Relativistic corrections to the quark-anti-quark static potential with gradient flow

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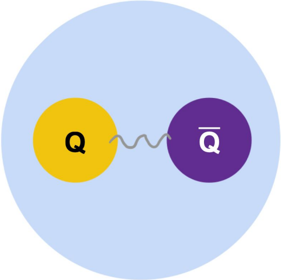
Potential corrections

$$V_{Q\bar{Q}}(r) = V^{(0)}_{Q\bar{Q}}(r) + \frac{1}{m_Q} V^{(1)}_{Q\bar{Q}}(r) + \frac{1}{m_Q^2}\left(V^{(2)}_{Q\bar{Q},\mathrm{SD}}(r) + V^{(2)}_{Q\bar{Q},\mathrm{SI}}(r)\right) + \mathcal{O}(m_Q^{-3})$$

[Eichten, Feinberg, 1981], [Barchielli et al. 1988], [Pineda, Vairo, 2001], [Brambilla, 2022]

- Corrections known up to N^{3}LO in perturbation theory, but computations in lattice QCD face difficulties. [Buchmüller, Ng, Tye, 1981], [Peset, Pineda, Stahlhofen, 2016], [Bali, Schilling, Wachter, 1997], [Koma, Koma, 2007]
- **Difficulties** UV-noise, renormalization and matching can be adressed with gradient flow. [Lüscher, 2010], [Brambilla, Leino, Mayer-Steudte, Vairo, 2023], [Brambilla, Wang, 2023]
- **Analogous** expressions recently derived for spin-dependent hybrid potentials. [Brambilla, Lai, Segovia, Castellà, 2020], [Soto, Castellà, 2021], [Soto, Valls, 2023]



Potentials in terms of generalized Wilson loops

$$V_{p^2} = \frac{1}{2}\left\{\boldsymbol{p}^2, (\mathcal{I}_2(E_z(t,0)E_z(0,0)) + \mathcal{I}_2(E_z(t,r)E_z(0,0)))\right\},$$

$$V_{LS} = \epsilon_{ijz}\frac{c_F(\mu)}{2r}\left(2\mathcal{I}_1(B_i(t,0)E_j(0,0)) + \mathcal{I}_1(B_i(t,r)E_j(0,0))\right)\boldsymbol{LS},$$

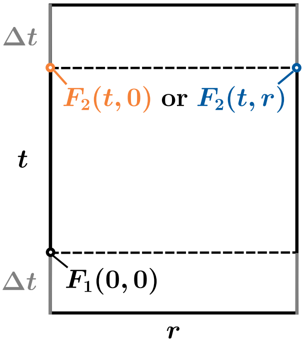
$$V_{S^2} = \frac{2c_F^2(\mu)}{3}\sum_i \left(\mathcal{I}_0(B_i(t,r)B_i(0,0))\right)(\boldsymbol{S}_1\boldsymbol{S}_2), \ \ldots$$

where

$$\mathcal{I}_n(F_2(t,r_2)F_1(0,r_1)) = \lim_{T\to\infty}\int_0^T dt\ t^n\ \langle\langle g^2 F_2(t,r_2)F_1(0,r_1)\rangle\rangle_{(c)},$$

$$\langle\langle g^2 F_2(t,r_2)F_1(0,r_1)\rangle\rangle = \lim_{\Delta t\to\infty}\langle g^2 F_2(t,r_2)F_1(0,r_1)\rangle_W/\langle 1\rangle_W.$$

Generalized Wilson loop $\langle F_2(t)F_1(0)\rangle_W$:



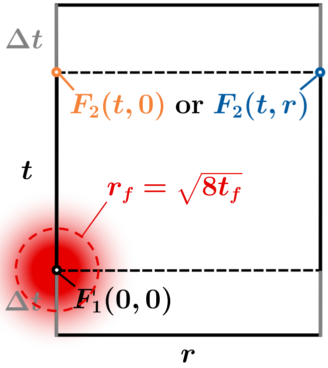
$\boldsymbol{r} = (0,0,r)$; $c_F(\mu)$: Matching coefficient for B-insertions;
$n = 0,1,2$.

