Towards more accurate $D_{(s)} \to \pi(K)$ and $B_{(s)} \to \pi(K)$ Form Factors

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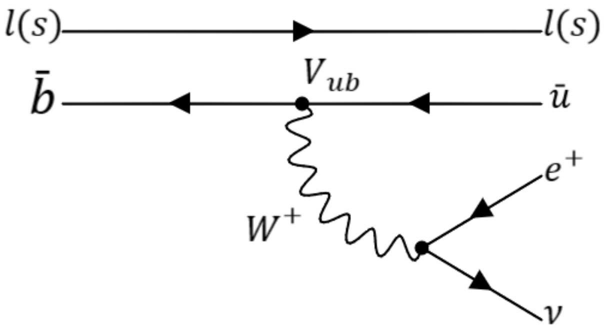
Form Factor Motivation

CKM matrix elements calculated from kinematic form factors.

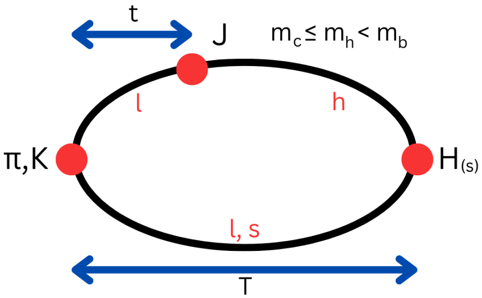
$$\frac{d\Gamma}{dq^2} \propto |V_{CKM}|^2 \times |f(q^2)|^2,$$

$|V_{cd}| = 0.221 \pm 0.004, |V_{ub}| = (3.82 \pm 0.20) \times 10^{-3}$[PDG, 2024].



Heavy HISQ

- Generic heavy-quark simulated at various masses.
- Allows extrapolation to physical b-quark mass.



- $N_f = 2 + 1 + 1$ MILC-HISQ gluon fields [1004.0342], [1212.4768].
- Fully relativistic, nearly full kinematic range.

Gluon Field Ensembles

Set	$\approx a$(fm)	m_s/m_l	$N_x^3 \times N_t$	am_h range	$\lvert\vec{p}^{\,\pi,K}_{\max}\rvert$ [MeV/c^2]	T range
f-5	0.09	5	$32^3 \times 96$	0.450-0.8	311	15-24
f-phys	0.09	27	$64^3 \times 96$	0.433-0.8	330	15-24
sf-5	0.06	5	$48^3 \times 144$	0.274-0.8	622	22-31
sf-phys	0.06	27	$96^3 \times 192$	0.2585-0.8	648	22-31
uf-5	0.04	5	$64^3 \times 192$	0.194-0.8	583	29-44

- Per ensemble we set smallest $am_h \approx am_c^{\text{phys}}$.
- Coming soon:
  - coarser ensembles, $a \approx 0.12$fm, 0.15fm.
  - $am_h = am_b^{\text{phys}}$ on select finer ensembles.
- Max q^2 on uf5 $\approx 20\text{GeV}^2$.

Correlator Fit Equations

Two-point correlator fit equation (e.g. π):

$$C_2^{\pi}(t) = \sum_{i=0}^{N_{\text{exp}}-1} \Big[|A_i^{\pi,n}|^2 (e^{-E_i^{\pi,n} t} + e^{-E_i^{\pi,n}(N_t - t)}) - (-1)^t |A_i^{\pi,o}|^2 (e^{-E_i^{\pi,o} t} + e^{-E_i^{\pi,o}(N_t - t)} \Big].$$

Three-point correlator fit equation (e.g. $H \to \pi$):

$$C_3^{\pi,H}(t,T) = \sum_{i,j=0}^{N_{\text{exp}}-1} \Big[A_i^{\pi,n} J_{ij}^{nn} A_j^{H,n} e^{-E_i^{\pi,n} t} e^{-E_j^{H,n}(T-t)} - (-1)^{(T-t)} A_i^{\pi,n} J_{ij}^{no} A_j^{H,o} e^{-E_i^{\pi,n} t} e^{-E_j^{H,o}(T-t)} - (-1)^t A_i^{\pi,o} J_{ij}^{on} A_j^{H,n} e^{-E_i^{\pi,o} t} e^{-E_j^{H,n}(T-t)} + (-1)^T A_i^{\pi,o} J_{ij}^{oo} A_j^{H,o} e^{-E_i^{\pi,o} t} e^{-E_j^{H,o}(T-t)} \Big].$$

