Finite-volume formalism for physical processes with an electroweak loop integral

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2/8/2024

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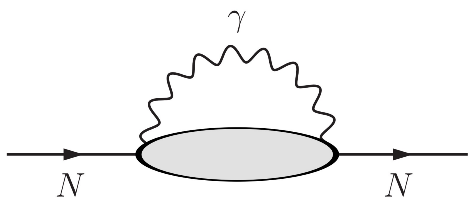
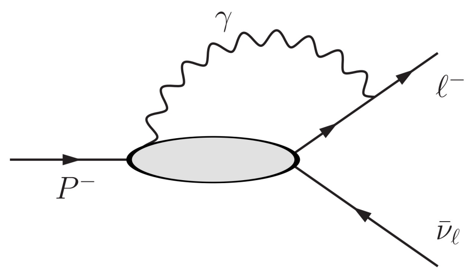
Brookhaven National Laboratory

Processes with an electroweak loop integral

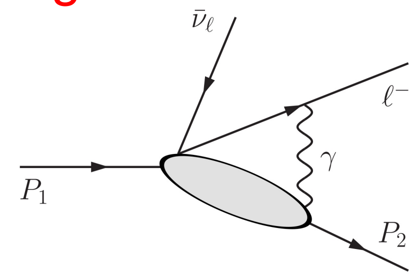
- In QED corrections and rare decays, electroweak propagators and hadronic matrix elements often form a loop integral structure



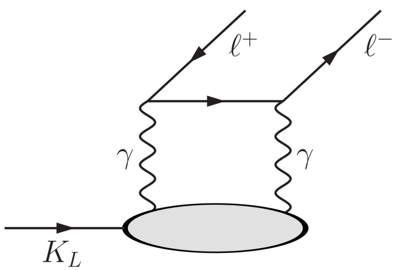
(a) QED self-energy of hadron N



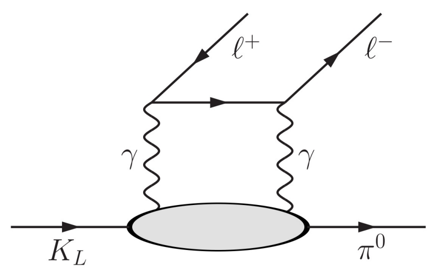
(b) Radiative correction: $P^- \to \ell^- \bar{\nu}_\ell$



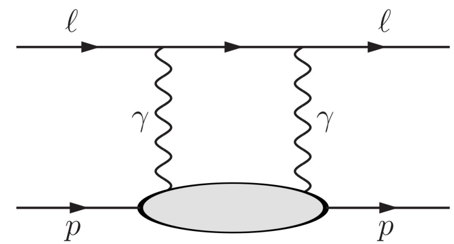
(c) Radiative correction: $P_1 \to P_2 \ell^- \bar{\nu}_\ell$



(d) 2γ exchange: $K_L \to \ell^+ \ell^-$



(e) 2γ exchange: $K_L \to \pi \ell^+ \ell^-$



(f) 2γ exchange: $N - \ell$ scattering

Processes with an electroweak loop integral

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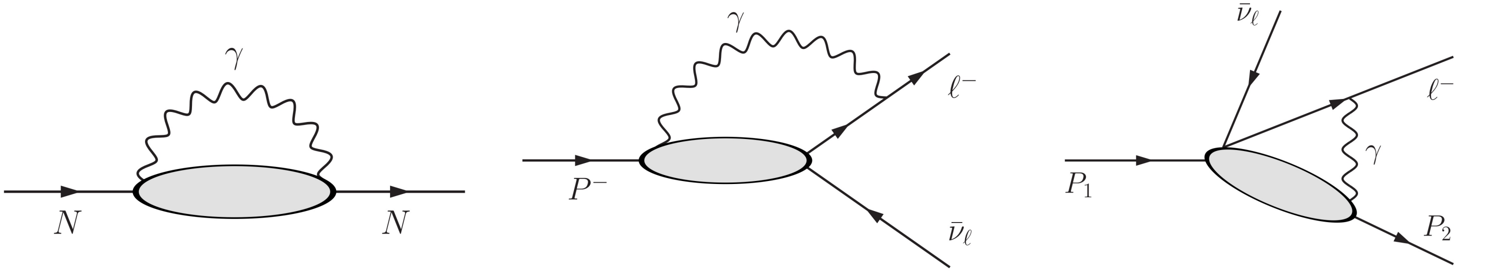