Gradient Flow

- Lattice-$F_{\mu\nu}$ requires renormalization, e.g. clover definition $F_{\mu\nu} = (\Pi_{\mu\nu} - \Pi^\dagger_{\mu\nu})/2$ via $\bar{F}_{\mu\nu} = (\Pi_{\mu\nu} + \Pi^\dagger_{\mu\nu})/2$. [Huntley, Michael, 1986]
- Flow equation:

$$\dot{B}_\mu = D_\nu G_{\mu\nu}, \qquad B_\mu|_{t=0} = A_\mu \,,$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \quad D_\mu = \partial_\mu + [B_\mu, \cdot]\,.$$

- Fields at flow time $t_f > 0$ are **smooth** and do not require additional **renormalization**. [Lüscher, 2010], [Shindler, 2023], [Brambilla, Leino, Mayer-Steudte, Vairo, 2023]
- Related flow radius $r_f = \sqrt{8t_f}$ acts as regulator, makes **continuum extrapolation** possible. [Brambilla, Wang, 2023]
- $\sqrt{8t_f}$ must not be too large, but at least $\gtrsim 1a$. [Eller, Moore, 2018], [Brambilla, Leino, Mayer-Steudte, Vairo, 2023]



Lattice setup

Ensembles:

Ensemble name	β	TL^3/a^4	a [fm]	$N_{\mathrm{Conf.}}$
A	6.284	$48 \cdot 24^3$	0.06	10000
B	6.451	$60 \cdot 30^3$	0.048	4400
C	6.594	$72 \cdot 36^3$	0.04	2200

- Generation of ensembles: CL2QCD [Sciarra et al., 2021]
- Error propagation and fitting: pyerrors [Joswig, Kuberski, Kuhlmann, Neuendorf, 2024], [Wolff, 2004], [Ramos, 2019]
- Flow times were set to $\sqrt{8t_f}/a = 0.6, 0.8, 0.96, 1.12$ for Ensemble C and held fixed in physical units for Ensembles A and B.

Static potential without improvement

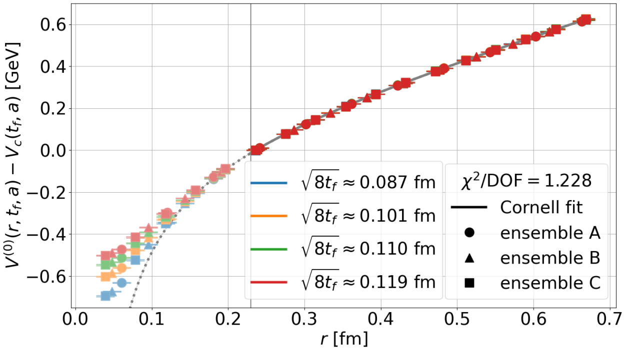


Figure: Static potential from all ensembles. Mass shifts V_c has been subtracted. Physical scale is set using $r_0 = 0.5$ fm.

- Errors due to discretization and gradient flow particularly large at $r < 2\sqrt{8t_f}$.
- t_f-dependence becomes negligible for sufficiently large r.
- **Reduce discretization errors using tree level improvement.**

Tree level improvement of $V^{(0)}$

Global fit ansatz: [Bali, Schilling, Wachter, 1997], [Schlosser, Wagner, 2021]

$$V^{(0)}_{\mathrm{fit}}(r, t_f, a) = -\frac{c^{(0)}}{r} + \sigma r + V_c(a, t_f) + \tilde{c}^{(0)} \left(4\pi G(\boldsymbol{r}, t_f) - \frac{1}{r}\right) ,$$

where $V_c(a, t_f)$ is a constant shift and $G(\boldsymbol{r}, t_f)$ the lattice propagator in **gradient flow**:
[Brambilla, Leino, Mayer-Steudte, Vairo, 2023]

$$G(\boldsymbol{r}, t_f) = \int_{[-\pi,\pi)^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\boldsymbol{p}\boldsymbol{r}} \frac{\exp(-2t_f \sum_i (2\sin(p_i/2))^2)}{\sum_i (2\sin(p_i/2))^2} .$$

What is plotted are the Cornell potential and the corrected data points:

$$V^{(0)}(r, t_f = 0) = -\frac{c^{(0)}}{r} + \sigma r ,$$

$$V^{(0)}_{\mathrm{lat,corr.}}(r, t_f, a) = V^{(0)}_{\mathrm{lat,meas.}}(r, t_f, a) - V_c(a, t_f) - \tilde{c}^{(0)} \left(4\pi G(\boldsymbol{r}, t_f) - \frac{1}{r}\right) .$$

Static potential

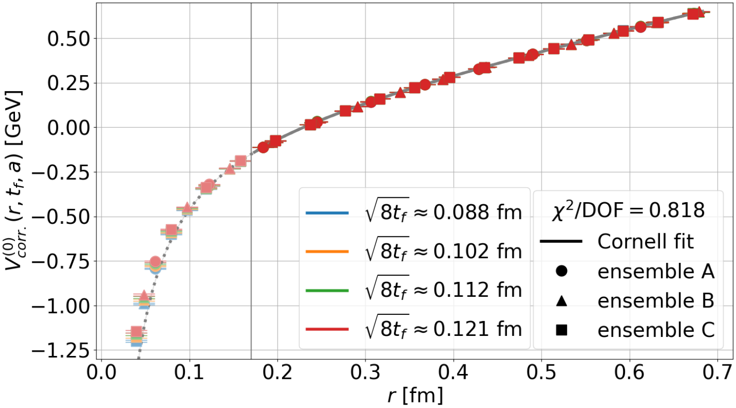


Figure: Static potential from all ensembles. Mass shifts V_c, as well as a- and t_f-dependence at tree level, have been removed. Physical scale is set using $r_0 = 0.5$ fm.

