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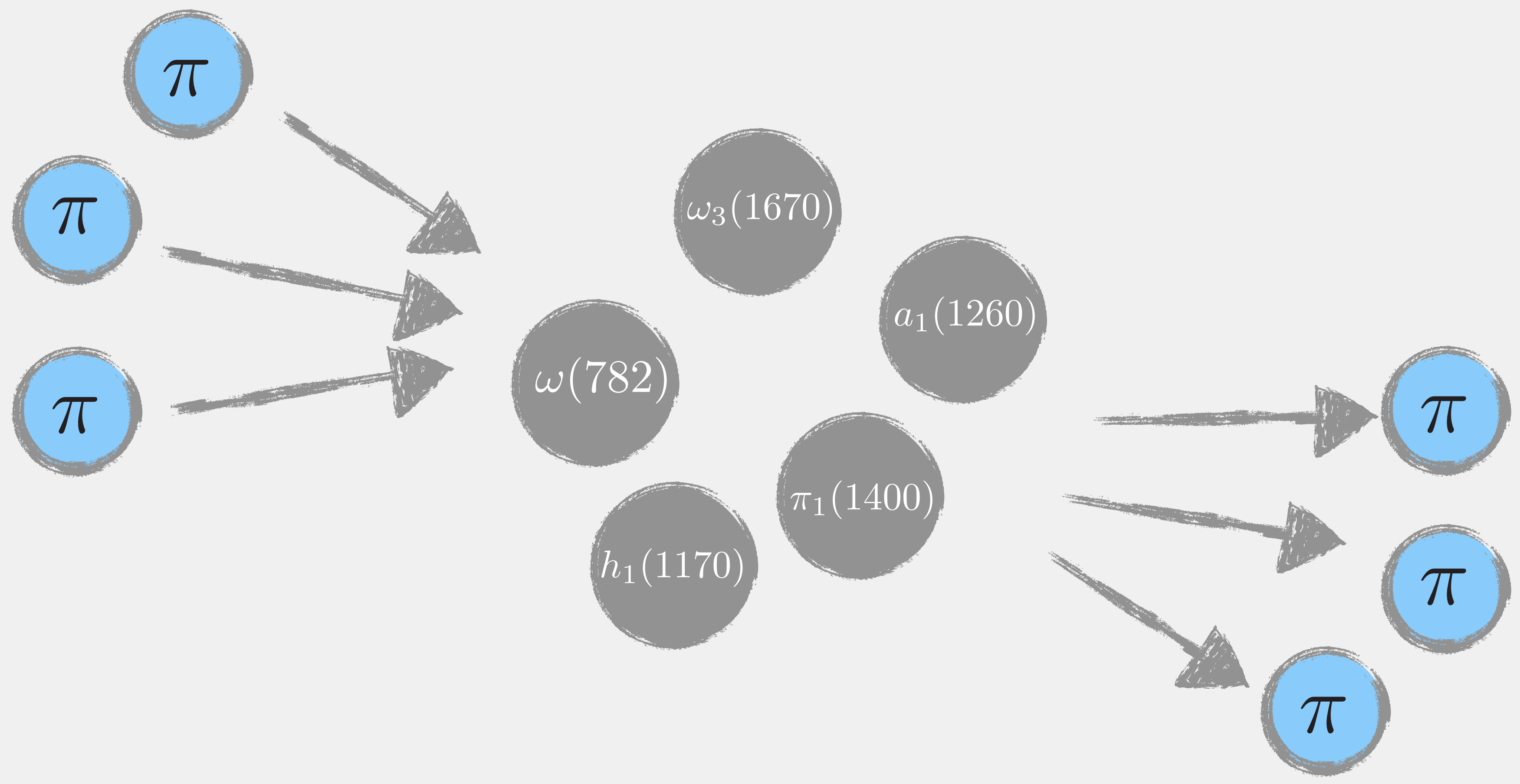
Implementing the relativistic-field-theory finite-volume formalism across all three-pion isospins

Based on work with: Dr. Maxwell T. Hansen

Table of Contents

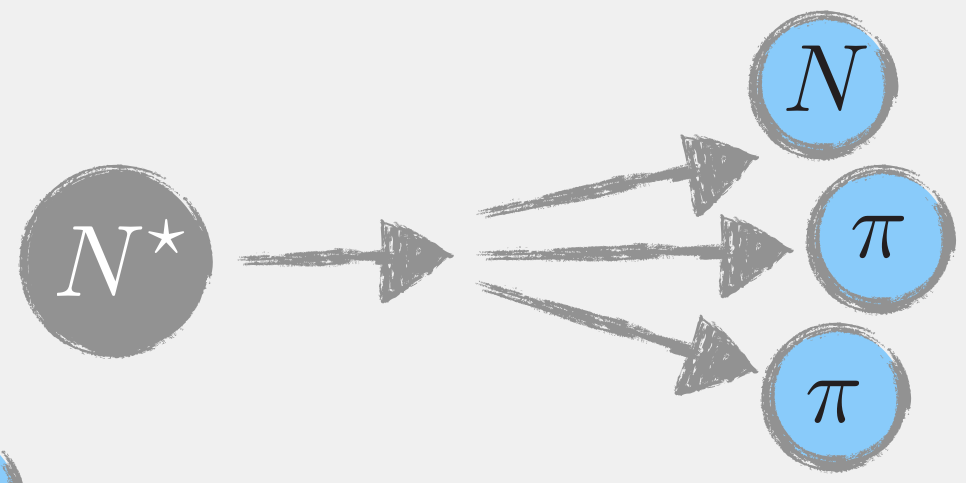
Motivation

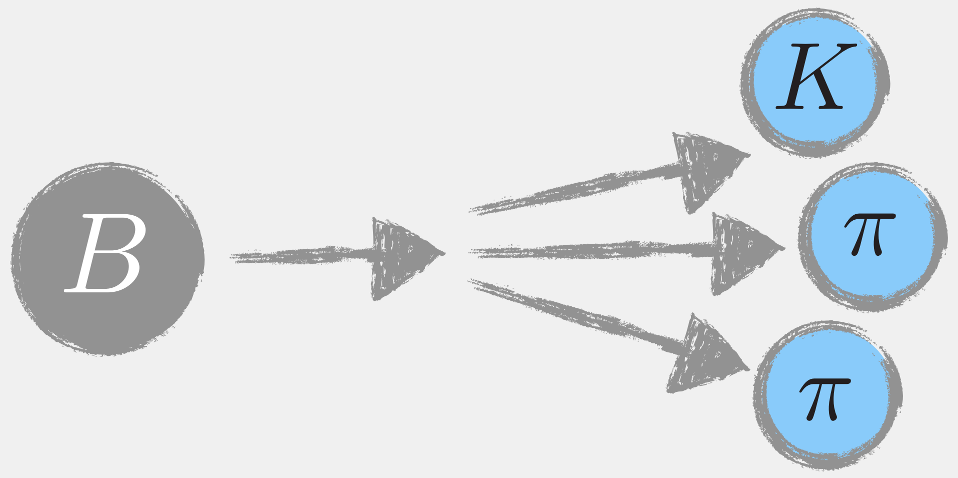
- Many interesting three-pion resonances.



- Better understand contributions to the hadronic vacuum polarization of the muon $(g-2)_{\mathrm{HVP}}$.

- Stepping stone up for more complicated resonances and weak decays:





- Frontier of LQCD = rigorously analyzing three-pion energies (especially in resonance channels).

- This talk = no lattice Monte Carlo, but implementation of finite-volume formalism.

