Three-particle formalism for multiple channels: the ηππ + KKπ system in isosymmetric QCD

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Based on work with Zack Draper, arXiv:2403.20064 (JHEP)

S.R.Sharpe, "Three-particle formalism for multiple channels…," LATTICE 2024, 7/29/2024

Status of 3-particle formalism [References at end]

- 3 identical spinless particles
	- Applications: $3\pi^+$, $3K^+$, as well as ϕ^4 theory [see talk by Fernando Romero-López (later this session)]
- Mixing of two- and three-particle channels for identical spinless particles
	- Step on the way to $N(1440) \rightarrow N\pi$, $N\pi\pi$, etc.
- 3 degenerate but distinguishable spinless particles, e.g 3π with isospin 0, 1, 2, 3
	- Potential applications: $\omega(782)$, $a_1(1260)$, $h_1(1170)$, $\pi(1300)$, ...
- 3 nondegenerate spinless particles
	- Potential applications: $D_s^+ D^0 \pi^-$
- 2 identical +1 different spinless particles
	- Applications: $\pi^+\pi^+K^+, K^+K^+\pi^+$ [see talk by Fernando Romero-López (later this session)]
- 3 identical spin- $\frac{1}{2}$ particles
	- Potential applications: $3n,~3p,~3\Lambda$ [see talk by Wilder Schaaf (Tuesday 11:35am "Structure of Hadrons..")
- *DDπ* for all isospins (also $BB\pi$, $KK\pi$)
	- Potential applications: $T_{cc}\to D^*D$ incorporating LH cut [see talk by Sebastian Dawid (later this session)]

What is missing?

- Non-indentical spin-1/₂ particles, e.g. ppn , nnp
	- Underway with Zack Draper, Max Hansen, & Fernando Romero-López
- Systems with mixed spins, e.g. $NN\pi$ with $I = 0,1,2$
	- Underway with Sebastian Dawid, Max Hansen, & Fernando Romero-López
- Systems with mixed spins that mix with 2-particle channels, e.g. Roper $\rightarrow p\pi\pi + p\pi$
	- Underway with Sebastian Dawid, Max Hansen, & Fernando Romero-López
- Multiple nondegenerate 3-particle channels, e.g. $b_1(1235)$, $\eta(1295) \to \pi\pi\eta + K\overline{K}\pi$
	- This talk!
- Combinations of the above
	- Should be straightforward…

- We work in isosymmetric QCD, so G parity is a good symmetry (also on the lattice)
	- $\pi \pi \eta$ has $G = +$, $K\overline{K}\pi$ has $G = \pm$ (depending on isospin & symmetry of $K\overline{K}$ pair)
	- We project on the $G = +$ sector, to avoid mixing with 3π

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- We consider unnatural $J^P(0^-, 1^+, 2^-, ...)$ to avoid mixing with 2π
	- In finite volume, must restrict to irreps that are not subduced from natural J^P
		- ★ **E.g.** A_{1u} in rest frame $(J^P = 0^-, 4^-, ...)$ and T_{1g} in rest frame $(J^P = 1^+, 3^+, ...)$

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- Allowed values of total isospin are $I = 0, 1, 2$
	- $I = 0$ contains $J^{PC} = 0^{-+}$ resonance $\eta(1295)$
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- Caveat: these resonances also decay to 4π , which we do not include, and could decay to (but are not seen in) $6\pi, 8\pi, 4\pi + \eta, ...$

Method of derivation

- Work in RFT approach [Hansen & SRS, 2014, …]
- Use time-ordered perturbation theory (TOPT) based method [Blanton & SRS, 2020]
	- Combines 2+1 formalism (for *ππη*) with nonidentical formalism (for *KKπ*)

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Method of derivation leaving the form of all equations unchanged. In the following, we assume that this rotation

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- Use time-ordered perturbation theory (TOPT) based method [Blanton & SRS, 2020] as in *Materia discussed*, the poles in the poles in a *in a letter the portugal* to the point of Ω SF and the point of Use time-ordered perturbation theory (TOPT) based method plant
	- Combines 2+1 formalism (for $\pi \pi \eta$) with nonidentical formalism (for $K\overline{K}\pi$) \mathcal{L} . The mansimally \bullet Combines 2+1 formalism (for
		- Result takes standard form $\frac{1}{2}$

