Three-particle formalism for multiple channels: the $\eta\pi\pi$ + $KK\pi$ system in isosymmetric QCD

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Based on work with Zack Draper, arXiv:2403.20064 (JHEP)

Status of 3-particle formalism
[References at end]

- 3 identical spinless particles
  - Applications: $3\pi^+,\ 3K^+$, as well as ϕ^4 theory [see talk by Fernando Romero-López (later this session)]
- Mixing of two- and three-particle channels for identical spinless particles
  - Step on the way to $N(1440) \to N\pi,\ N\pi\pi$, etc.
- 3 degenerate but distinguishable spinless particles, e.g 3π with isospin 0, 1, 2, 3
  - Potential applications: $\omega(782),\ a_1(1260),\ h_1(1170),\ \pi(1300), \ldots$
- 3 nondegenerate spinless particles
  - Potential applications: $D_s^+ D^0 \pi^-$
- 2 identical +1 different spinless particles
  - Applications: $\pi^+\pi^+K^+, K^+K^+\pi^+$ [see talk by Fernando Romero-López (later this session)]
- 3 identical spin-½ particles
  - Potential applications: $3n,\ 3p,\ 3\Lambda$ [see talk by Wilder Schaaf (Tuesday 11:35am "Structure of Hadrons..")
- $DD\pi$ for all isospins (also $BB\pi,\ KK\pi$)
  - Potential applications: $T_{cc} \to D^*D$ incorporating LH cut [see talk by Sebastian Dawid (later this session)]

What is missing?

- Non-indentical spin-½ particles, e.g. $ppn,\ nnp$
  - Underway with Zack Draper, Max Hansen, & Fernando Romero-López
- Systems with mixed spins, e.g. $NN\pi$ with $I = 0,1,2$
  - Underway with Sebastian Dawid, Max Hansen, & Fernando Romero-López
- Systems with mixed spins that mix with 2-particle channels, e.g. Roper $\rightarrow p\pi\pi + p\pi$
  - Underway with Sebastian Dawid, Max Hansen, & Fernando Romero-López
- Multiple nondegenerate 3-particle channels, e.g. $b_1(1235), \eta(1295) \rightarrow \pi\pi\eta + K\overline{K}\pi$
  - This talk!
- Combinations of the above
  - Should be straightforward…

Example: $\pi\pi\eta + K\overline{K}\pi$

- We work in isosymmetric QCD, so G parity is a good symmetry (also on the lattice)
  - $\pi\pi\eta$ has $G = +$, $K\overline{K}\pi$ has $G = \pm$ (depending on isospin & symmetry of $K\overline{K}$ pair)
  - We project on the $G = +$ sector, to avoid mixing with 3π

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- We consider unnatural $J^P(0^-, 1^+, 2^-, \ldots)$ to avoid mixing with 2π
  - In finite volume, must restrict to irreps that are not subduced from natural J^P
    - ✳ E.g. A_{1u} in rest frame ($J^P = 0^-, 4^-, \ldots$) and T_{1g} in rest frame ($J^P = 1^+, 3^+, \ldots$)

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- Allowed values of total isospin are $I = 0,\ 1,\ 2$
  - $I = 0$ contains $J^{PC} = 0^{-+}$ resonance $\eta(1295)$
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  - $I = 0$ contains $J^{PC} = 0^{-+}$ resonance $\eta(1295)$
  - $I = 1$ contains $J^{PC} = 1^{+-}$ resonance $b_1(1235)$
- Caveat: these resonances also decay to 4π, which we do not include, and could decay to (but are not seen in) $6\pi, 8\pi, 4\pi + \eta, \ldots$

Method of derivation

- Work in RFT approach [Hansen & SRS, 2014, …]
- Use time-ordered perturbation theory (TOPT) based method [Blanton & SRS, 2020]
  - Combines 2+1 formalism (for $\pi\pi\eta$) with nonidentical formalism (for $K\overline{K}\pi$)