Relativistic corrections

Preliminary

[Bali, Schilling, Wachter, 1997], [Brambilla, Groher, Martinez, Vairo, 2014]

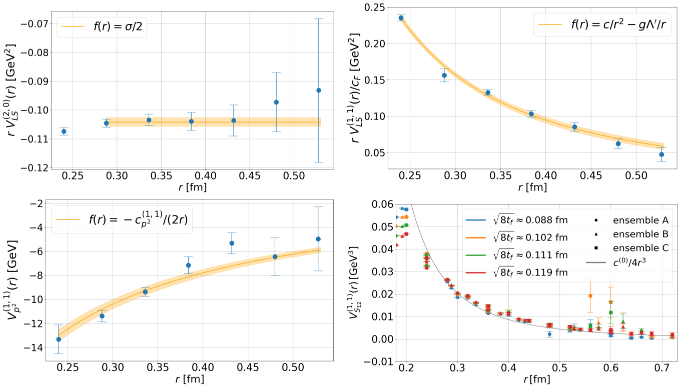


Figure: $V_{LS}^{(2,0)}$, $V_{LS}^{(1,1)}$ and $V_{p^2}^{(1,1)}$ from ensemble B at $\sqrt{8t_f} \approx 0.1$ fm. Bottom right: $V_{S_{12}}^{(1,1)}$ from all ensembles and flow times.

Required steps:

- Compute correlators from lattice ensembles.
- Fit ansatz $\langle\langle F_2(t)F_1(0)\rangle\rangle \sim \sum_k D_{12}^{(k)} e^{-(E_k - E_{\Sigma_g^+})t}$ for every set of (a, t_f, r) and determine t-integral analytically.
- Determine suitable fit range such, that $\sigma_{\text{stat.}}$ is small and results are stable.

Gromes and BBMP relations

Preliminary

[Gromes, 1984], [Barchielli, Brambilla, Prosperi, 1988]

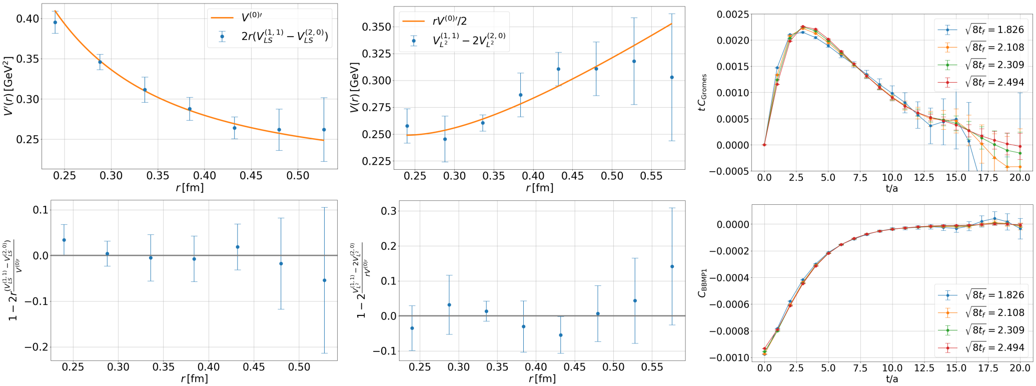


Figure: Gromes (left) and first BBMP relation (center) at $\sqrt{8t_f} \approx 0.1$ fm, as well as respective correlators (right), all from ensemble B ($a \approx 0.048$ fm).

Tree level improvement of field-strength correlators

Preliminary

Is it possible, to improve field strength correlators using tree level expressions? E.g. via

$$C_{\mathrm{impr.}}(r,t,t_f) = C_{\mathrm{meas.}}(r,t,t_f) - \frac{4\pi\tilde{c}^{(0)}}{C_F g^2}\left(C^{\mathrm{lat.}}_{\mathrm{tree}}(r,t,t_f) - C^{\mathrm{cont.}}_{\mathrm{tree}}(r,t)\right) .$$

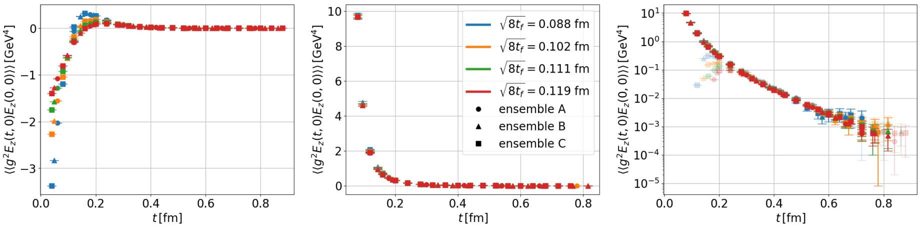


Figure: Measured results (left) and tree level improved results (center) for $\langle\langle E_z(t,0)E_z(0,0)\rangle\rangle$ at various flow times from all ensembles in physical units, using $r \approx 0.24\,\mathrm{fm}$. Right: Measured results (transparent points) and improved results (opaque points) plotted with logarithmic scale.