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New features appearance of *S^D* follows from eq. (3.15). N N N T 2 T N S $\widehat{F}^{[I]} = \widehat{F}^{[I]} = \widehat{F}^{[I]} = \widehat{F}^{[I]}$ \mathbf{A} $\mathbf{$ label distinct isospin channels. The symmetry factor of 1*/*2 is explained in BS2, and the and **b** the contract appear here show \mathbf{h} that \mathbf{h} is \mathbf{h} in \mathbf{h} in \mathbf{h} in \mathbf{h} in \mathbf{h} New features and Cov is the symmetry factor of 1^{/2} Cov $\$

choices in eq. (2.1) as well as the conventions for momentum labels in eq. (3.24). The *a*

- choices in eq. (2.1) as well as the conventions for momentum labels in eq. (3.24). The *a* $\begin{array}{cc} \begin{array}{cc} 1 & 3 \end{array} & 3 & \begin{array}{cc} 3 & -1 \end{array} & (\widehat{\mathcal{K}}_{2,L}^{[I]})^{-1} + \end{array}$ $\det \left[1+\widehat{F}_3^{[I]}\widehat{\mathcal{K}}_{\text{df},3}^{[I]}\right.$ $= 0,$ $\widehat{K}_{2,L}^{[I]})^{-1}$ + (*K*'[*I*] ²*,L*)≠¹ ⁺ *^F*'[*I*] *G F*'[*I*] *.* (3.50) $\frac{1}{\sqrt{2}}\left[1 + \hat{E}^{[I]}\hat{\mathcal{E}}^{[I]} \right]$ <u>o</u> $\widehat{F}_3^{[I]} = \frac{\widehat{F}^{[I]}}{3} - \widehat{F}^{[I]} \frac{1}{(\widehat{\mathcal{K}}_{\alpha}^{[I]})^{-1}}$ $(\widehat{\mathcal{K}}_{2,L}^{[I]})^{-1} + \widehat{F}_G^{[I]}$ $\widehat{F}^{[I]}$.
- Multiple non-degenerate channels (notation is $[[pair-a, pair-b]_{I_{ab}}$, spectator]_{*I*}) *L* æ Œ limit; this, however, will be sucient for our purposes in the following. **3.3** Converting to the total is the tot <u>L E E R E Limited for our purposes in the following</u> with the following with the following with the following with the following with the isospin blocks, beginning with the isospin blocks, beginning with the isospin block where the explicit forms for the isospin blocks, beginning with the isospin blocks, beginning with those for $\overline{2}$, $\overline{2}$

 $I=2$ has 5: $[[K\bar{K}]_1\pi]_2, [[K\pi]_{3/2}\bar{K}]_2, [[\bar{K}\pi]_{3/2}K]_2, [[\pi\pi]_2\eta]_2, [[\pi\eta]_1\pi]_2,$ \mathbf{S} **J.** $\left[[\mathbf{I} \mathbf{I} \mathbf{I}] \mathbf{I}^n \$ η ₁ π ₂, *^F*'[*I*=2] *^G* =

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이 [[*fifi*]1*÷*]1*,* [[*fi÷*]1*fi*]1*,* $\int_{\mathbb{R}} [f(x, y)] \cdot f(x, y) \cdot$ $[[K\bar{K}]_1\pi]_1, [[K\bar{K}]_0\pi]_1, [[K\pi]_{3/2}\bar{K}]_1, [[K\pi]_{1/2}\bar{K}]_1, [[\bar{K}\pi]_{3/2}K]_1, [[\bar{K}\pi]_{1/2}K]_1$ $[[\pi\pi]_1\eta]_1, [[\pi\eta]_1\pi]_1,$ $\frac{1}{\sqrt{2\pi}}$

- $I=0$ has 5: $[[K\bar K]_1\pi]_0, [[K\pi]_{1/2}\bar K]_0, [[\bar K\pi]_{1/2}K]_0, [[\pi\pi]_0\eta]_0, [[\pi\eta]_1\pi]_0$ primary particle in the pair. The first five elements have *I* = 2, the next eight have *I* = 1, primary particle in the pair. The first five elements have *I* = 2, the next eight have *I* = 1,
- Kinematic factors are channel-dependent find that all matrices block diagonalize according to isospin. find the factor's are channer-dependent. where, as before, as before, the final entry corresponds to the spectator, and the first entry to the final en