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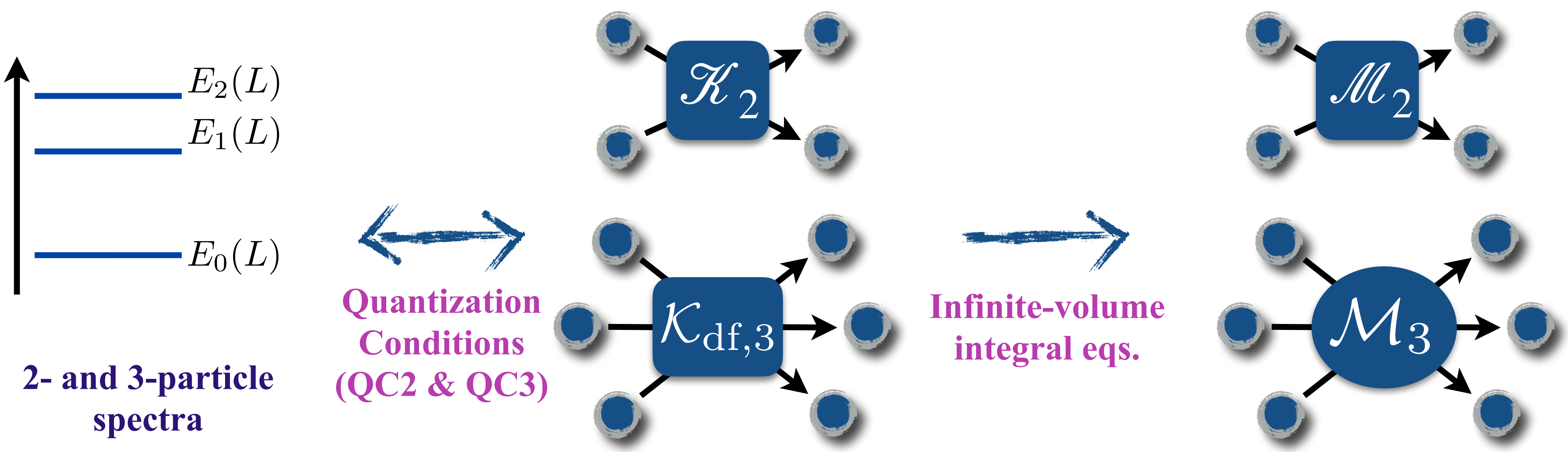

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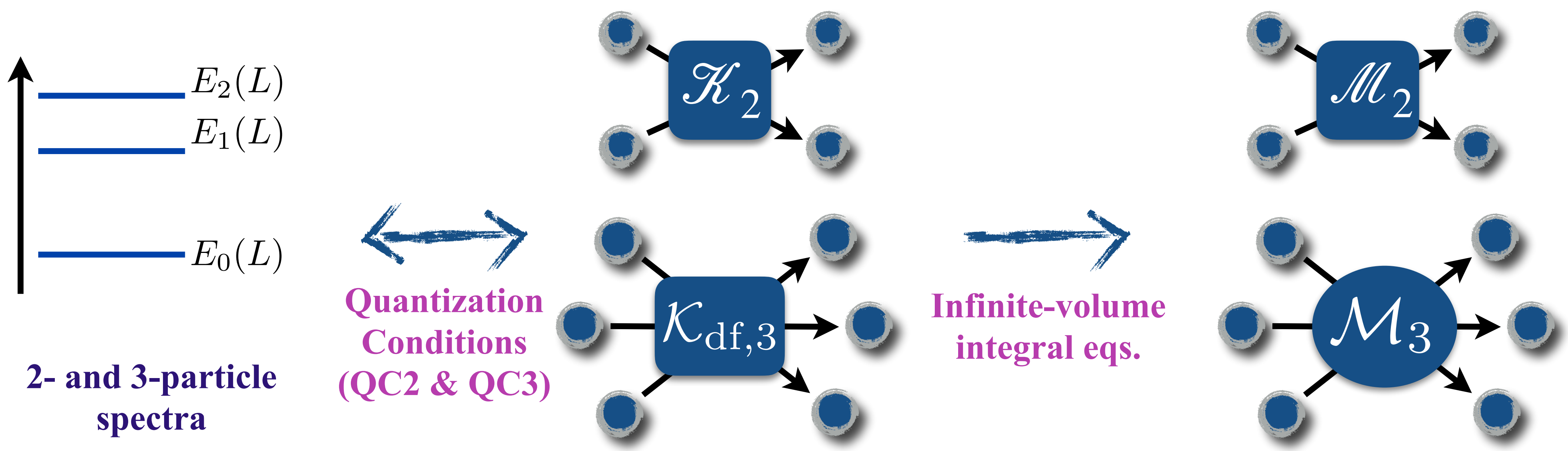