Form Factor Equations

Matrix element for ground-state lattice current J_{00}^{nn} (e.g. $H \to \pi$):

$$\langle \pi | J_{\text{latt}} | H \rangle = 2 Z_{\text{disc}} \sqrt{M_H E_\pi} \times J_{00}^{nn}.$$

Form factor relations to lattice matrix elements:

Scalar: $$\langle \pi | S_{\text{latt}} | H \rangle = f_0(q^2) \times \frac{M_H^2 - M_\pi^2}{m_h - m_u},$$

Vector: $$Z_V \langle \pi | V_{\text{latt}}^{\mu} | \hat{H} \rangle = f_+(q^2) \left(p_H^\mu + p_\pi^\mu - \frac{M_H^2 - M_\pi^2}{q^2} q^\mu \right) + f_0(q^2) \frac{M_H^2 - M_\pi^2}{q^2} q^\mu$$

Tensor: $$Z_T(\mu) \langle \hat{\pi} | T_{\text{latt}}^{k0} | \hat{H} \rangle = f_T(q^2, \mu) \times \frac{2 i M_H p_\pi^k}{M_H^2 + M_\pi^2}.$$

Note: $\hat{H}$ and $\hat{\pi}$ denote local non-Goldstone pseudoscalars. Z terms calculated in [1211.6966], [1008.4562], [1305.1462], [2008.02024]

Fitting Obstacles

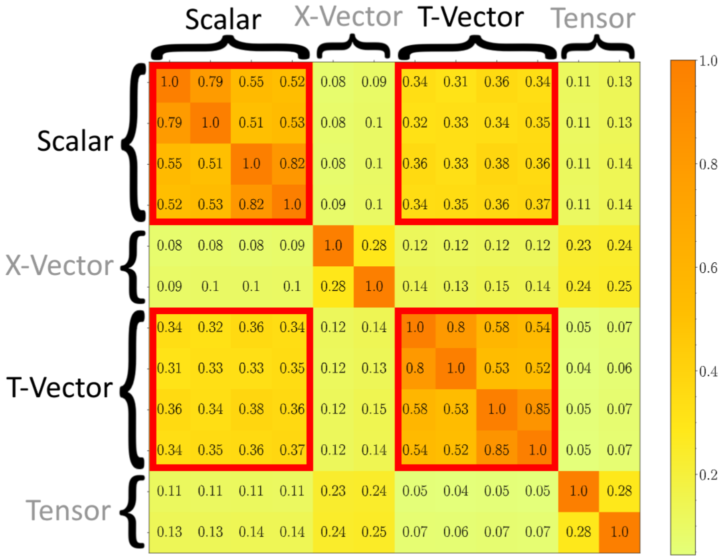
Fitting requires inverting correlation matrix. For $H \to \pi$, per ensemble we fit over:

- $4 \times am_h$,
- $5 \times \theta$, where $\theta = |a\overrightarrow{p}_{\pi,K}| \times \frac{N_x}{\sqrt{3}\pi}$,
- $4 \times T$,
- $4\times$ spin-taste copies for H from using local current operators,
- $4\times$ 3-point current components: (scalar, temporal vector, spacial vector, tensor),

... and the $H_s \to K$ alternative of each of the above.

		Scalar				X-Vector		T-Vector				Tensor	
Scalar		1.0	0.79	0.55	0.52	0.08	0.09	0.34	0.31	0.36	0.34	0.11	0.13
		0.79	1.0	0.51	0.53	0.08	0.1	0.32	0.33	0.34	0.35	0.11	0.13
		0.55	0.51	1.0	0.82	0.08	0.1	0.36	0.33	0.38	0.36	0.11	0.14
		0.52	0.53	0.82	1.0	0.09	0.1	0.34	0.35	0.36	0.37	0.11	0.14
X-Vector		0.08	0.08	0.08	0.09	1.0	0.28	0.12	0.12	0.12	0.12	0.23	0.24
		0.09	0.1	0.1	0.1	0.28	1.0	0.14	0.13	0.15	0.14	0.24	0.25
T-Vector		0.34	0.32	0.36	0.34	0.12	0.14	1.0	0.8	0.58	0.54	0.05	0.07
		0.31	0.33	0.33	0.35	0.12	0.13	0.8	1.0	0.53	0.52	0.04	0.06
		0.36	0.34	0.38	0.36	0.12	0.15	0.58	0.53	1.0	0.85	0.05	0.07
		0.34	0.35	0.36	0.37	0.12	0.14	0.54	0.52	0.85	1.0	0.05	0.07
Tensor		0.11	0.11	0.11	0.11	0.23	0.24	0.05	0.04	0.05	0.05	1.0	0.28
		0.13	0.13	0.14	0.14	0.24	0.25	0.07	0.06	0.07	0.07	0.28	1.0