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• Multiple non-degenerate channels (notation is $[[pair-a, pair-b]_{I_{ab}}$, spectator]_{*I*}) *L* æ Œ limit; this, however, will be sucient for our purposes in the following. **3.3** Converting to the total is the tot *L* æ Œ limit; this, however, will be sucient for our purposes in the following. 0 0 ^Ô2*P¸G*˜Õ*fi÷P^e ^F*˜Õ*fi* ⁺ *^G*˜Õ*fifi* Q *^PeKKK,I* ¯ =1 ²*,L ^P^e* 0 0 *^PeKfi÷*¡*KK,I* ¯ =1 R *K*'[*I*=2] *^PeKKK,I* ¯ =1 $\left[\text{cfactor}\right]_I$) where the explicit forms for the isospin blocks, beginning with the isospin blocks, beginning with those for $\overline{2}$, $\overline{2}$

 $I=2$ has 5: $[[K\bar{K}]_1\pi]_2, [[K\pi]_{3/2}\bar{K}]_2, [[\bar{K}\pi]_{3/2}K]_2, [[\pi\pi]_2\eta]_2, [[\pi\eta]_1\pi]_2,$ $I = 2$ has 5: $\lfloor [KK]_1 \pi \rfloor_2$, $\lfloor [K \pi]_{3/2} K \rfloor_2$, $\lfloor [K \pi]_{3/2} K \rfloor_2$, $\lfloor [\pi \pi]_2 \eta \rfloor_2$, $\lfloor [\pi \eta]_1 \pi \rfloor_2$, 2 has 5: $[[K\bar{K}]_1\pi]_2, [[K\pi]_{3/2}\bar{K}]_2, [[\bar{K}]_2$ 5: $[[K\bar{K}]_1\pi]_2, [[K\pi]_{3/2}\bar{K}]_2, [[\bar{K}\pi]_{3/2}K]_2$ ²*,L* 0 $\mathcal{L}[2,][\pi\pi]_2\eta|_2, [[\pi\eta]_1\pi]_2,$ $\frac{c}{c}$ 0 0 ¹ *^F*'[*I*=2] η ₁ π ₂, *^F*'[*I*=2] *^G* =

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이 [[*fifi*]1*÷*]1*,* [[*fi÷*]1*fi*]1*,* $\int_{\mathbb{R}} [f(x, y)] \cdot f(x, y) \cdot$ $\begin{split} \mathbf{C} \quad [[K\bar{K}]_1\pi]_1, [[K\bar{K}]_0\pi]_1, [[K\pi]_{3/2}\bar{K}]_1, [[K\pi]_{1/2}\bar{K}]_1, [[\bar{K}\pi]_{3/2}K]_1, [[\bar{K}\pi]_{1/2}K]_1 \end{split}$ $[[\pi\pi]_1\eta]_1, [[\pi\eta]_1\pi]_1,$ \int Several comments are in order. First, we note that the upper 2 ◊ 2 block in *^F*'[*I*=2] $\sum_{i=1}^{n}$ [*I*² $\binom{n}{1/2}$ **i**₁, [*I*² $\binom{n}{3/2}$ *i*₁, *[I*² $\binom{n}{1/2}$ *i*₁ \overline{r} $\overline{$ $\frac{1}{\sqrt{2\pi}}$

 $I = 0$ has 5: $[[K\bar{K}]_1\pi]_0, [[K\pi]_{1/2}\bar{K}]_0, [[\bar{K}\pi]_{1/2}K]_0, [[\pi\pi]_0\eta]_0, [[\pi\eta]_1\pi]_0$ primary particle in the pair. The first five elements have *I* = 2, the next eight have *I* = 1, primary particle in the pair. The first five elements have *I* = 2, the next eight have *I* = 1, Second, we could replace *PeF*Â*fiP^e* with *PeF*Â*fi*, due to the properties of *F*Â, as explained in $a_1 = a_1 + a_2 = a_3 + a_4 = a_5$ of the top-left entry of the top-left ent

- Kinematic factors are channel-dependent find that all matrices block diagonalize according to isospin. find the factor's are channer-dependent. where, as before, as before, the final entry corresponds to the spectator, and the first entry to the final en a *V*inometic fosters are shannel dopendent **•** Kinematic factors are channel-dependent in *^K*'[*I*=2] ²*,L* . Finally, the factors of *^P^e* acting on the odiagonal *fi÷* ¡ *KK* entries in *^K*'[*I*=2] can be dropped, since the *G* = + *fi÷* state only couples to the *KK* state with even partial
- T_{max} $C = \pm$ reduces channel count to 4.6 and 4.8 g n onto $G = +$ reduces channel count to 4, 6, and 4, e.g. onto $G = +$ reduces channel count to 4,6, and 4, e.g. m_{tot} of m_{tot} is concess channel count to π , σ , and π , σ . g . find that $G = +$ reduces channel count to 4, 6, and 4, e. The unitary matrix *C*(18) chæiso that converts between bases is given in appendix E. We • Projection onto $G = +$ reduces channel count to 4, 6, and 4, e.g. The results for *I* = 0 are similar The results for *I* = 0 are similar