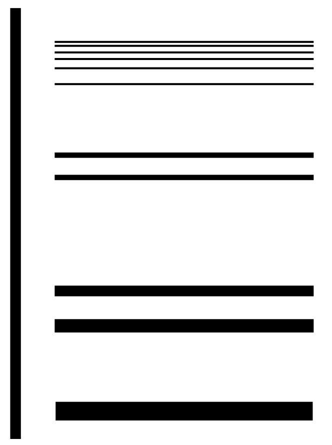
RFT three-pion finite-volume formalism

Quantization Condition

Integral Equations

Finite-volume energies

E



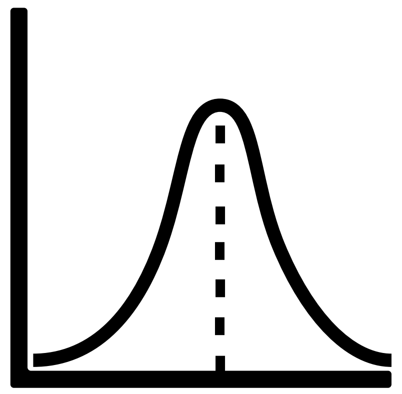
The K-matrix

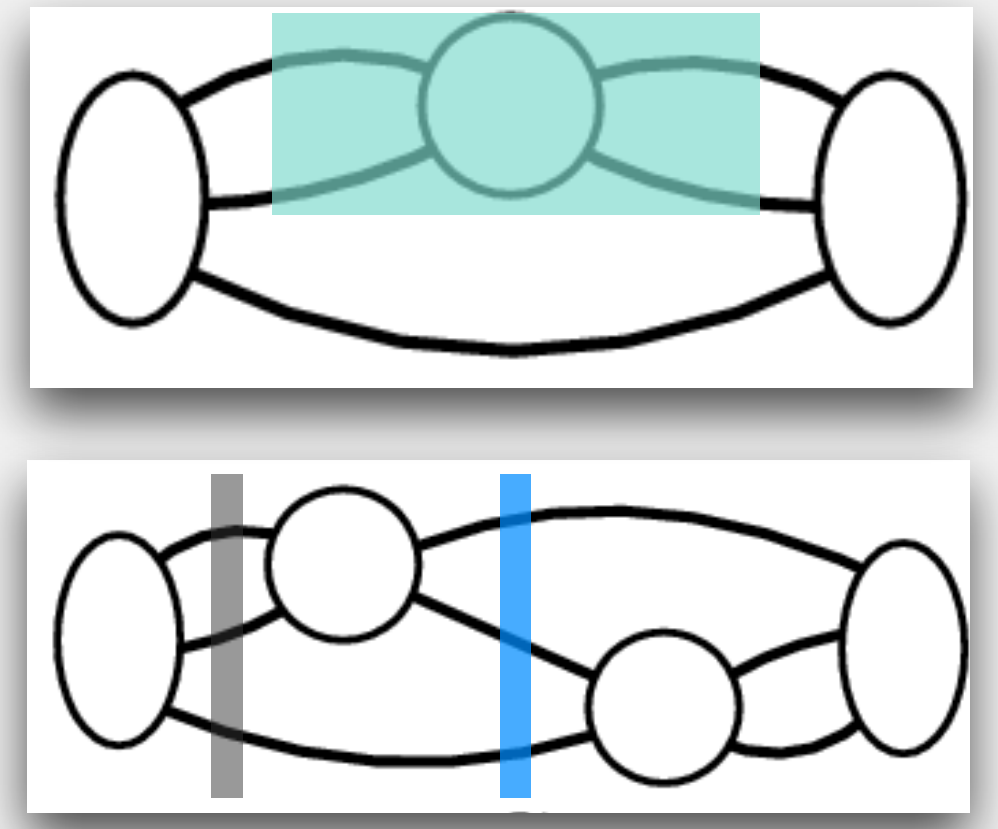
$$\det[\boldsymbol{F}_3^{-1}(E,\boldsymbol{P},L) + \boldsymbol{\mathcal{K}}_{df,3}(E^\star)] = 0$$

where:

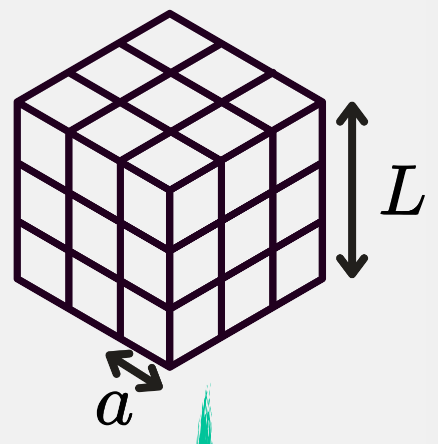
$$\boldsymbol{F}_3 \equiv \frac{\boldsymbol{F}}{3} + \boldsymbol{F}\boldsymbol{\mathcal{K}}_2[1-(\boldsymbol{F}+\boldsymbol{G})\boldsymbol{\mathcal{K}}_2]^{-1}\boldsymbol{F}$$

Scattering Amplitude

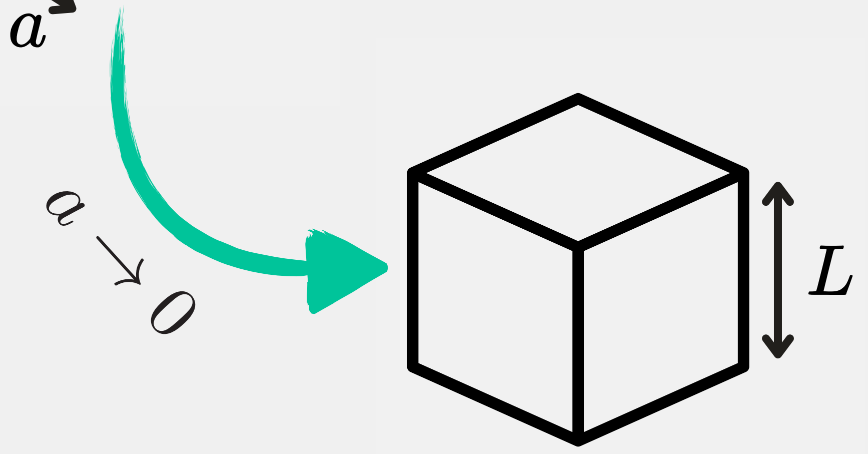




• F Romero-Lopez (2021) •



$$\boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{n}$$



- Different methods of deriving the finite-volume formalism.

  • Döring, Mai (2017) • • Hammer, Pang, Rusetsky (2017) •

- Our work uses the relativistic-field-theory (RFT) approach.

  • Hansen, Sharpe (2014) •

- Matrices on tensor-product space:

$$(\textit{flavor space}) \otimes (\textit{spectator space}) \otimes (\textit{angular-momentum space})$$

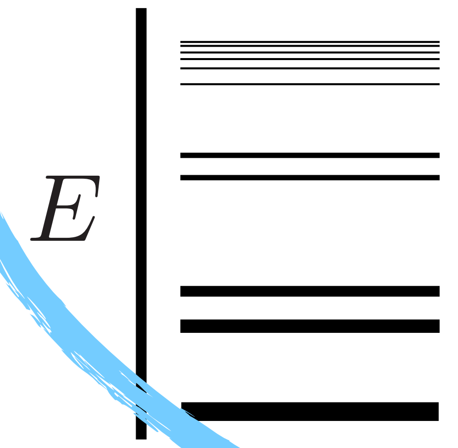
- Holds up to exponentially suppressed term $\mathcal{O}(e^{-mL})$.

- Requires parametrization of both $\boldsymbol{\mathcal{K}}_2$ and $\boldsymbol{\mathcal{K}}_{df,3}$ (e.g. using polynomials and poles or EFT).

RFT three-pion finite-volume formalism

Quantization Condition — Integral Equations

Finite-volume energies



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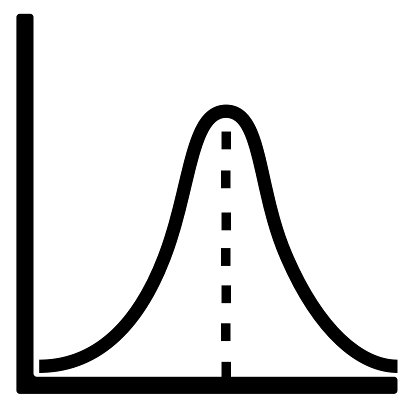