QC3: $\det\left[1 + \widehat{F}_3^{[I]}\widehat{\mathcal{K}}_{\mathrm{df},3}^{[I]}\right] = 0\,,$

$$\widehat{F}_3^{[I]} = \frac{\widehat{F}^{[I]}}{3} - \widehat{F}^{[I]}\frac{1}{(\widehat{\mathcal{K}}_{2,L}^{[I]})^{-1} + \widehat{F}_G^{[I]}}\widehat{F}^{[I]}$$

Matrix indices are: channel, $\mathbf{k}$, ℓm

New features

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- Multiple non-degenerate channels (notation is $[[\text{pair-}a,\ \text{pair-}b]_{I_{ab}}, \text{spectator}]_I$)

$I=2$ has 5: $[[K\bar{K}]_1\pi]_2, [[K\pi]_{3/2}\bar{K}]_2, [[\bar{K}\pi]_{3/2}K]_2, [[\pi\pi]_2\eta]_2, [[\pi\eta]_1\pi]_2,$

$I=1$ has 8: $[[K\bar{K}]_1\pi]_1, [[K\bar{K}]_0\pi]_1, [[K\pi]_{3/2}\bar{K}]_1, [[K\pi]_{1/2}\bar{K}]_1, [[\bar{K}\pi]_{3/2}K]_1, [[\bar{K}\pi]_{1/2}K]_1$
$[[\pi\pi]_1\eta]_1, [[\pi\eta]_1\pi]_1,$

$I=0$ has 5: $[[K\bar{K}]_1\pi]_0, [[K\pi]_{1/2}\bar{K}]_0, [[\bar{K}\pi]_{1/2}K]_0, [[\pi\pi]_0\eta]_0, [[\pi\eta]_1\pi]_0$

  - Kinematic factors are channel-dependent

New features

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  - Kinematic factors are channel-dependent

- Projection onto $G=+$ reduces channel count to 4, 6, and 4, e.g.

$$\widehat{F}_G^{[I=0]}\to\begin{pmatrix} P_e\tilde{F}^{\pi}P_e & -\sqrt{2}P_e\tilde{G}^{\pi K}P_\ell & 0 & 0\\ -\sqrt{2}P_\ell\tilde{G}^{K\pi}P_e & \tilde{F}^K+\tilde{G}^{KK} & 0 & 0\\ 0 & 0 & P_e\tilde{F}'^{\eta}P_e & \sqrt{2}P_e\tilde{G}'^{\eta\pi}P_\ell\\ 0 & 0 & \sqrt{2}P_\ell\tilde{G}'^{\pi\eta}P_e & \tilde{F}'^{\pi}+\tilde{G}'^{\pi\pi}\end{pmatrix}$$