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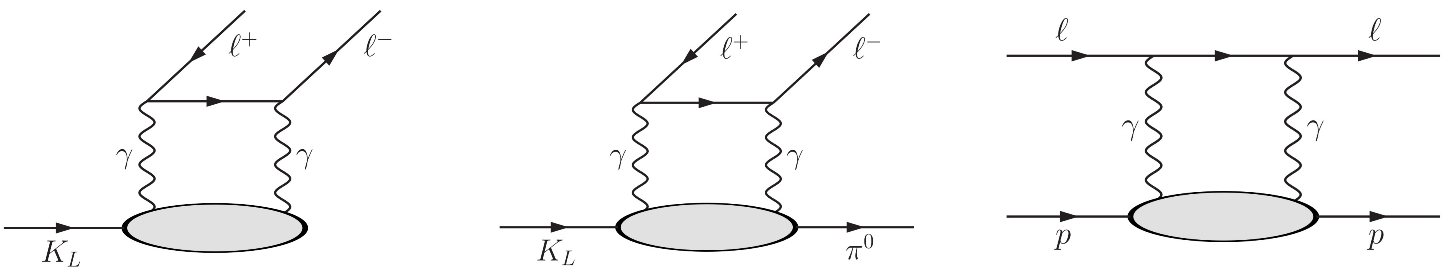

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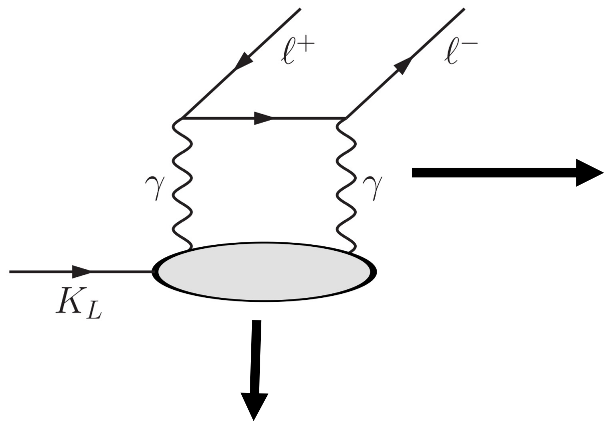
- Important: High-order EW interaction is sensitive to new physics signal (CKM, CP violation, FCNC).

- But difficult: large uncertainties from long-distance hadronic matrix elements.

→ Lattice QCD

Challenge on lattice: FV effects

- FV effects from both



EW part:
massless photons,
IR singularities

hadronic part:
$\pi\pi$ intermediate states

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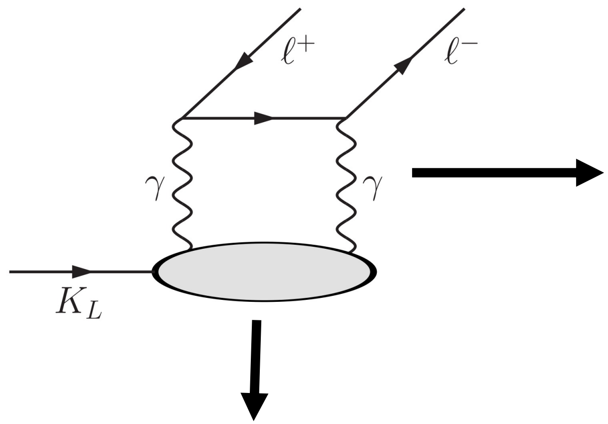

