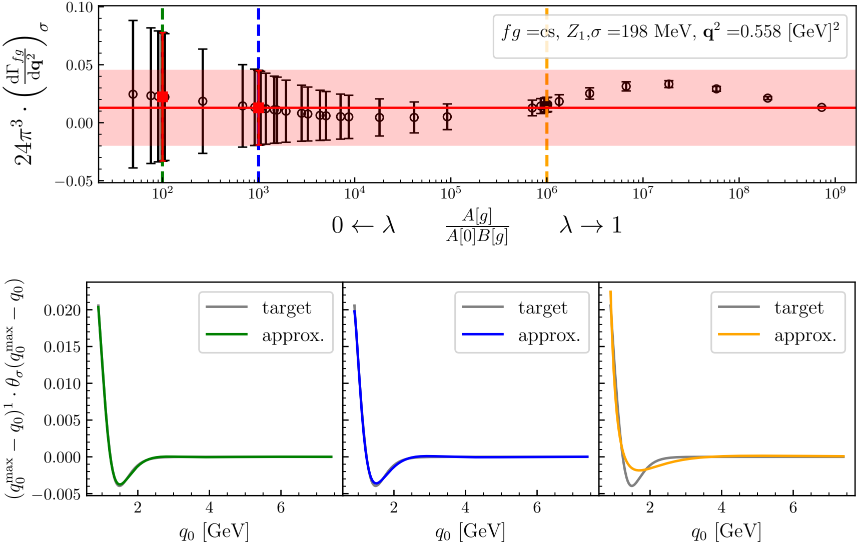
$$Y_4 = -m_{D_s} \sum_{i} \hat{q}^i H^{0i} = \left(1 - \frac{q_0}{m_{D_s}}\right) |\mathbf{q}| h_3 + |\mathbf{q}| h_4$$

$$Y_5 = \frac{i m_{D_s}}{2} \sum_{ijk} \varepsilon^{ijk} \hat{q}^k H^{ij} = |\mathbf{q}| h_5$$

$$\hat{n}^2 = 1$$

$$\hat{n} \cdot \mathbf{q} = 0$$

$$\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$$



fg =cs, Z_1,σ =198 MeV, $\mathbf{q}^2$ =0.558 [GeV]^2
$24\pi^3 \cdot \left(\frac{d\Gamma_{fg}}{d\mathbf{q}^2}\right)_\sigma$
$0 \leftarrow \lambda$
$\frac{A[g]}{A[0]B[g]}$
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$(q_0^{\max} - q_0)^1 \cdot \theta_\sigma(q_0^{\max} - q_0)$
target
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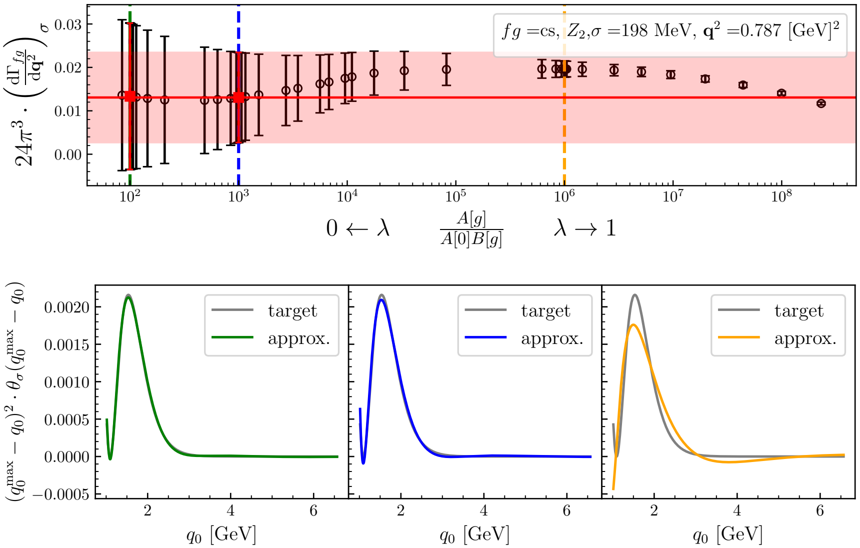

fg =cs, Z_2,σ =198 MeV, $\mathbf{q}^2$ =0.787 [GeV]^2
$24\pi^3 \cdot \left(\frac{d\Gamma_{fg}}{d\mathbf{q}^2}\right)_\sigma$
$0 \leftarrow \lambda$
$\frac{A[g]}{A[0]B[g]}$
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$(q_0^{\max} - q_0)^2 \cdot \theta_\sigma(q_0^{\max} - q_0)$
target
approx.
target
approx.
target
approx.
q_0 [GeV]
q_0 [GeV]
q_0 [GeV]

Production line

This is repeated for:

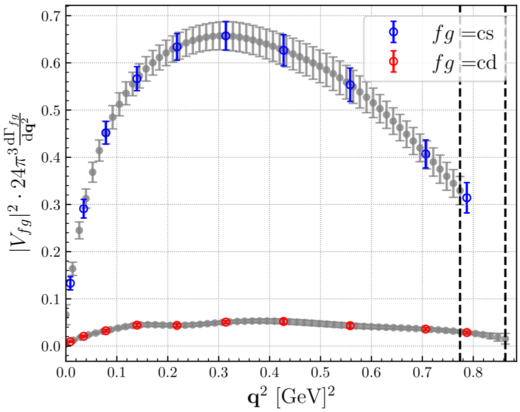
- 2 channels, third coming
- Z_0, Z_1 and Z_2
- 2 smearing kernels
- $\mathcal{O}(10)$ vales of σ
- 10 values of $\mathbf{q}^2$
- for each ensemble