$$\widehat{\mathcal{K}}_{2,L}^{[I=0]}\to\begin{pmatrix} P_e\overline{\mathcal{K}}_{2,L}^{K\bar{K},I=1}P_e & 0 & 0 & P_e\overline{\mathcal{K}}_{2,L}^{\pi\eta\leftrightarrow K\bar{K},I=1}\\ 0 & \overline{\mathcal{K}}_{2,L}^{K\pi,I=1/2} & 0 & 0\\ 0 & 0 & \frac{1}{2}\overline{\mathcal{K}}_{2,L}^{\pi\pi,I=0} & 0\\ \overline{\mathcal{K}}_{2,L}^{\pi\eta\leftrightarrow K\bar{K},I=1}P_e & 0 & 0 & \overline{\mathcal{K}}_{2,L}^{\pi\eta,I=1}\end{pmatrix}$$

Form of $\mathscr{K}_{\rm df,3}$: example of $I=0$

- Two underlying channels: $a)\ [[K\bar{K}]_1\pi]_0,\ \ b)\ [[\pi\pi]_0\eta]_0$
- Assuming PT symmetry, there are 3 independent amplitudes: $aa, ab=ba, bb$
- Convert to 4-d matrix form using projection operators:

$$\widehat{\mathcal{K}}_{\rm df,3}^{[I=0]} = \sum_{x,y\in\{a,b\}} \boldsymbol{\mathcal{Y}}^{[I=0],x} \circ \mathcal{K}_{\rm df,3}^{[I=0],xy}(\{\boldsymbol{p}\},\{\boldsymbol{k}\}) \circ \boldsymbol{\mathcal{Y}}^{[I=0],y\dagger}$$

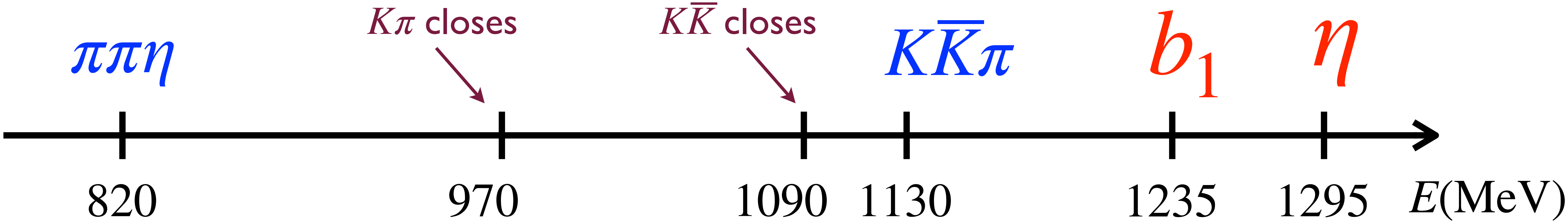
$$\boldsymbol{\mathcal{Y}}^{[I=0],a\dagger} \to \left(\tfrac{1}{2}(\boldsymbol{\mathcal{Y}}^{[kab]\dagger}_{(312)} + \boldsymbol{\mathcal{Y}}^{[kab]\dagger}_{(321)}), -\sqrt{\tfrac{1}{2}}(\boldsymbol{\mathcal{Y}}^{[kab]\dagger}_{(213)} + \boldsymbol{\mathcal{Y}}^{[kab]\dagger}_{(123)}),\ 0,\ 0\right),$$

$$\boldsymbol{\mathcal{Y}}^{[I=0],b\dagger} \to \left(0,\ 0,\ \sqrt{\tfrac{1}{2}}\boldsymbol{\mathcal{Y}}^{[kab]\dagger}_{(312)},\ \boldsymbol{\mathcal{Y}}^{[kab]\dagger}_{(213)}\right),$$

$$\left[\boldsymbol{\mathcal{Y}}^{[kab]}_{\boldsymbol{\sigma}} \circ g\right]_{k\ell m} = \frac{1}{4\pi}\int d\Omega_{a^*} Y_{\ell m}(\hat{a}^*) g(\{p_i\})\Big|_{p_{\sigma(1)}\to k,\ p_{\sigma(2)}\to a,\ p_{\sigma(3)}\to b}$$