$$
\hat{F}_{G}^{[I=0]} \rightarrow \begin{pmatrix}\nP_{e}\tilde{F}^{\pi}P_{e} & -\sqrt{2}P_{e}\tilde{G}^{\pi K}P_{\ell} & 0 & 0 \\
-\sqrt{2}P_{\ell}\tilde{G}^{K\pi}P_{e} & \tilde{F}^{K} + \tilde{G}^{KK} & 0 & 0 \\
0 & 0 & P_{e}\tilde{F}^{\eta\eta}P_{e} & \sqrt{2}P_{e}\tilde{G}^{\eta\pi}P_{\ell} \\
0 & 0 & \sqrt{2}P_{\ell}\tilde{G}^{\eta\pi\eta}P_{e} & \tilde{F}^{\eta\pi} + \tilde{G}^{\eta\pi} \\
\tilde{K}_{2,L}^{[I=0]} \rightarrow \begin{pmatrix}\nP_{e}\overline{K}_{2,L}^{K\bar{K},I=1}P_{e} & 0 & 0 & P_{e}\overline{K}_{2,L}^{\pi\eta\leftrightarrow K\bar{K},I=1} \\
0 & \overline{K}_{2,L}^{K\pi,I=1/2} & 0 & 0 \\
0 & 0 & \frac{1}{2}\overline{K}_{2,L}^{\pi\pi,I=0} & 0 \\
0 & 0 & \frac{1}{2}\overline{K}_{2,L}^{\pi\pi,I=0} & 0 \\
\overline{K}_{2,L}^{\pi\eta\leftrightarrow K\bar{K},I=1}P_{e} & 0 & 0 & \overline{K}_{2,L}^{\pi\eta,I=1}\n\end{pmatrix}\n\end{pmatrix}
$$

S.R.Sharpe, "Three-particle formalism for multiple channels…," LATTICE 2024, 7/29/2024 /9 /9 – 11 – and analogous comments apply. and analogous community community applications apply. The community community community community applications applications applications of the community community community community community community community communit

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$\hat{F}_3^{[I]} = \frac{\hat{F}^{[I]}}{3} - \hat{F}^{[I]} \frac{1}{(\hat{K}_{2,L}^{[I]})^{-1} + \hat{F}_G^{[I]}} \hat{F}^{[I]}$		

Form of $\mathcal{K}_{df,3}$: example of $I=0$ with the conjugate vectors given similarly. The superscripts *a, b* here refer to the two inde- Form of $\mathcal{K}_{\mathsf{a} \mathsf{b} \mathsf{c}}$: example of $I=0$ *K Kfi*. Process **finally for the process** *fift***¹. PT in process** *fight***₁. PT is a fightpoon of** $\frac{d}{dx}$ **in the process of** $\frac{d}{dx}$ **is a fightpoon of** $\frac{d}{dx}$ **is a fightpoon of** $\frac{d}{dx}$ **is a fightpoon of ** f *^PeKKK,I* ¯ =1 ²*,L ^P^e* 0 0 00 *^PeKfi÷*¡*KK,I* ¯ =1 2*,L* \int / df.3[.] CACILIPIC ULL — C 0 00 *^KKfi,I*=1*/*² operator *^X ‡* [*kab*] acts on a vector *fk¸^m* as Ë *X ‡* [*kab*] ¶ *f* È \mathcal{P} ^{*p*} d *<i>I*, **³** iii D **I**3 = 0 components of the following seven channels, the following seven channels, the following seven channels, the following seven channels, and the following seven channels, and the following seven channels, and the fol *DI* U df ², *df*₃. CXaIIIPIC