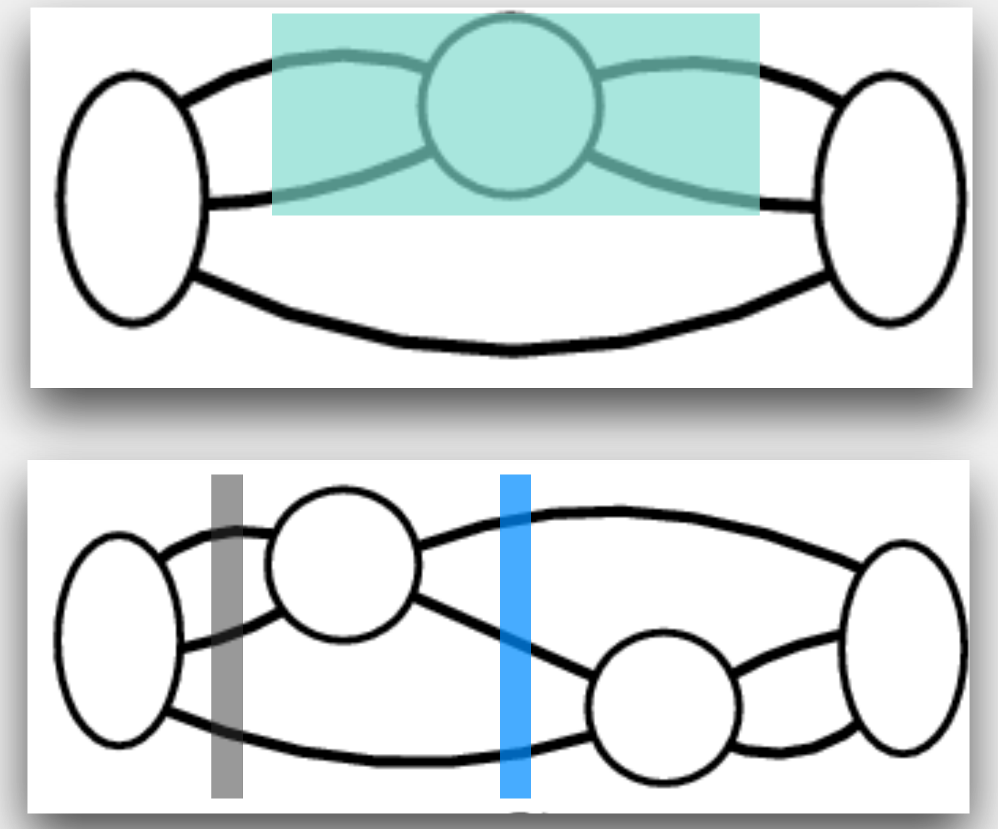

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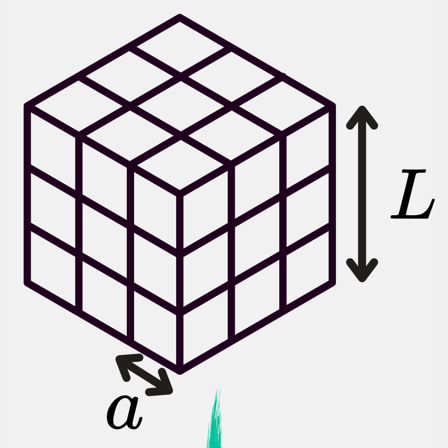
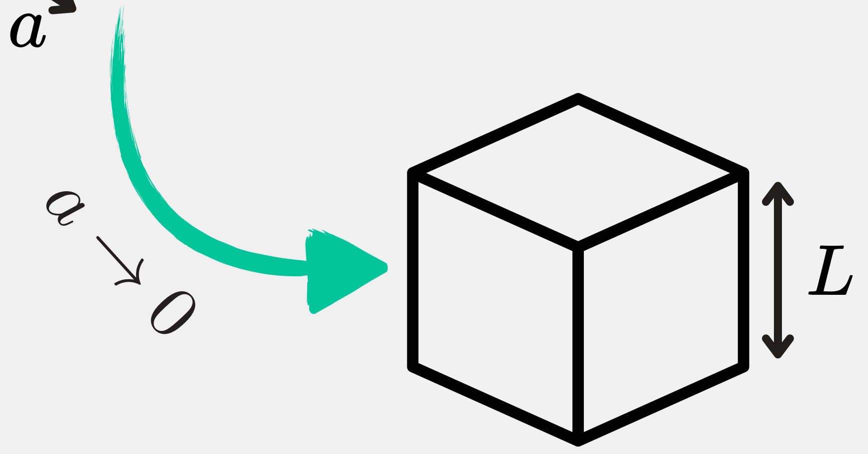


$$\boldsymbol{P}=\frac{2\pi}{L}\boldsymbol{n}$$



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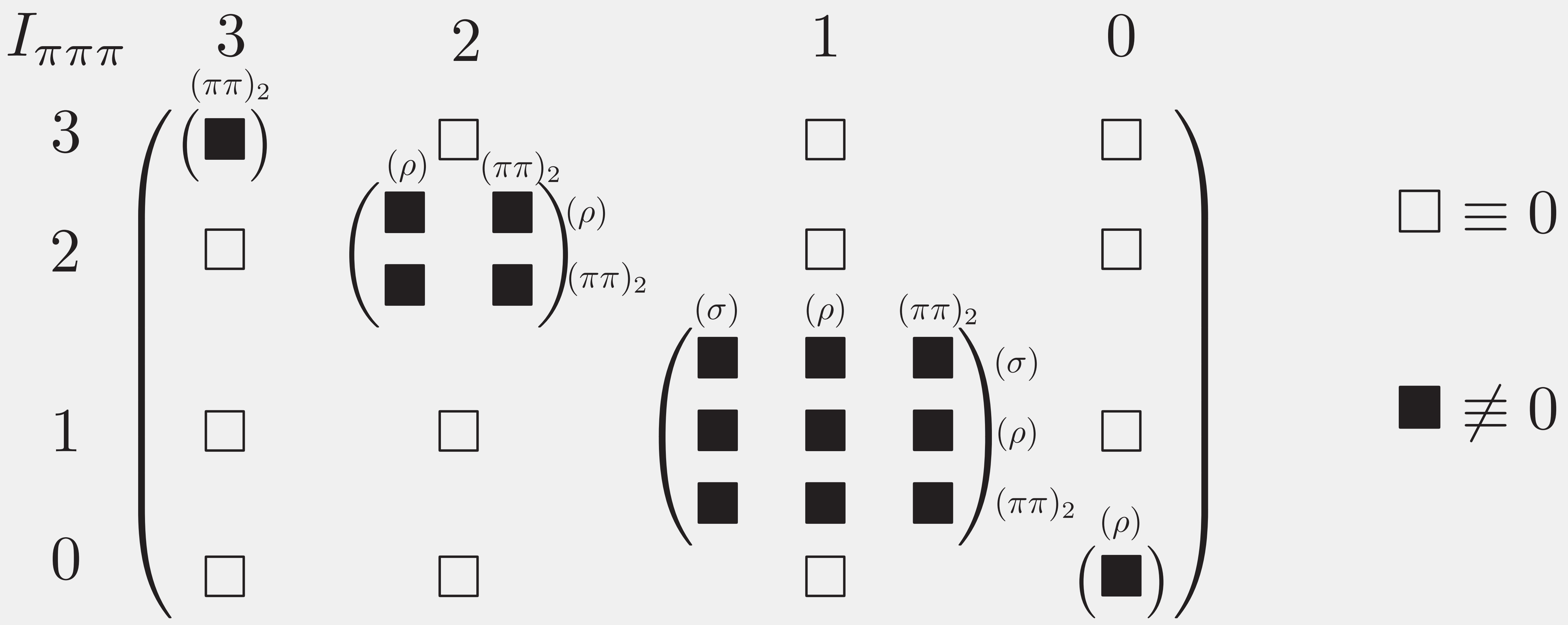
Isospin decomposition

• Hansen, Romero-López, Sharpe (2020) •

$$\begin{aligned}1\otimes 1\otimes 1 &= (0\oplus 1\oplus 2)\otimes 1\\ &= (1\oplus 2\oplus 3)_{I_{\pi\pi=2}} \oplus (0\oplus 1\oplus 2)_{I_{\pi\pi=1(\rho)}} \oplus 1_{I_{\pi\pi=0(\sigma)}}\end{aligned}$$

Block diagonalize quantization condition to four sectors



Irrep projection

P	G(|G|)	Irrep (dimension)
[000]	O$_h$ (48)	$A_1^{\pm}(1)$
		$A_2^{\pm}(1)$
		$E^{\pm}(2)$
		$T_1^{\pm}(3)$
		$T_2^{\pm}(3)$
[001]	Dic$_4$ (8)	$A_1(1)$
		$A_2(1)$
		$B_1(1)$
		$B_2(1)$
		$E_2(2)$
[011]	Dic$_2$ (4)	$A_1(1)$
		$A_2(1)$
		$B_1(1)$
		$B_2(1)$

Using “am-pi-ell” public Python package (available on GitHub)

Implementation of the quantization condition

- Features of ***ampyL***:
  - Calculates non-interacting energies, with multiplicities for a given finite-volume irrep and isospin:

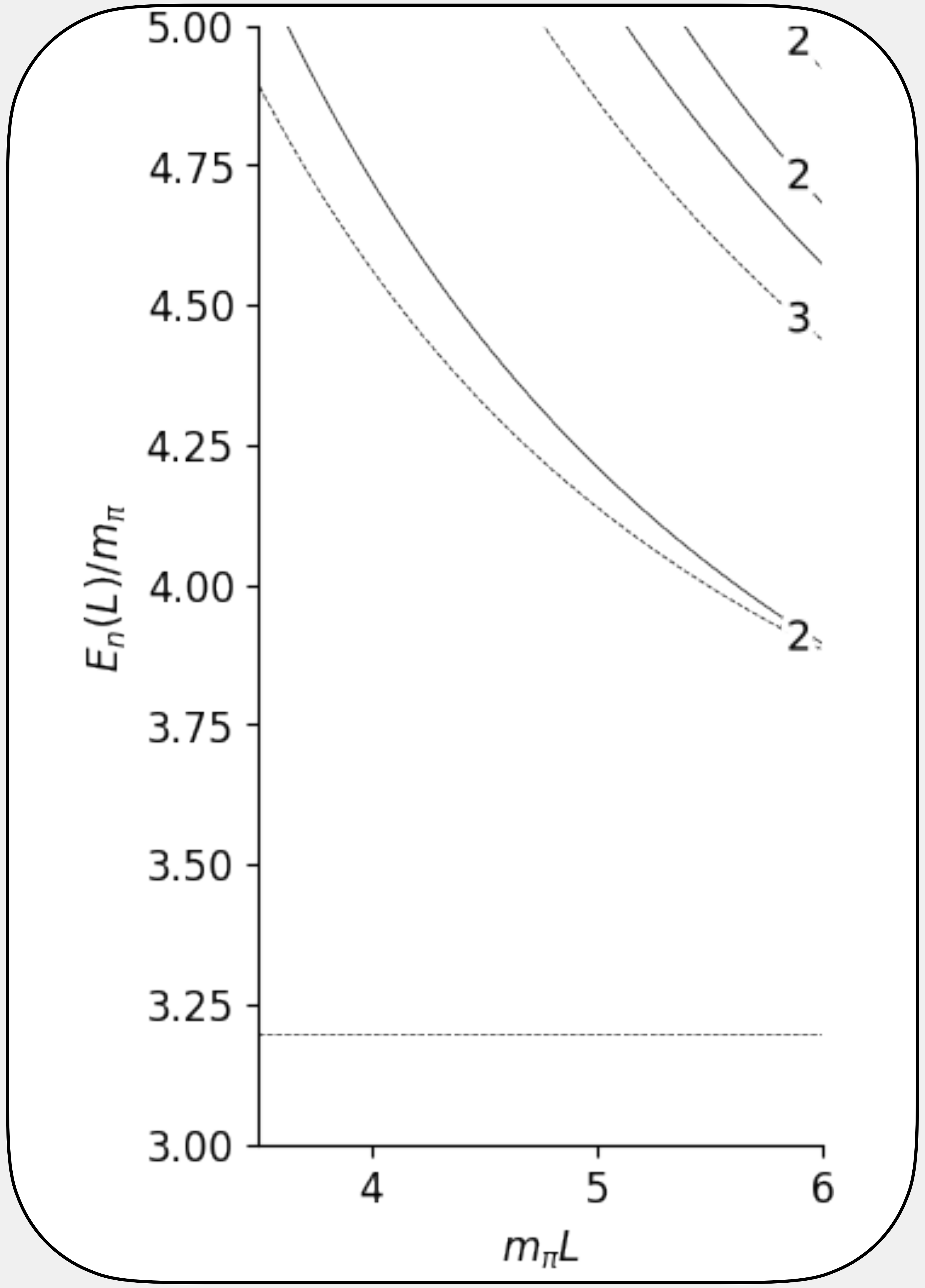
$$E^{\{\boldsymbol{n}\}}(L) = \sum_{i=1}^{3} \sqrt{m_\pi^2 + (2\pi/L)^2 \boldsymbol{n}_i^2}$$

  - Builds and projects the quantization condition for a given irrep and isospin.
  - Pre-computes spectator-momentum space matrices to accelerate evaluation.
  - Uses splines on irrep-projected building blocks to accelerate evaluation.
  - Various strategies for efficient root finding.

$S(\boldsymbol{n})$	$N_{\pi_0\pi_0\pi_0}$	$N_{\pi_+\pi_-\pi_0}$	$I_{\pi\pi\pi}=3$	$I_{\pi\pi\pi}=2$	$I_{\pi\pi\pi}=1$	$I_{\pi\pi\pi}=0$
$\{000\}$	1	1	1	0	1	0
$\{011\}$	3	18	3	6	9	3
$\{022\}$	6	36	6	12	18	6
$\{112\}$	12	72	12	24	36	12

Example spectra of three pions with isospin 2

$I_{\pi\pi\pi} = 2 \quad \boldsymbol{P}^2 = 0 \quad (T_1^+)$



(n_ρ^2, n_π^2)	irrep
$(0,0)$	T_1^+
$(1,1)$	$A_1^-, E^-, T_1^-, T_2^-, 2T_1^+, T_2^+$
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$\vdots$	$\vdots$

$S(\boldsymbol{n})$	irrep
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$(0,1,1)$	A_1^-, E^-, T_1^+
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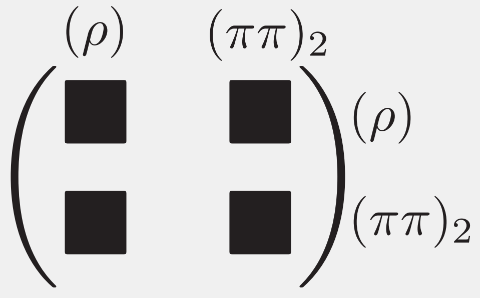

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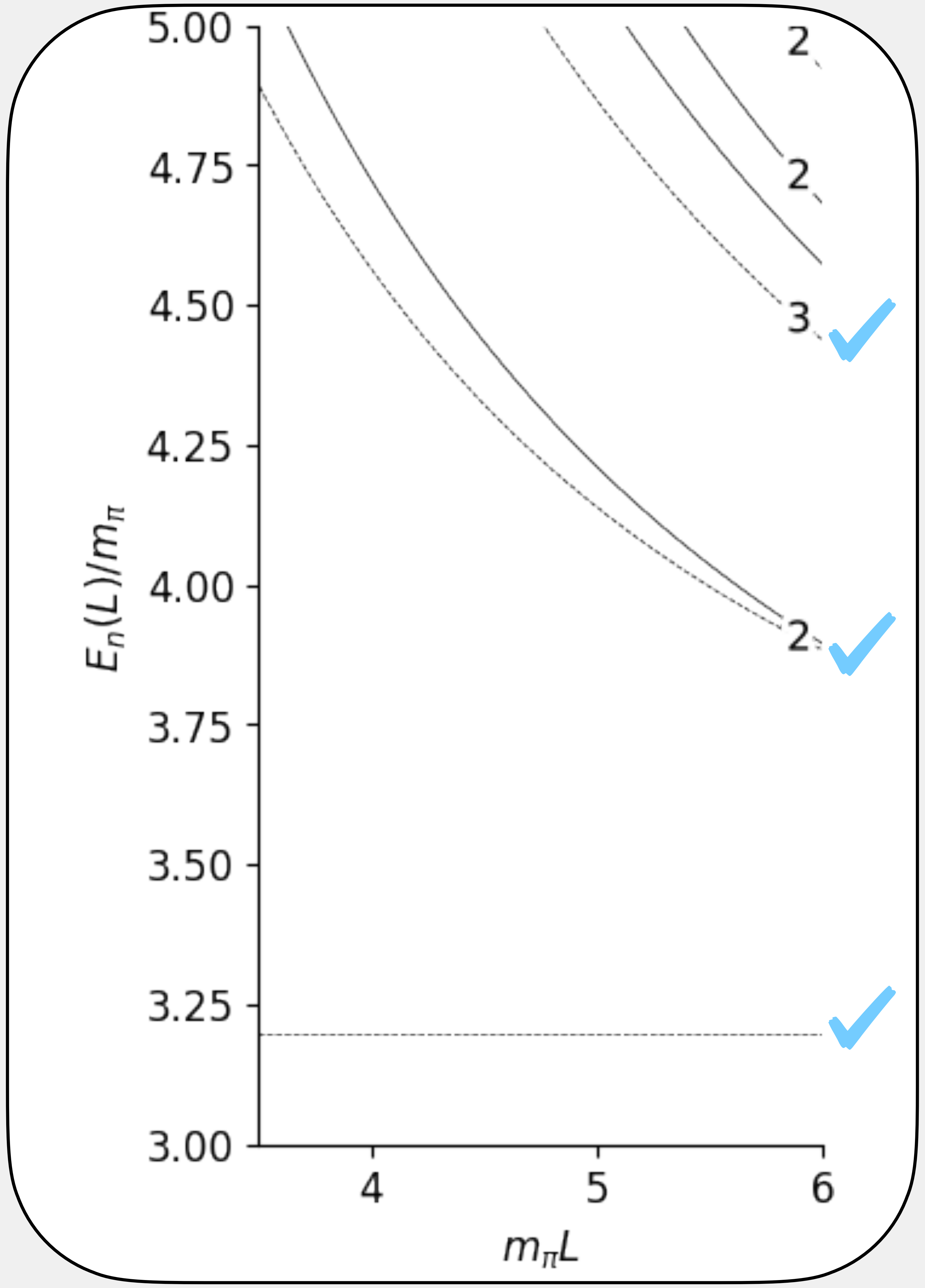

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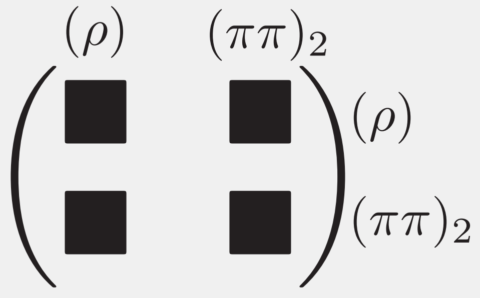