- Can expand underlying amplitudes in a threshold expansion, or assume a pole form in the presence of a resonance—both are constrained by symmetries

Reduction to single channel



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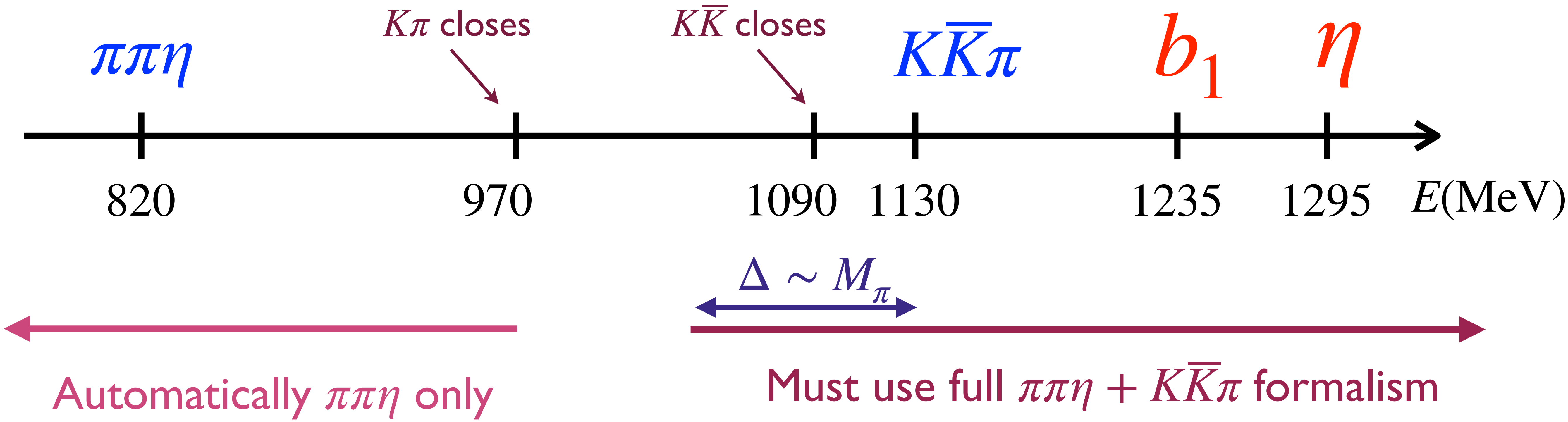