- Two underlying channels: matrices, as claimed above. Finally, we note that, the note that, to obtain the results, we have used the results, we α **I** is a finite of α is summetric under the interchange of α is summetric the two pions. Two underlying channels: $\quad a) \, \, [[K\bar K]_1\pi]_0, \,\, b) \,\, [[\pi\pi]_0\eta]_0$ **a** *a*^{f} (f f) (f f f) (f f f) (I ahannels: $a)$ $[[K\bar{K}]_{1}\pi]_{0}, \; b)$ $[[\pi\pi]_{0}\eta]_{0}$;
- Assuming PT symmetry, there are 3 independent amplitudes: *aa*, *ab* = *ba*, *bb* S *Suming* PT symmetry, there are 3 independent amplitudes: aa , $ab = ba$, bb Assuming F Γ symmetry, there are 3 independent amplitudes. $\ddot{a}u, \dot{a}v = v\dot{a}, \dot{v}\dot{v}$ $\mathop{\mathsf{mmetry}},$ there are 3 independent amplitudes: $aa,ab=ba,bb$ boosted to the CMF of the nonspectator pair, is set to the direction of *p‡*² . For three on- λ 3 and μ ₂ and μ *px*₂ and *p*^{*x*} and *p*² and *p*² and *b* μ ² μ $\bullet\,$ Assuming PT symmetry, there are 3 independent amplitudes: $aa, ab = ba, bb$
- Convert to 4-d matrix form using projection operators: The result for the *I* = 0 block takes the same form as for *I* = 2, t-a matr σ ³ . The result is a function of the result is a function of the three ones parameters ρ Convert to 4-d matrix form using projection *A* e Convert to 4-d matrix form using projection operators: decomposing under isospin, the symmetric and antisymmetric parts of *fi*+*fi*[≠] are treated

$$
\hat{\mathcal{K}}_{\text{df},3}^{[I=0]} = \sum_{x,y \in \{a,b\}} \mathbf{y}^{[I=0],x} \circ \mathcal{K}_{\text{df},3}^{[I=0],xy}(\{\boldsymbol{p}\},\{\boldsymbol{k}\}) \circ \mathbf{y}^{[I=0],y\dagger}
$$
\n
$$
\mathbf{y}^{[I=0],a\dagger} \rightarrow \left(\frac{1}{2}(\mathbf{y}_{(312)}^{[kab]\dagger} + \mathbf{y}_{(321)}^{[kab]\dagger}), -\sqrt{\frac{1}{2}}(\mathbf{y}_{(213)}^{[kab]\dagger} + \mathbf{y}_{(123)}^{[kab]\dagger}), 0, 0\right),
$$
\n
$$
\mathbf{y}^{[I=0],b\dagger} \rightarrow \left(0, 0, \sqrt{\frac{1}{2}}\mathbf{y}_{(312)}^{[kab]\dagger}, \mathbf{y}_{(213)}^{[kab]\dagger}\right),
$$
\n
$$
\left[\mathbf{y}_{\sigma}^{[kab]} \circ g\right]_{k\ell m} = \frac{1}{4\pi} \int d\Omega_{a^*} Y_{\ell m}(\hat{a}^*) g(\{p_i\}) \Big|_{p_{\sigma(1)} \rightarrow k, \ p_{\sigma(2)} \rightarrow a, \ p_{\sigma(3)} \rightarrow b}
$$

 \cdot Can expand underlying amplitudes in a threshold expansion, or assume a pole form in the *^Y*[*I*=1]*,x* ¶ *^K*[*I*=1]*,xy* presence of a resonance—both are constrained by symmetries $\frac{1}{\sqrt{2\pi}}$ momentum, leaving *p‡*² and *p‡*³ to be the remaining pair. We boost to the CMF of this Lan expand underlying amplitudes in a threshold expansion, or assume a pole **f 6** Can expand underlying amplitudes in a threshold expansion, or assume a pole form in the presence of a resonance—both are constrained by symmetries • Can expand underlying amplitudes in a threshold expansion, or assume a pole form in the

Reduction to single channel

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Automatically *ππη* only

Must use full $\pi \pi \eta + K \overline{K} \pi$ formalism

Reduction to single channel

- Intermediate region where can use either $\pi \pi \eta$ only or $\pi \pi \eta + K \overline{K} \pi$ formalism
- Two 3-particle channel formalism must reduce to that for single channel if integrate out offshell $K\overline{K}\pi$ intermediate states
- We have shown explicitly how this works—provides a cross-check on formalism

Summary & Outlook

- Generalization of QC3 to two 3-particle channels is relatively straightforward
	- New feature of G-parity projection not directly related to multiple channels
	- Decoupling of channel as drop below threshold is understood
- Extending formalism to include additional 3-particle channels will be simple
- Application to LQCD lies some way in the future
	- Working with heavier-than-physical pion masses will reduce issue of neglected four-pion channels
- Main present motivation was to understand/extend formalism

Thanks Any questions?