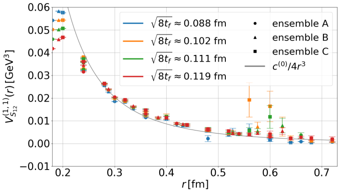
Conclusions

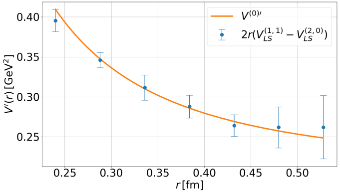
Summary

- $\mathcal{O}(1/m_Q^2)$-corrections to the static $Q\bar{Q}$-potential have been computed at various t_fs for distances up to $\gtrsim 0.5$ fm.
- Results in agreement with Gromes and BBMP relations.
- Leading flow and discretization effects can be effectively removed using tree level improvement.

Outlook

- Further investigate $a \to 0$ and $t_f \to 0$ limits with tree level improvement.
- Obtain results at larger separations.
- Determine quarkonium masses using potential results in a Schrödinger equation.





Thank you for your attention!

Tree level improvement of Wilson loops with field strength insertions I

At tree level, what remains is: [Bali, Schilling, Wachter, 1997]

$$\frac{\langle g^2 G_{\mu\nu}(z,t_f) G_{\rho\sigma}(0,t_f)\rangle_W}{\langle 1\rangle_W} = g^2 \langle G_{\mu\nu}(z,t_f) G_{\rho\sigma}(0,t_f)\rangle + \mathcal{O}(g^3)\,.$$

The measured result can then be improved via

$$\begin{aligned}\langle\langle g^2 G_{\mu\nu}(z,t_f) G_{\rho\sigma}(0,t_f)\rangle\rangle_{\mathrm{impr.}} =& \langle\langle g^2 G_{\mu\nu}(z,t_f) G_{\rho\sigma}(0,t_f)\rangle\rangle_{\mathrm{meas.}} \\ &- \frac{4\pi\tilde{c}}{C_f g^2}\left(\langle\langle g^2 G_{\mu\nu}(z,t_f) G_{\rho\sigma}(0,t_f)\rangle\rangle^{\mathrm{lat.}}_{\mathrm{tree}} - \langle\langle g^2 F_{\mu\nu}(z) F_{\rho\sigma}(0)\rangle\rangle^{\mathrm{cont.}}_{\mathrm{tree}}\right)\end{aligned}$$

where

$$\langle\langle g^2 F_{\mu\nu}(z) F_{\rho\sigma}(0)\rangle\rangle^{\mathrm{cont.}}_{\mathrm{tree}} = C_F g^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{ipz-2t_f p^2}}{p^2}\left(-p_\mu p_\rho \delta_{\nu\sigma} + p_\mu p_\sigma \delta_{\nu\rho} + p_\nu p_\rho \delta_{\mu\sigma} - p_\nu p_\sigma \delta_{\mu\rho}\right)\,,$$

Tree level improvement of Wilson loops with field strength insertions II

Cf. [Fritzsch, Ramos, 2013]

$$
\begin{aligned}
&\langle\langle g^2 G^{\mathrm{clover}}_{\mu\nu}(z,t_f)\,G^{\mathrm{clover}}_{\rho\sigma}(0,t_f)\rangle\rangle^{\mathrm{lat.}}_{\mathrm{tree}} \\
&\quad = \frac{1}{(T/a)\cdot(L/a)^3}\sum_{p\neq 0}\frac{e^{ipz-2t_f\hat{p}^2}}{\hat{p}^2}\Big(\sin(p_\mu)\sin(p_\rho)\cos(p_\nu/2)\cos(p_\sigma/2)\delta_{\nu\sigma} \\
&\qquad + \sin(p_\nu)\sin(p_\sigma)\cos(p_\mu/2)\cos(p_\rho/2)\delta_{\mu\rho} \\
&\qquad - \sin(p_\mu)\sin(p_\sigma)\cos(p_\nu/2)\cos(p_\rho/2)\delta_{\nu\rho} \\
&\qquad - \sin(p_\nu)\sin(p_\rho)\cos(p_\mu/2)\cos(p_\sigma/2)\delta_{\mu\sigma}\Big)\,.
\end{aligned}
$$