H to pi, f-5 ensemble, Sample Fit Correlation Matrix



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	0.31	0.33	0.33	0.35	0.12	0.13	0.8	1.0	0.53	0.52	0.04	0.06
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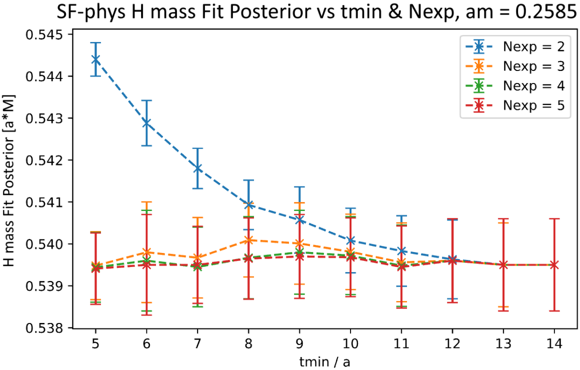
Tensor	0.11	0.11	0.11	0.11	0.23	0.24	0.05	0.04	0.05	0.05	1.0	0.28
	0.13	0.13	0.14	0.14	0.24	0.25	0.07	0.06	0.07	0.07	0.28	1.0

H to pi, f-5 ensemble, Sample Fit Correlation Matrix

N_{exp} and t_{min} Testing

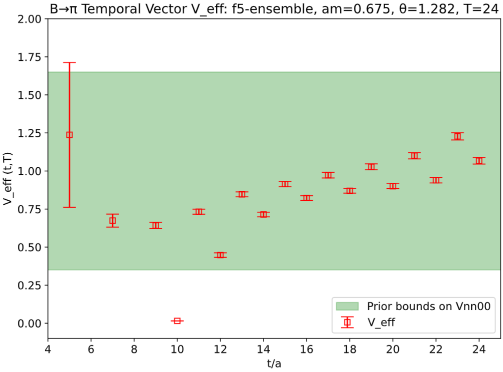


- Uncertainty decreases at smaller t_{min}/a.
- Posterior central value saturates at higher N_{exp}.
- Here I choose $t_{min}/a = 7$.
- Empirical Bayes testing favors $N_{exp} = 4$ across all ensembles.

Priors and Bayesian Fitting

$P_i \pm \sigma_i =$ Fit Parameter,
$\tilde{P}_i \pm \tilde{\sigma}_i =$ Prior on Parameter,

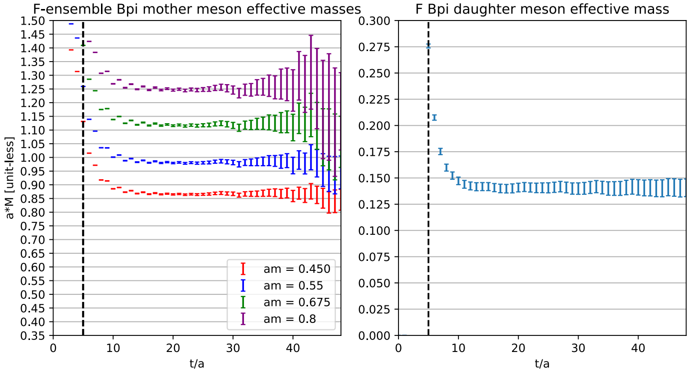
$\chi^2 \rightarrow \chi^2_{\text{aug}} = \chi^2 + \chi^2_{\text{prior}}$,
$\chi^2_{\text{prior}} = \sum_i \left(\frac{P_i - \tilde{P}_i}{\tilde{\sigma}_i}\right)$.



Priors $\tilde{P}_i \pm \tilde{\sigma}_i$ are initially set from M_{eff} and A_{eff} plots, refined through Empirical Bayesian Analysis.