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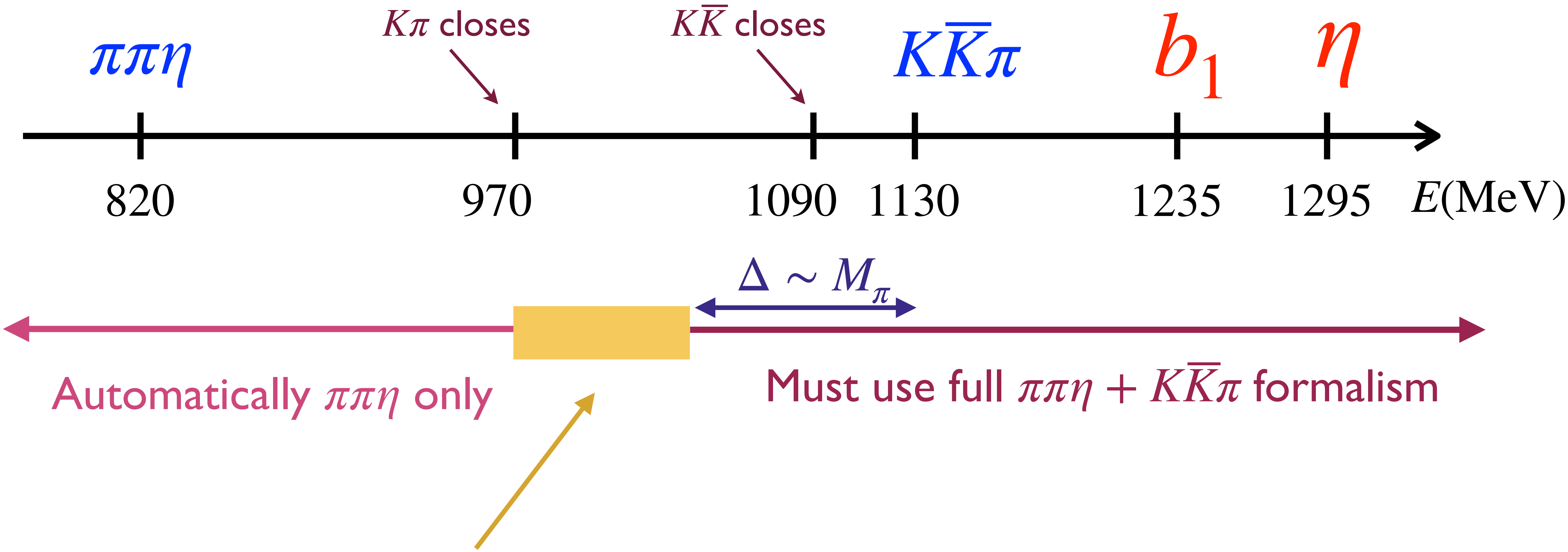

- Intermediate region where can use either $\pi\pi\eta$ only or $\pi\pi\eta + K\overline{K}\pi$ formalism
- Two 3-particle channel formalism must reduce to that for single channel if integrate out off-shell $K\overline{K}\pi$ intermediate states
- We have shown explicitly how this works—provides a cross-check on formalism

Summary & Outlook

- Generalization of QC3 to two 3-particle channels is relatively straightforward
    - New feature of G-parity projection not directly related to multiple channels
    - Decoupling of channel as drop below threshold is understood
- Extending formalism to include additional 3-particle channels will be simple
- Application to LQCD lies some way in the future
    - Working with heavier-than-physical pion masses will reduce issue of neglected four-pion channels
- Main present motivation was to understand/extend formalism

Thanks
Any questions?

References

(Highly-)selected 2-particle refs

★ Original papers

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- L. Lellouch & M. Lüscher, Commun.Math.Phys. 219 (2001) 31-44; arXiv:hep-lat/0003023 [Determined LL factors relating finite- and infinite-volume matrix elements]

★ Generalizations

- C. Kim, C. Sachrajda, & SRS, Nucl.Phys.B 727 (2005) 218-243; arXiv:hep-lat/0507006 [QFT-based approach; LL factors in moving frames]
- R. Briceño, Phys.Rev.D 89 (2014) 7, 074507; arXiv:1401.3312 [QC2 for arbitrary spin]

RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

<u>Raúl Briceño, Max Hansen & SRS</u>:

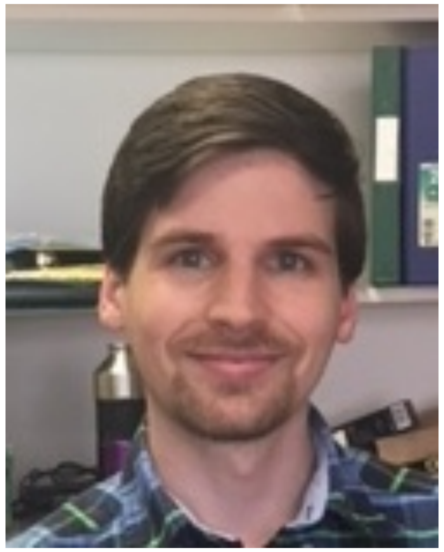
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