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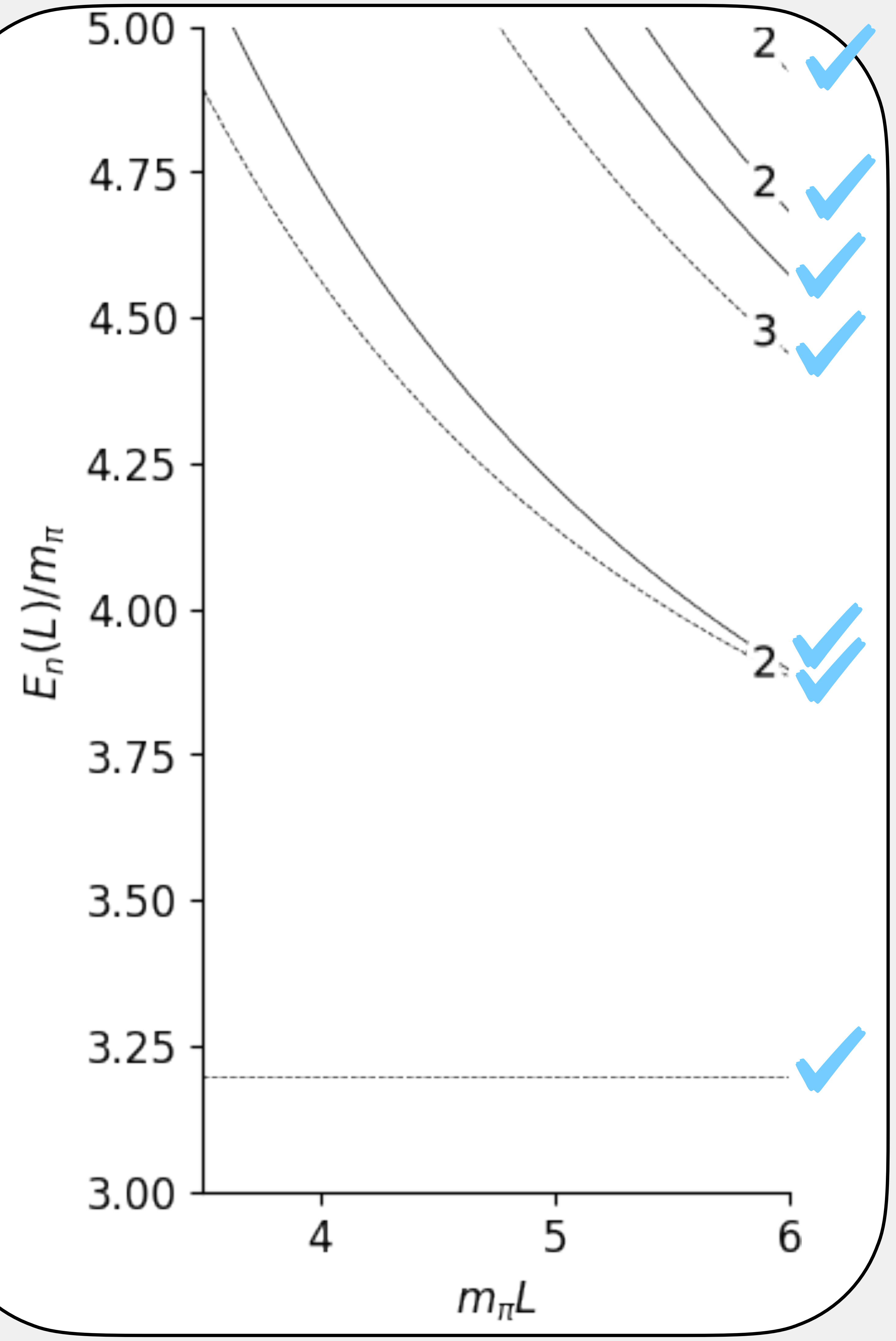

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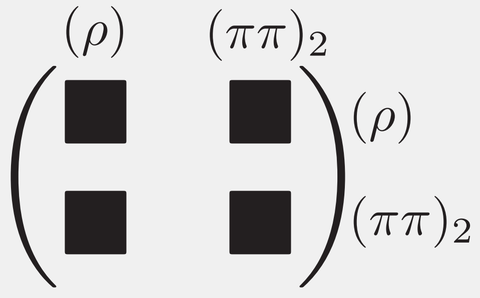

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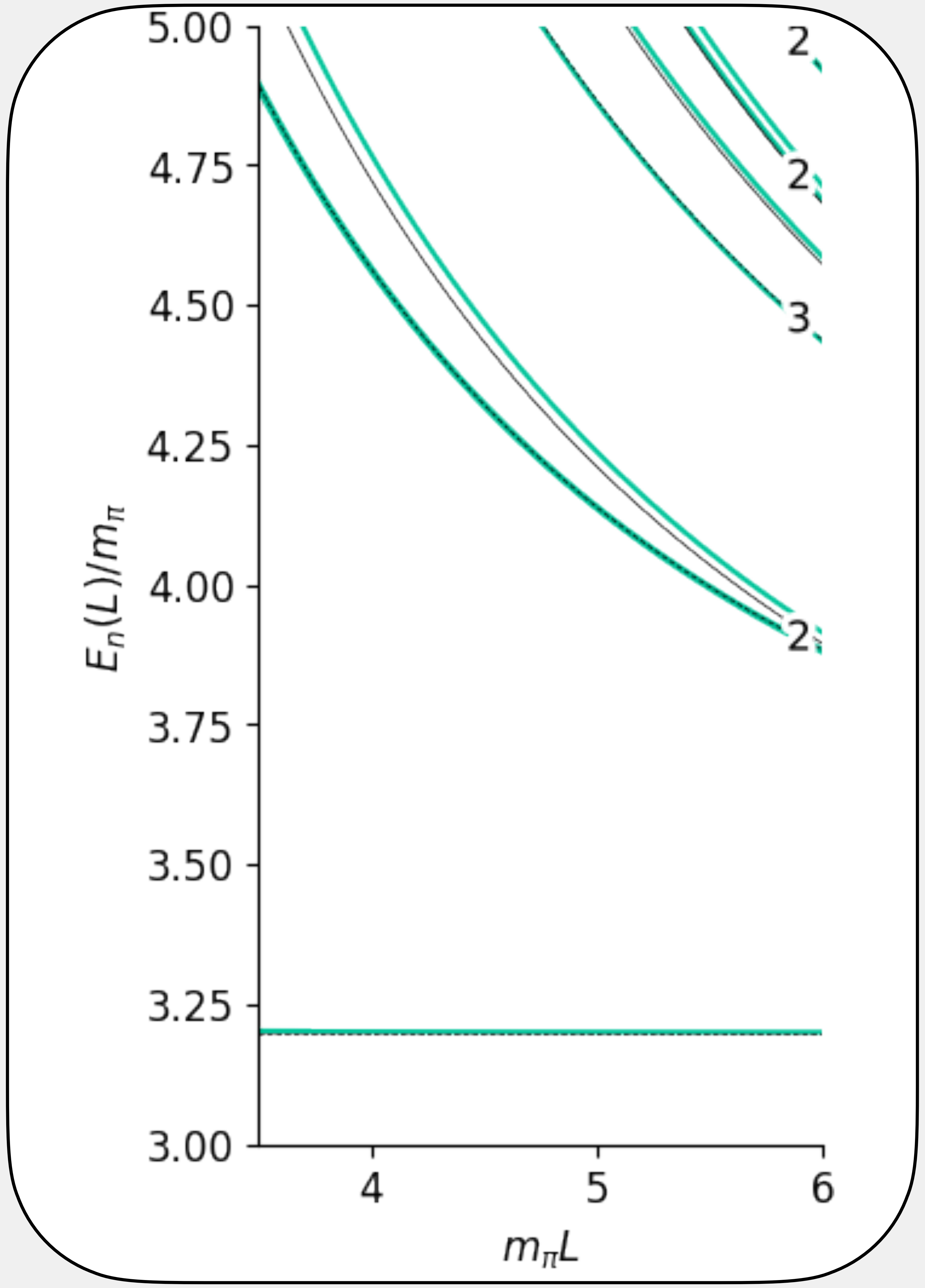

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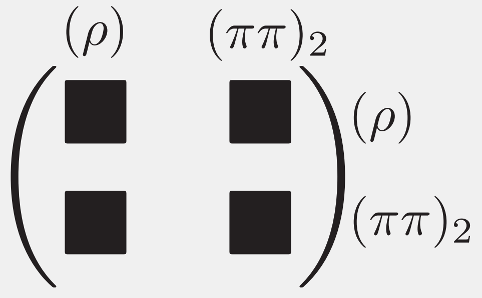

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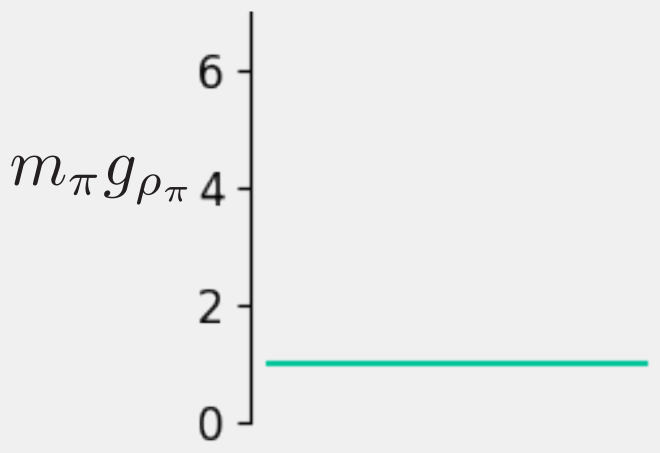

$$\left(\frac{q}{m_\pi}\right)^3 \cot\delta = \frac{6\pi}{g_{\rho\pi}^2}\,\frac{m_\rho^2 - E^2}{E m_\pi}\,\frac{E^2}{m_\rho^2} \qquad K_3 = 0$$