S.R.Sharpe, "Three-particle formalism for multiple channels...," LATTICE 2024, 7/29/2024

References

(Highly-)selected 2-particle refs

★ Original papers

- ^M. Lüscher, Commun.Math.Phys. 105 (1986) 153-188; Nucl.Phys.B 364 (1991) 237-251 [Derived QC2 using NRQM and proving relation to QFT]
- ^L. Lellouch & M. Lüscher, Commun.Math.Phys. 219 (2001) 31-44; arXiv:hep-lat/0003023 [Determined LL factors relating finite- and infinite-volume matrix elements]

★ Generalizations

- ^C. Kim, C. Sachrajda, & SRS, Nucl.Phys.B 727 (2005) 218-243; arXiv:hep-lat/0507006 [QFT-based approach; LL factors in moving frames]
- ^R. Briceño, Phys.Rev.D 89 (2014) 7, 074507; arXiv:1401.3312 [QC2 for arbitrary spin]

RFT 3-particle papers

Max Hansen & SRS:

 arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

 arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box," arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

S. Sharpe, ``Multiparticle scattering from LQCD," Amplitudes24, 6/12/24

Raúl Briceño, Max Hansen & SRS:

"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17] "Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18] "Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

"Testing the threshold expansion for three-particle energies at fourth order in φ⁴ theory," arXiv:1707.04279 (PRD) [SPT17]

Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

"I=3 three-pion scattering amplitude from lattice QCD," arXiv:[1909.02973](https://arxiv.org/abs/1909.02973) (PRL) [BRS-PRL19]

S. Sharpe, ``Multiparticle scattering from LQCD,'' Amplitudes24, 6/12/24 14/15 **"Implementing the three-particle quantization condition** for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)

Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms,'' arXiv:1905.12007 (PRD)

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Other work

★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), $\frac{2009.04931}{2}$ $\frac{2009.04931}{2}$ $\frac{2009.04931}{2}$, PRL [Calculating $3\pi^{+}$ spectrum and using to determine three-particle scattering amplitude]
- ^A. Jackura et al., [2010.09820](https://arxiv.org/abs/2010.09820), PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam and R. Briceño, [2303.04394](https://arxiv.org/abs/2303.04394) [Analytic continuation of 3-particle amplitudes]
- ^A. Jackura, 2208.10587, PRD [3-body scattering and quantization conditions from S-matrix unitarity]

★ Reviews

- A. Rusetsky, [1911.01253](https://arxiv.org/abs/1911.01253) [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, [2103.00577](https://arxiv.org/abs/2103.00577) [Review of formalisms and chiral extrapolations]
- ^F. Romero-López, 2112.05170, [Three-[particle scattering amplitudes from lattice QCD](https://inspirehep.net/literature/1987770)]

★ Other numerical simulations

- F. Romero-López, A. Rusetsky, C. Urbach, $\frac{1806.02367}{1806.02367}$, JHEP [2- & 3-body interactions in φ^4 theory]
- M. Fischer et al., $\frac{2008.03035}{2}$ $\frac{2008.03035}{2}$ $\frac{2008.03035}{2}$, Eur.Phys.J.C $[2\pi^+ \otimes 3\pi^+]$ at physical masses]
- M. Garofolo et al., $\frac{2211.05605}{211.05605}$ $\frac{2211.05605}{211.05605}$ $\frac{2211.05605}{211.05605}$, JHEP [3-body resonances in φ^4 theory]

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Other work

★ Other RFT (and related) derivations

- ^A. Jackura, 2208.10587, PRD [3-body scattering and quantization conditions from S-matrix unitarity]
- ^R. Briceño, A. Jackura, D. Pefkou & F. Romero-López, 2402.12167, JHEP [Electroweak three-body decays in the presence of two- and three-body bound states]

★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), $\frac{2009.04931}{2}$ $\frac{2009.04931}{2}$ $\frac{2009.04931}{2}$, PRL [Calculating $3\pi^{+}$ spectrum and using to determine three-particle scattering amplitude]
- ^A. Jackura et al., [2010.09820](https://arxiv.org/abs/2010.09820), PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam & R. Briceño, [2303.04394](https://arxiv.org/abs/2303.04394), PRD [Analytic continuation of 3-particle amplitudes]
- ^S. Dawid, Md. Islam, R. Briceño, & A. Jackura, 2309.01732 [Evolution of Efimov States]
- ^A. Jackura & R. Briceño, 2312.00625 [Partial-wave projection of the one-particle exchange in three-body scattering amplitudes]

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★ ^NREFT approach Other work