<u>SRS</u>

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



<u>Tyler Blanton, Fernando Romero-López & SRS:</u>

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

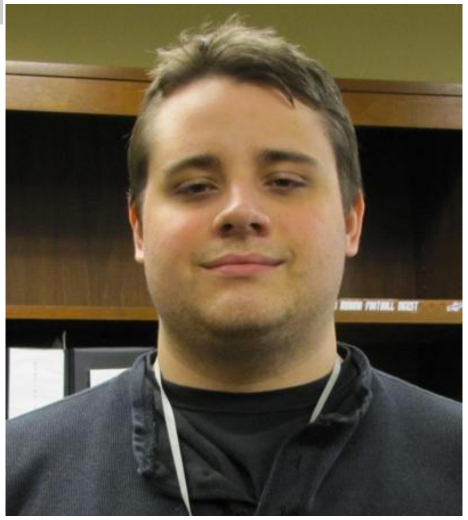
"Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

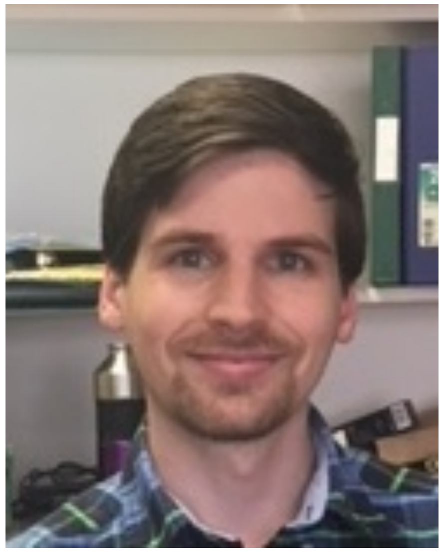
"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)

<u>Tyler Blanton & SRS:</u>



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<u>Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

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<u>Zack Draper, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

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Zach Draper, Max Hansen, Fernando Romero-López & SRS:

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Zach Draper & SRS:

"Three-particle formalism for multiple channels: the $\eta\pi\pi + K\overline{K}\pi$ system in isosymmetric QCD," arXiv:2403.20064 (JHEP)

Max Hansen, Fernando Romero-López & SRS:

"Incorporating $DD\pi$ effects and left-hand cuts in lattice studies of the $T_{cc}(3875)^+$," arXiv:2401.06609 (JHEP)

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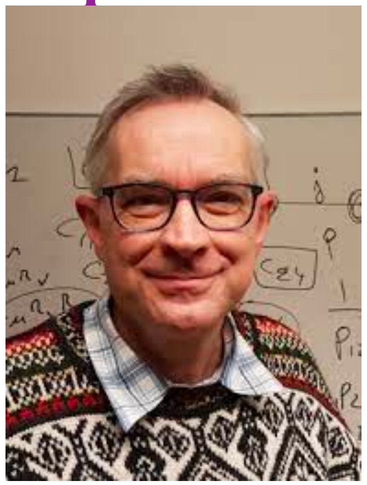

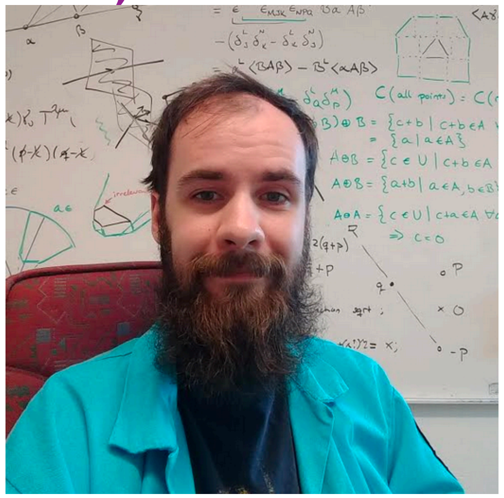

Other work

★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]
- A. Jackura et al., 2010.09820 , PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam and R. Briceño, 2303.04394 [Analytic continuation of 3-particle amplitudes]
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- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