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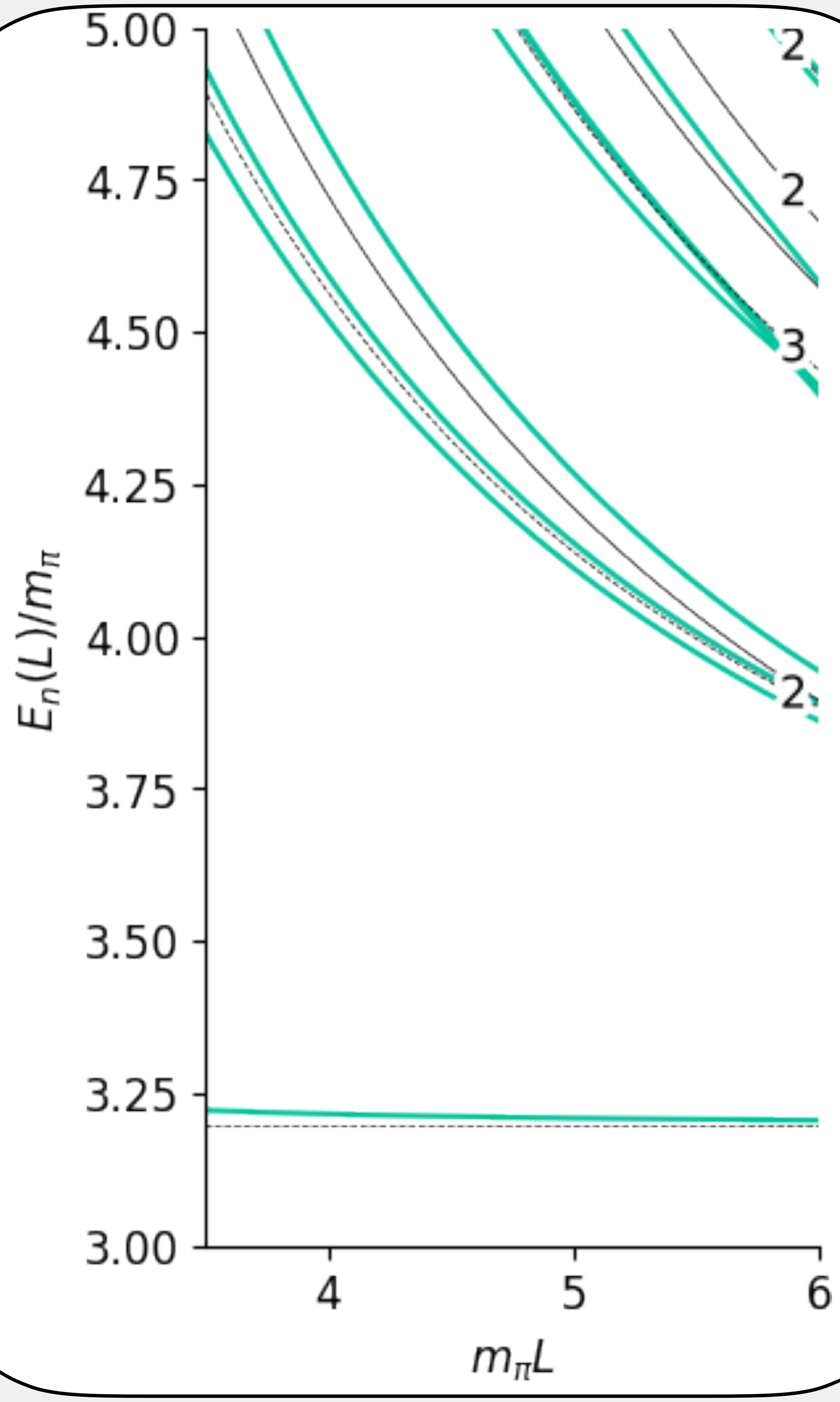

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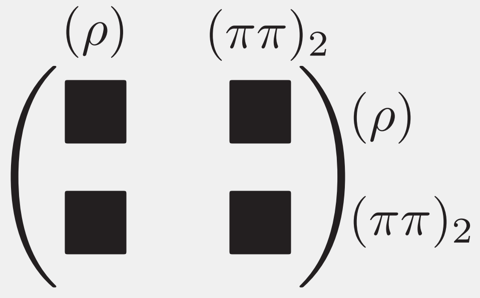

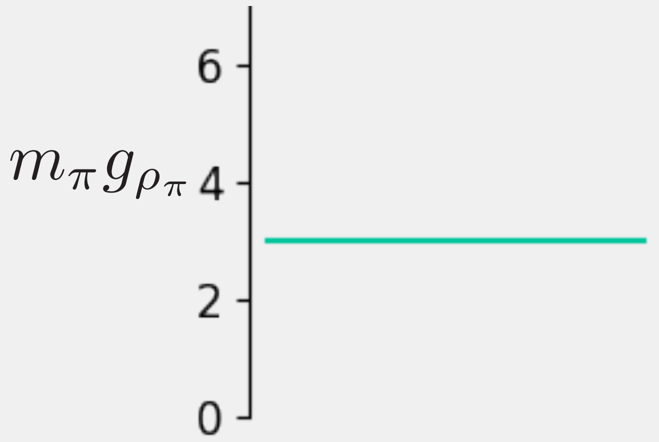

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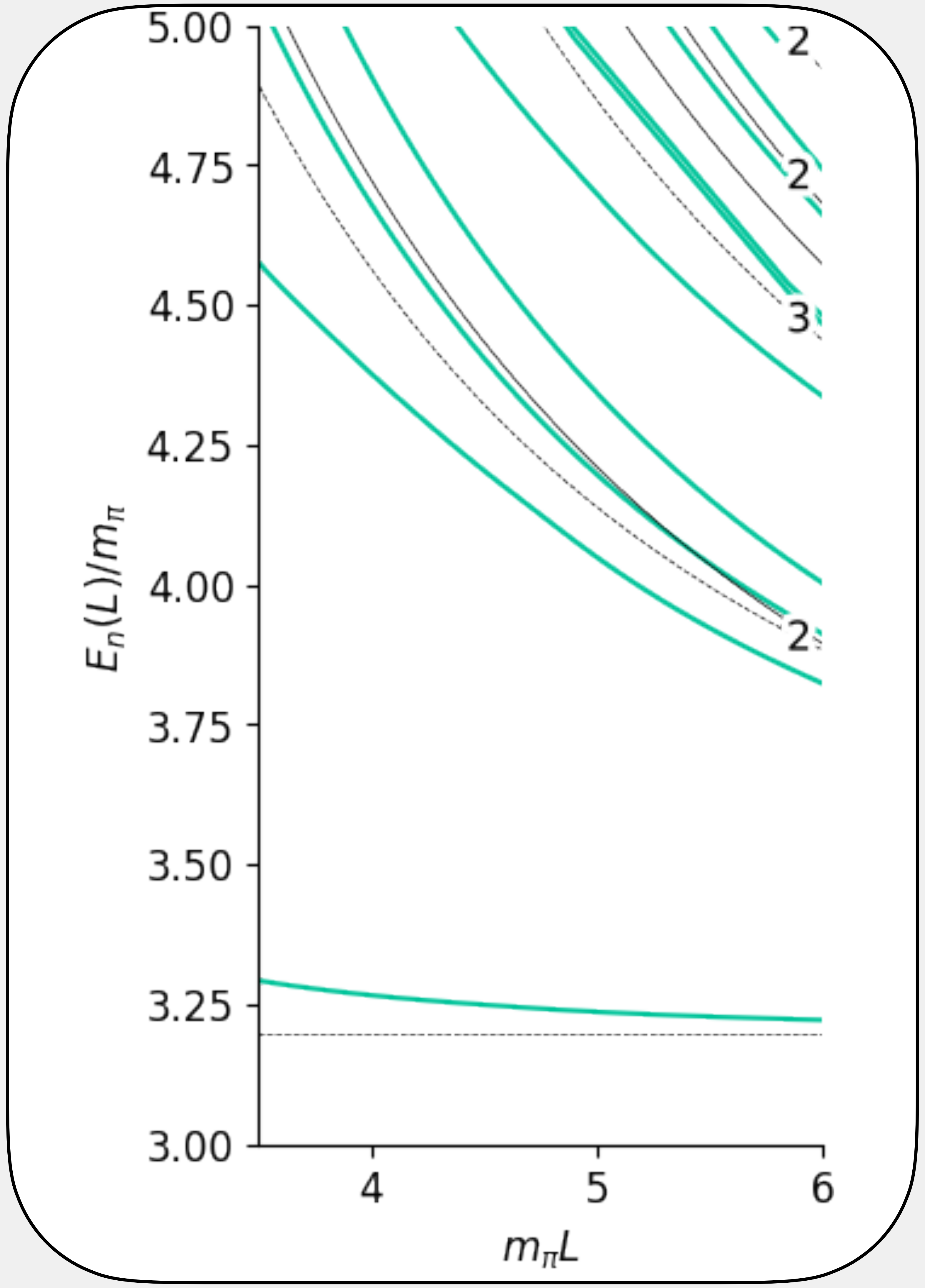