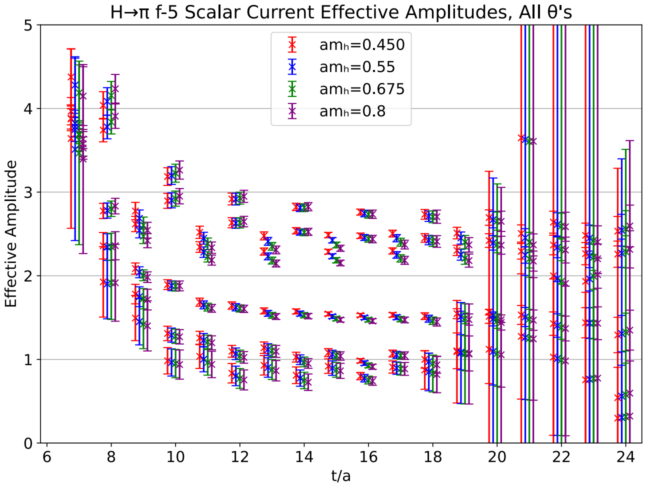
Setting Priors

Some parameters warrant precise priors with narrow widths:

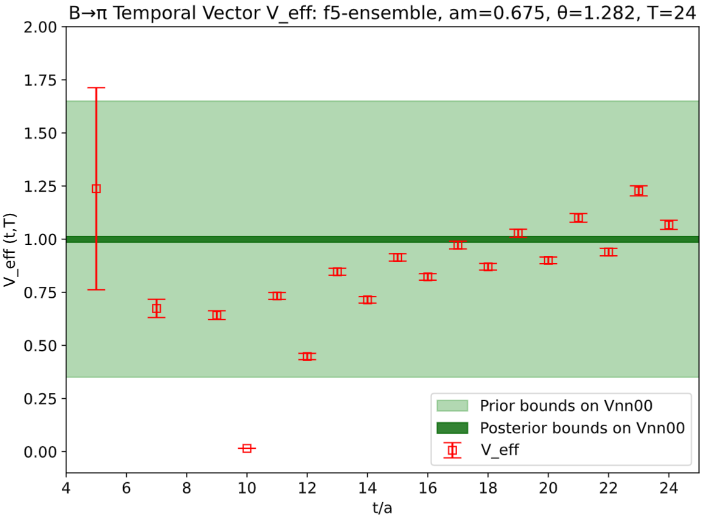


Setting Priors

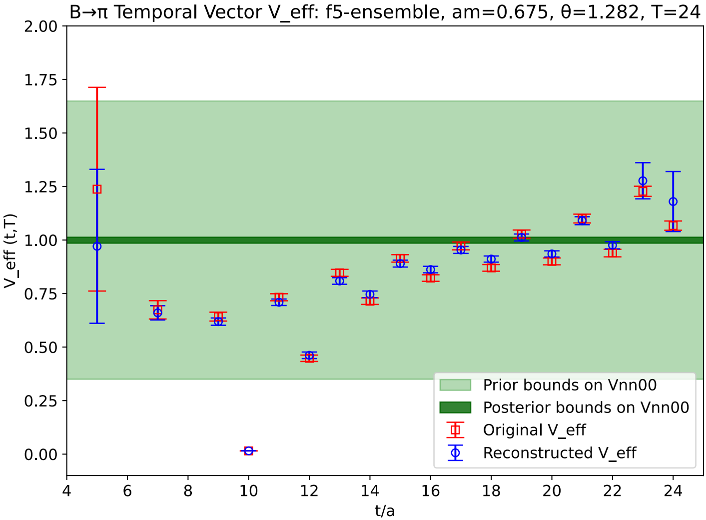
Others warrant conservative priors with broad widths:



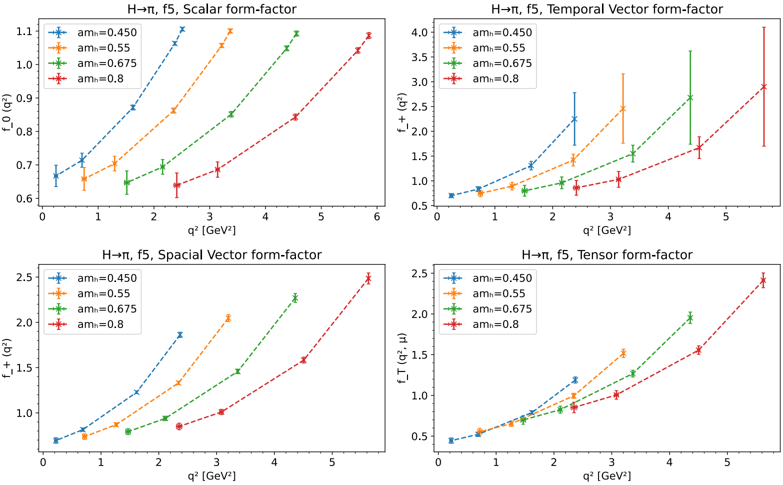
Sample Fit Result - Prior vs. Posterior Bounds