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Alternate 3-particle approaches

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- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144, PRD [determining $3\pi^+$ interaction from LQCD spectrum]
- M. Mai et al., 2107.03973, PRL [three-body dynamics of the $a_1(1260)$ from LQCD]
- D. Dasadivan et al., 2112.03355, PRD [pole position of $a_1(1260)$ in a unitary framework]
- D. Seivert, M. Mai, U-G. Meißner, 2212.02171, JHEP [Particle-dimer approach for the Roper resonance]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]

Backup slides

Forms of F and G

- Symmetric form of QC3 takes the by-now familiar form

$$\prod_{I\in\{0,1,2\}} \det_{i,\boldsymbol{k},\ell,m} \left[1 + \widehat{\mathcal{K}}^{[I]}_{\mathrm{df},3}\ \widehat{F}^{[I]}_3\right] = 0$$

$$\widehat{F}^{[I]}_3 \equiv \frac{\widehat{F}^{[I]}}{3} - \widehat{F}^{[I]} \frac{1}{1+\widehat{\mathcal{M}}^{[I]}_{2,L}\widehat{G}^{[I]}} \widehat{\mathcal{M}}^{[I]}_{2,L}\widehat{F}^{[I]}, \qquad \widehat{\mathcal{M}}^{[I]}_{2,L} \equiv \frac{1}{\widehat{\mathcal{K}}^{[I]-1}_{2,L} + \widehat{F}^{[I]}}$$

$$\widehat{F}^{[I=0]} = \mathrm{diag}\ \left(\widetilde{F}^{D}, \widetilde{F}^{\pi}\right) \qquad \widehat{G}^{[I=0]} = \begin{pmatrix} G^{DD} & \sqrt{2}P_\ell G^{D\pi} \\ \sqrt{2}G^{\pi D}P_\ell & 0 \end{pmatrix}$$

$$\left[\widetilde{F}^{(i)}\right]_{p'\ell'm';p\ell m} = \delta_{\boldsymbol{p}'\boldsymbol{p}} \frac{H^{(i)}(\boldsymbol{p})}{2\omega^{(i)}_p L^3} \left[\frac{1}{L^3}\sum_{\boldsymbol{a}}^{\mathrm{UV}} - \mathrm{PV}\int^{\mathrm{UV}} \frac{d^3a}{(2\pi)^3}\right] \left[\frac{\mathcal{Y}_{\ell'm'}(\boldsymbol{a}^{*(i,j,p)})}{(q^{*(i)}_{2,p'})^{\ell'}} \frac{1}{4\omega^{(j)}_a\omega^{(k)}_b(E-\omega^{(i)}_p-\omega^{(j)}_a-\omega^{(k)}_b)} \frac{\mathcal{Y}_{\ell m}(\boldsymbol{a}^{*(i,j,p)})}{(q^{*(i)}_{2,p})^{\ell}}\right]$$

$$\left[\widetilde{G}^{(ij)}\right]_{p\ell'm';r\ell m} = \frac{1}{2\omega^{(i)}_p L^3} \frac{\mathcal{Y}_{\ell'm'}(\boldsymbol{r}^{*(i,j,p)})}{(q^{*(i)}_{2,p})^{\ell'}} \frac{H^{(i)}(\boldsymbol{p})H^{(j)}(\boldsymbol{r})}{b^2_{ij}-m^2_k} \frac{\mathcal{Y}_{\ell m}(\boldsymbol{p}^{*(j,i,r)})}{(q^{*(j)}_{2,r})^{\ell}} \frac{1}{2\omega^{(j)}_r L^3},$$

where $b_{ij} = (E - \omega^{(i)}_p - \omega^{(j)}_r, \boldsymbol{P} - \boldsymbol{p} - \boldsymbol{r})$.