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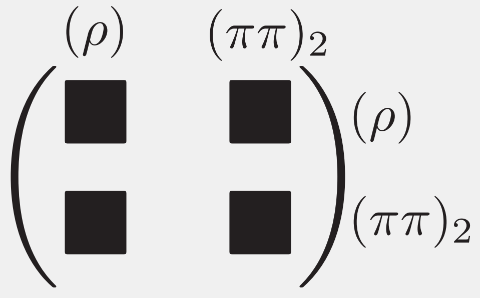

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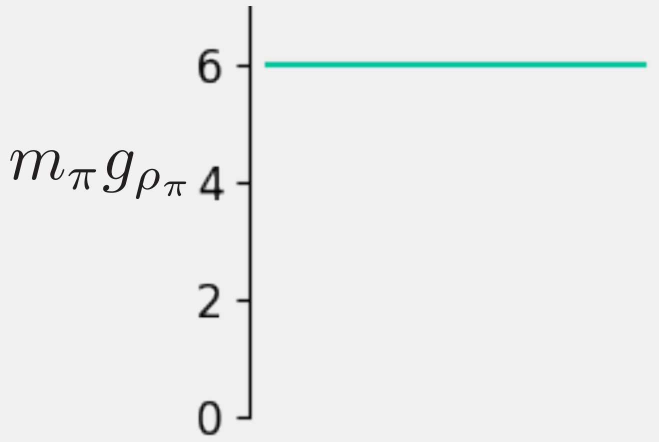

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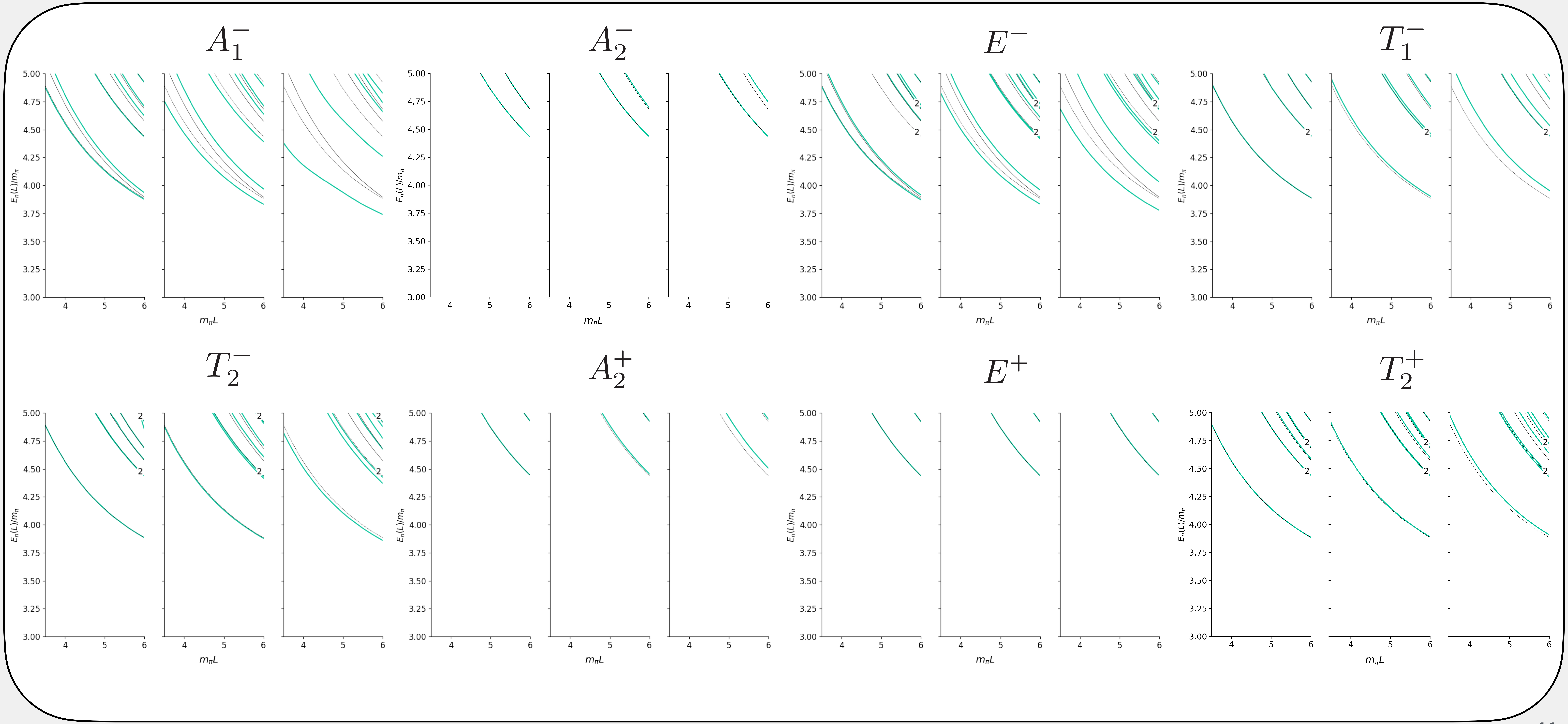
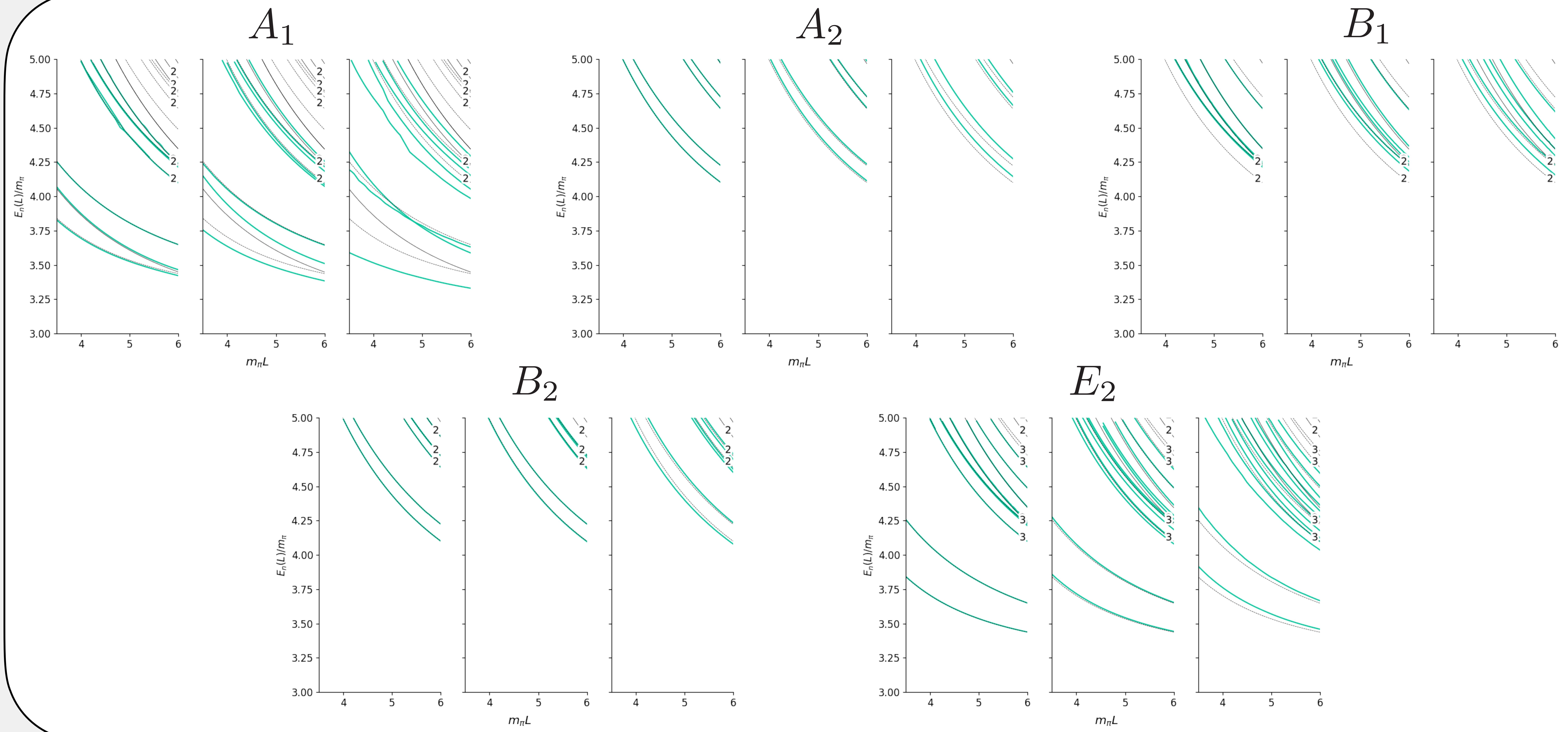
Example spectra of three pions with isospin 2