Sample Fit Result - Reconstructed Effective Amplitude

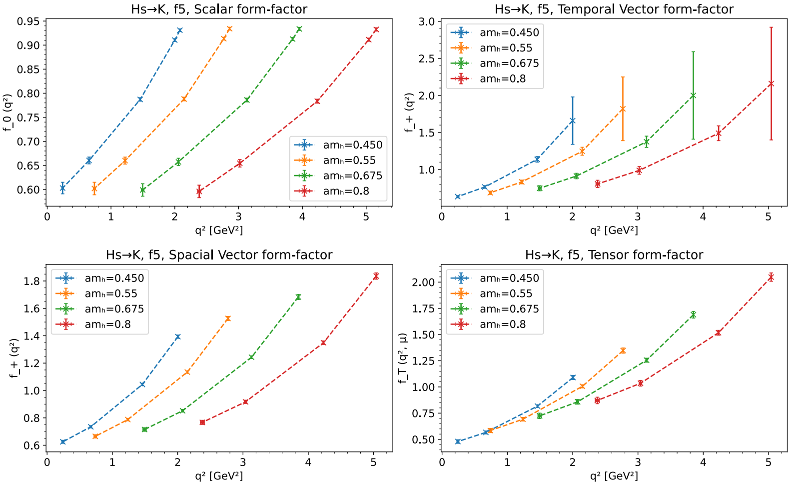


$H \to \pi$ ($a = 0.09$fm, $m_s/m_l = 5$) form factors



Results shown are preliminary. $am_c^{\text{phys}} \equiv am_h^{\text{min}} = 0.450$

$H_s \to K$ ($a = 0.09$fm, $m_s/m_l = 5$) form factors



Results shown are preliminary. $am_c^{\text{phys}} \equiv am_h^{\text{min}} = 0.450$

Recap and Outlook

Key Points

- Kinematic form factors $\rightarrow |V_{ub}|$ and $|V_{cd}|$.
- First use of heavy-HISQ method for $H_{(s)} \rightarrow \pi(K)$ across wide kinematic range.
- Bayesian Statistics integral to fitting method (priors).

Next steps:

- Fit refinement:
  - SVD cut analysis,
  - Empirical Bayes testing of 3pt priors,
  - Stability testing.
- Modified z-expansion:
  - Physical b-quark mass extrapolation,
  - Continuum limit extrapolation.

Back-Up: Renormalization

- Z_{disc}: tree level discretization correction starting at $(am_h)^4$ [Monahan, Shigemitsu, Horgan, 1211.6966].
- Z_V: derived from the partially conserved vector current relation [Na *et al.*, 1008.4562],[Koponen *et al.*, 1305.1462].

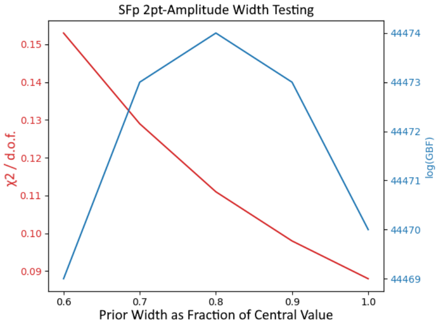
$$\text{For} \quad H \to \pi : \quad Z_V = \left| \frac{(m_h - m_l)\langle \pi | S | H \rangle}{(M_H - M_\pi)\langle \pi | V^0 | H \rangle} \right|_{q^2 = q^2_{\text{max}}}$$

- Z_T: tensor current renormalization at energy scale = 4.8GeV $\approx m_b$ [Hatton *et al.*, 2008.02024].

Back-Up: Empirical Bayes Analysis

Gaussian Bayes Factor (GBF) = probability density of randomly sampling the fit data from fit model (including priors). By construction it punishes over-fitting.

- Optimization: minimize χ^2_{aug}, maximise GBF.
- $\Delta log(\text{GBF}) \geq 3$ is considered significant. Increasing prior width artificially lowers χ^2_{aug}.
- Adding "noise" to priors restores $\chi^2/\text{d.o.f.} \approx 1$.



Back-Up: Sample Correlator Fitting