ETMC ensembles all close to physical point

ID	$L^3 \times T$	a [fm]	L [fm]
B48	$48^3 \times 96$	0.07951	3.82
B64	$64^3 \times 128$	0.07951	5.09
B96	$96^3 \times 192$	0.07951	7.63
C80	$80^3 \times 160$	0.06816	5.45
D96	$96^3 \times 92$	0.05688	5.46
E112	$112^3 \times 224$	0.04891	5.47

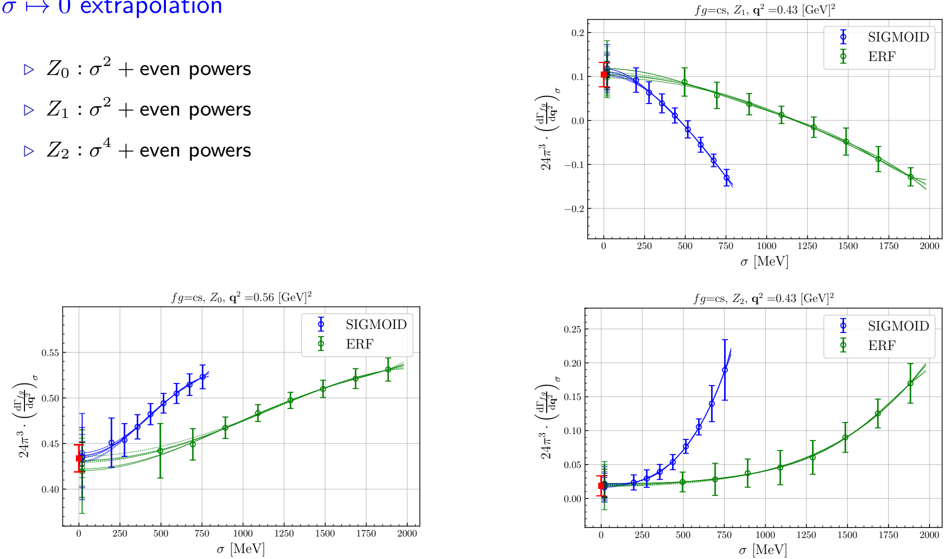
Final results

Spline interpolation + trapezoid integration

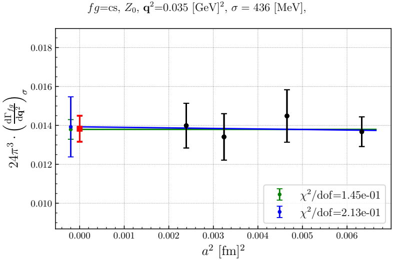


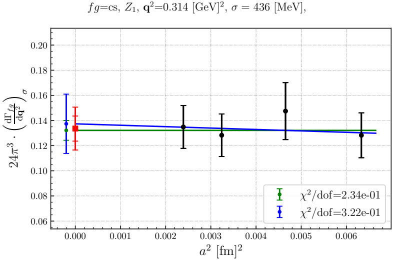
$\sigma \mapsto 0$ extrapolation

- $Z_0 : \sigma^2 +$ even powers
- $Z_1 : \sigma^2 +$ even powers
- $Z_2 : \sigma^4 +$ even powers

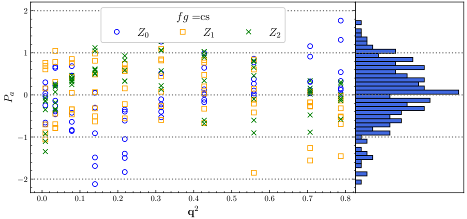


Continuum extrapolation





Pull of significance between finest lattice spacing and extrapolated point



Lepton moments

Everything presented in this talk applies straightforwardly to the Lepton moments

$$m_{D_s}^{1+n} \frac{\mathrm{d}M_{fg}^{n}}{\mathrm{d}\mathbf{q}^2} = \int \mathrm{d}q_0 \int \mathrm{d}E_\ell E_\ell^n \frac{\mathrm{d}\Gamma_{fg}}{\mathrm{d}q_0 \mathrm{d}\mathbf{q}^2 \mathrm{d}E_\ell}$$

The first lepton moment reads

$$96\pi^4 m_{D_s} \frac{\mathrm{d}M_{fg}^{(1)}}{\mathrm{d}\mathbf{q}^2} = \lim_{\sigma \mapsto 0} \sum_{n=0}^{3} \mathbf{q}^{4-n} \int_0^\infty \mathrm{d}q_0 (q_0^{\max} - q_0)^n \theta_\sigma (q_0^{\max} - q_0) Z_n^{(1)}$$

with

$$Z_0^{(1)} = Y_2 + Y_3 - 2Y_4$$

$$Z_1^{(1)} = -4Y_1 + Y_2 + 3Y_3 - 4Y_4 + 2Y_5$$

$$Z_2^{(1)} = -6Y_1 + 3Y_3 - 2Y_4 + Y_5$$

$$Z_3^{(1)} = -2Y_1 + Y_3$$

Exclusive ground-state contribution to Γ_{fg}

$$\frac{\mathrm{d}\Gamma_{fg}^{\mathrm{ex}}}{\mathrm{d}\mathbf{q}^2} = \frac{1}{24\pi^3}\frac{m_{D_s}}{q_0}|\mathrm{q}|^3 f_+^2(\mathbf{q}^2)$$

$f_+^2(\mathbf{q}^2)$ can be computed by fitting the leading exponential contribution to the correlation functions