$I_{\pi\pi\pi} = 2 \quad \boldsymbol{P}^2 = 0$



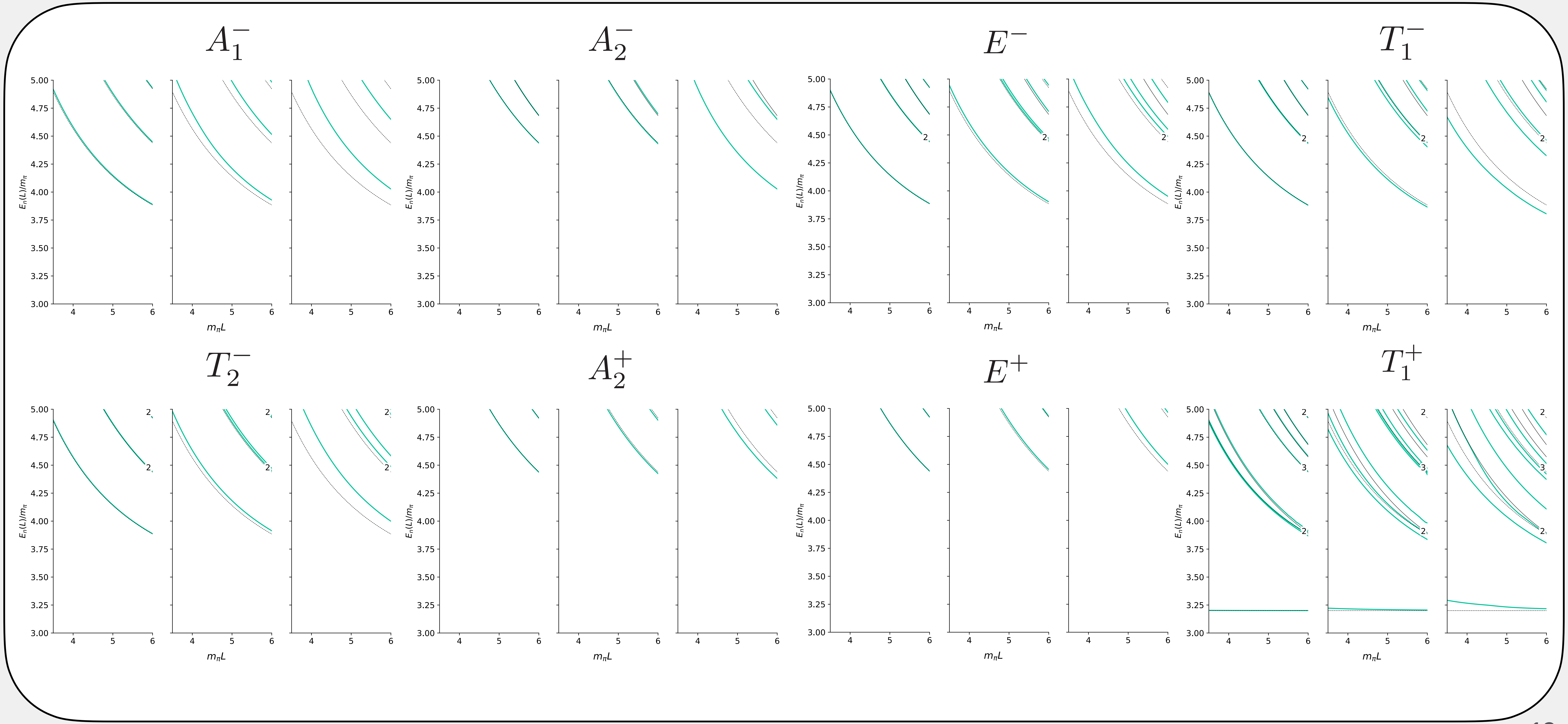
Example spectra of three pions with isospin 2

$I_{\pi\pi\pi} = 2 \qquad \boldsymbol{P}^2 = 1$

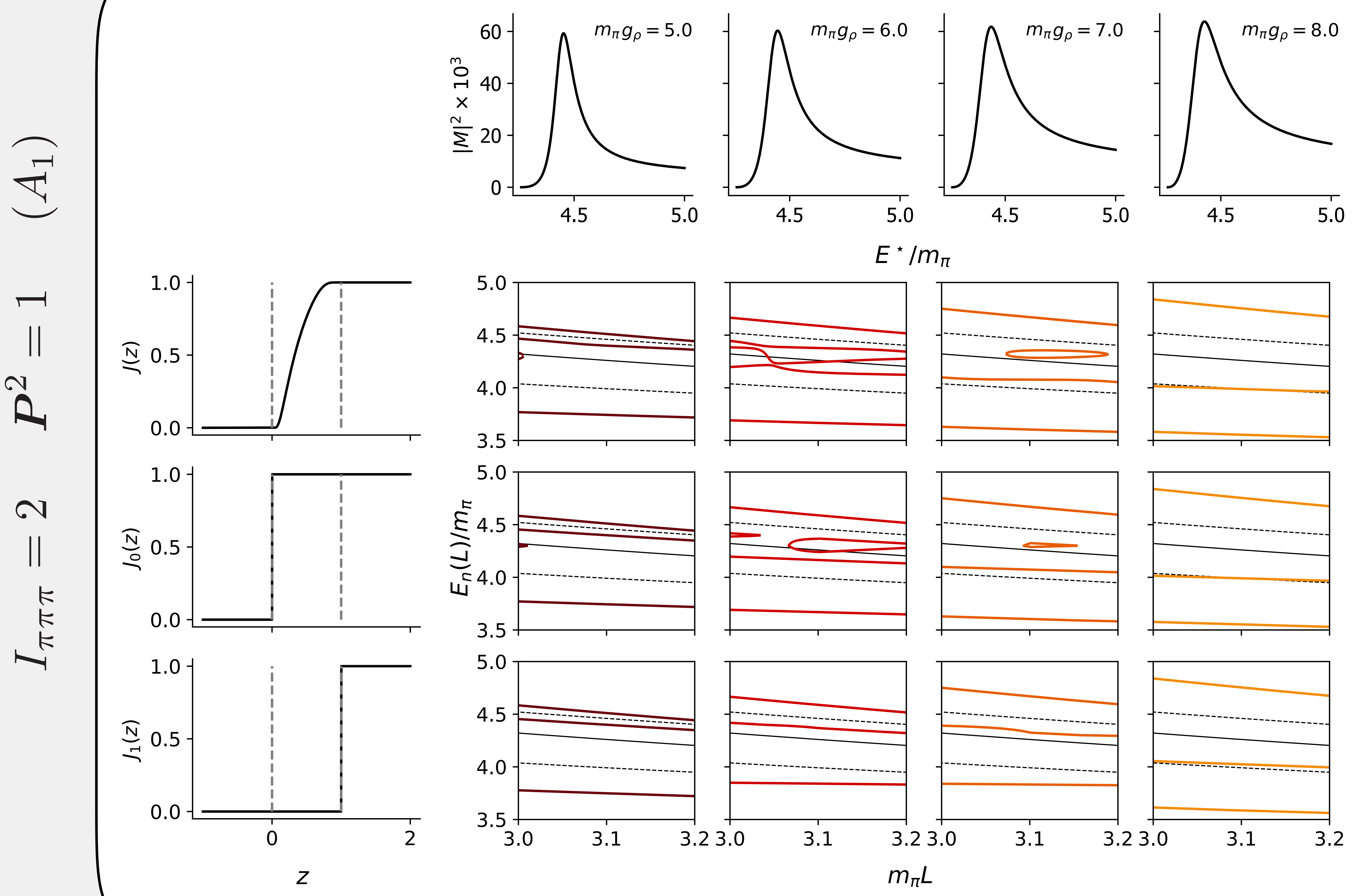


Example spectra of three pions with isospin 0

$I_{\pi\pi\pi} = 0 \quad \boldsymbol{P}^2 = 0$



Unphysical Solution



- Unphysical behaviour can arise for certain parameters, especially at small volumes.
- Cutoff function leading to enhanced neglected volume effects is one possible culprit:

$$H(\boldsymbol{k}) \equiv J(z),$$

$$z = \frac{E_{2,k}^{*2} - (1+\alpha)m^2}{(3-\alpha)m^2},$$

- Further investigation is required, but in this case, this is not too concerning for such small L.

Conclusion & outlook

Main takeaways:

- Quantitatively exploring RFT formalism for three pions with non-maximal isospin with realistic quantum numbers.
- The results can guide future lattice QCD calculations.

Future outlook:

- Fully implement decay formalism (e.g. $K \to \pi\pi\pi \quad \gamma^* \to \pi\pi\pi$).
- Implement other channels, including non-identical, non-degenerate particles and intrinsic spin.
- Further optimise ***ampyL*** library.
- Perform LQCD calculations of three-particle amplitudes.

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Higgs Center for Theoretical Physics
University of Edinburgh

THANK YOU FOR LISTENING.

All questions are welcome.