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, [1706.07700](http://arxiv.org/abs/arXiv:1706.07700), JHEP & 1707.02176, JHEP [Formalism & examples]
- ^M. Döring et al., [1802.03362](http://arxiv.org/abs/arXiv:1802.03362) , PRD [Numerical implementation]
- J.-Y. Pang et al., [1902.01111](http://arxiv.org/abs/arXiv:1902.01111), PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, [2011.14178](https://arxiv.org/abs/2011.14178), PRD [large volume expansion for I=1 three pion ground state]
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- ^F. Müller & A. Rusetsky, [2012.13957,](https://arxiv.org/abs/2012.13957) JHEP [Three-particle analog of Lellouch-Lüscher formula]
- J-Y. Pang, M. Ebert, H-W. Hammer, F. Müller, A. Rusetsky, [2204.04807](https://arxiv.org/abs/2204.04807), JHEP, [Spurious poles in a finite volume]
- ^F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, [2110.09351](https://arxiv.org/abs/2110.09351), JHEP [Relativistic-invariant formulation of the NREFT threeparticle quantization condition]
- ^J. Lozano, U. Meißner, F. Romero-López, A. Rusetsky & G. Schierholz, [2205.11316](https://arxiv.org/abs/2205.11316), JHEP [Resonance form factors from finite-volume correlation functions with the external field method]
- ^F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, [2211.10126](https://arxiv.org/abs/2211.10126), JHEP [3-particle Lellouch-Lüscher formalism in moving frames]
- R. Bubna, F. Müller, A. Rusetsky, [2304.13635](https://arxiv.org/abs/2304.13635) [Finite-volume energy shift of the three-nucleon ground state]
- \bullet J-Y. Pang, R. Bubna, F. Müller, A. Rusetsky, J-J. Wu, 2312.04391 [Lellouch-Lüscher factor for $K \to 3\pi$ decays]
- ^R. Bubna, H-W. Hammer, F. Müller, J-Y. Pang, A. Rusetsky, 2402.12985 [Lüscher equation with long range forces]

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Alternate 3-particle approaches

★ Finite-volume unitarity (FVU) approach

- ^M. Mai & M. Döring, [1709.08222](http://arxiv.org/abs/arXiv:1709.08222) , EPJA [formalism]
- M. Mai et al., [1706.06118,](http://arxiv.org/abs/arXiv:1706.06118) EPJA [unitary parametrization of M₃ involving R matrix; used in FVU approach]
- A. Jackura et al., [1809.10523](https://arxiv.org/abs/1809.10523), EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., <u>[1909.05749](https://arxiv.org/abs/1909.05749)</u>, PRD [applying FVU approach to 3π ⁺spectrum from Hanlon & Hörz]
- C. Culver et al., <u>1911.09047</u>, PRD [calculating 3π ⁺ spectrum and comparing with FVU predictions]
- A. Alexandru et al., [2009.12358](https://arxiv.org/abs/2009.12358), PRD [calculating 3K⁻ spectrum and comparing with FVU predictions]
- R. Brett et al., $\frac{2101.06144}{2}$ $\frac{2101.06144}{2}$ $\frac{2101.06144}{2}$, PRD [determining $3\pi^{+}$ interaction from LQCD spectrum]
- \bullet M. Mai et al., [2107.03973,](https://arxiv.org/abs/2107.03973) PRL [three-body dynamics of the $a_1(1260)$ from LQCD]
- D. Dasadivan et al., $\frac{2112.03355}{2}$ $\frac{2112.03355}{2}$ $\frac{2112.03355}{2}$, PRD [pole position of $a_1(1260)$ in a unitary framework]
- ^D. Seivert, M. Mai, U-G. Meißner, 2212.02171, JHEP [Particle-dimer approach for the Roper resonance]

★ HALQCD approach

• T. Doi et al. (HALQCD collab.), [1106.2276](http://arxiv.org/abs/arXiv:1106.2276), Prog. Theor. Phys. [3 nucleon potentials in NR regime]

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Backup slides

S.R.Sharpe, "Three-particle formalism for multiple channels...," LATTICE 2024, 7/29/2024

Forms of F and G seen below, the expressions presented here have a dimension-8 space. The reason is that ϵ in the *P* α borms of F and α Γ the system: Ÿ Ë 1 + *^K*'[*I*] df*,*³ *^F*'[*I*] È where \Box *I* labels the system of the system. The system of the sys blocks is as follows: 2 for *^I* = 0 and 2, and 4 for *^I* = 1. Parametrizations of *^K*'[*I*] df*,*³ in each ◊ U *^Y¸*Õ*m*Õ(*a*ú(*i,j,p*)) ! *q* ú(*i*) 2*,p*Õ "*¸*Õ 4*Ê*(*j*) *^a Ê*(*k*) *b* ! *^E*≠*Ê*(*i*) isospin channel will be discussed below. The explicit relation between *^F*'[*I*] ³ , the two-particle $K = \sum_{k=1}^{n}$ of the two-meson relative momentum in their CMF, and *a*ú(*i,j,p*) four momentum (*Ê*(*j*) *^a , a*) after boosting it with boost velocity *—*(*i*) *^p* ⁼ [≠](*^P* [≠] *^p*)*/*(*^E* [≠] *^Ê*(*j*)

• Symmetric form of QC3 takes the by-now familiar form the total isospin of the system: where the superindex \mathcal{I} is the system. The system of the system of the system. The dimensionality of the system of the system of the system of the system of the system. The dimensionality of the system of the system \bullet Symmetric form of QC3 takes the by-now familiar form \bullet $S_{\text{unmutation}}$ ϵ Ω *n* of QC3 ta orm of QC3 takes the by-now familiar form $\sum_{i=1}^{n}$ where $\sum_{i=1}^{n}$ is suit to same $\sum_{i=1}^{n}$

of each entry is

$$
\prod_{I \in \{0,1,2\}} \det_{i,k,\ell,m} \left[1 + \hat{\mathcal{K}}_{\text{df},3}^{[I]} \; \hat{F}_{3}^{[I]} \right] = 0
$$
\n
$$
\hat{F}_{3}^{[I]} = \frac{\hat{F}^{[I]}}{3} - \hat{F}^{[I]} \frac{1}{1 + \hat{\mathcal{M}}_{2,L}^{[I]} \hat{G}^{[I]}} \hat{\mathcal{M}}_{2,L}^{[I]} \hat{F}^{[I]}, \qquad \hat{\mathcal{M}}_{2,L}^{[I]} = \frac{1}{\hat{\mathcal{K}}_{2,L}^{[I]-1} + \hat{F}^{[I]}}
$$
\n
$$
\hat{F}^{[I=0]} = \text{diag } (\tilde{F}^{D}, \tilde{F}^{\pi}) \qquad \hat{G}^{[I=0]} = \begin{pmatrix} G^{DD} & \sqrt{2}P_{\ell}G^{D\pi} \\ \sqrt{2}G^{\pi D}P_{\ell} & 0 \end{pmatrix}
$$
\n
$$
\left[\tilde{F}^{(i)} \right]_{p'\ell'm':p\ell m} = \delta_{p'p} \frac{H^{(i)}(p)}{2\omega_{p}^{(i)}L^{3}} \left[\frac{1}{L^{3}} \sum_{a}^{UV} -\text{PV} \int^{UV} \frac{d^{3}a}{(2\pi)^{3}} \right] \left[\frac{\mathcal{Y}_{\ell m'}(a^{*(i,j,p)})}{(a_{2,p'}^{(i,p)})^{\ell}} \frac{1}{4\omega_{a}^{(j)}\omega_{b}^{(k)}(E - \omega_{p}^{(i)} - \omega_{a}^{(j)} - \omega_{b}^{(k)})} \frac{\mathcal{Y}_{\ell m}(a^{*(i,j,p)})}{(a_{2,p}^{*(j)})^{\ell}} \right]
$$
\n
$$
\left[\tilde{G}^{(ij)} \right]_{p\ell'm':r\ell m} = \frac{1}{2\omega_{p}^{(i)}L^{3}} \frac{\mathcal{Y}_{\ell m'}(r^{*(i,j,p)})}{(q_{2,p}^{*(i)})^{\ell}} \frac{H^{(i)}(p)H^{(j)}(r)}{b_{ij}^{2} - m_{k}^{2}} \frac{\mathcal{Y}_{\ell m}(p^{*(j,i,r)})}{(q_{2,r}^{*(j)})^{\ell}} \frac{1}{2\
$$

S.R.Sharpe, "Three-particle formalism for multiple channels…," LATTICE 2024, 7/29/2024 23 /9 The superindex in each of the matrix entries labels the spectator particle. The definition S.R.Sharpe, "Three-particle formalism for multiple channels...," LATTICE ^Ô2*GfiDP¸* ⁰ Q ≠1 3*GDD* ² Ô2 ³ *^GDD* ^Ò² ³*P¸GDfi* [≠] ^Ô *K*'[*I*=1] ²*,L* = diag ³ *KDfi,I*=3*/*² ²*,L , ^KDfi,I*=1*/*² ²*,L ,* 2 *KDD,I*=1 ²*,L ,* 2 *KDD,I*=0 ²*,L* ⁴ ²*,L* ⁴