EW part:
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hadronic part:
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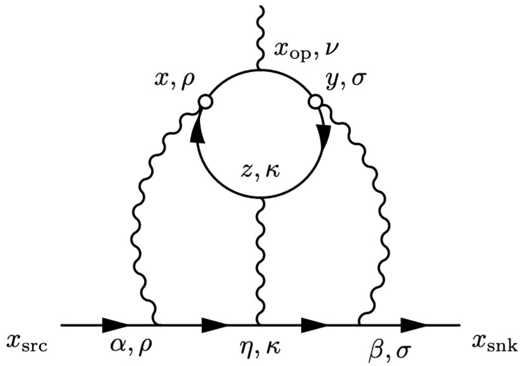
- $EW_\infty(QED_\infty)$ idea: using electroweak part as infinite-volume version and do the integration in coordinate space

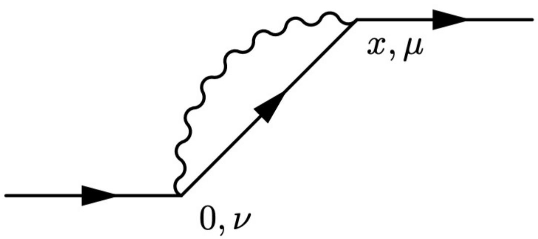
$$\int_V d^3x \int_{-t_s}^{t_s} d\tau \boxed{L^\infty(\tau, \boldsymbol{x})} H^{(L)}(\tau, \boldsymbol{x})$$

Idea of EW_∞ (QED_∞)

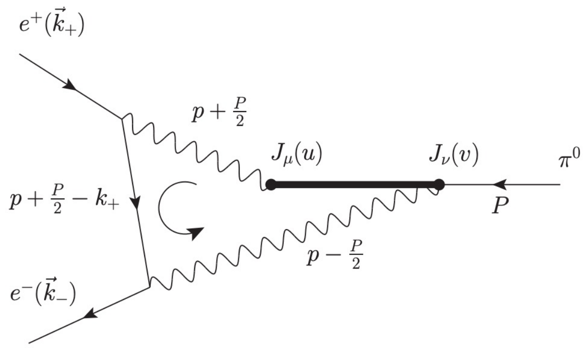
- Examples using $EW_\infty(QED_\infty)$:



HLbL
PRD 96 (2017) 3, 034515,
arxiv:1705.01067



QED self energy: IVR
without power-law FV errors
PRD 100 (2019) 9, 094509
arxiv:1812.09817



$\pi^0 \to e^+e^-$
PRL 130 (2023) 19, 191901
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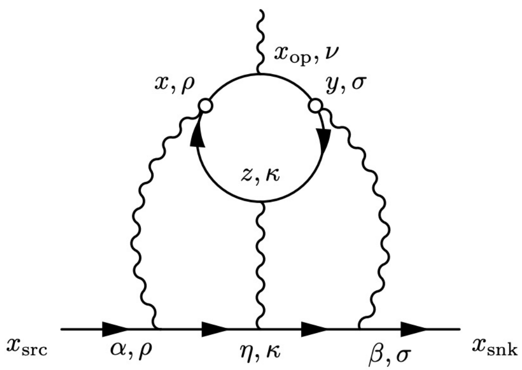

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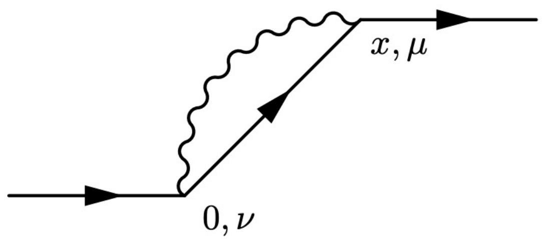

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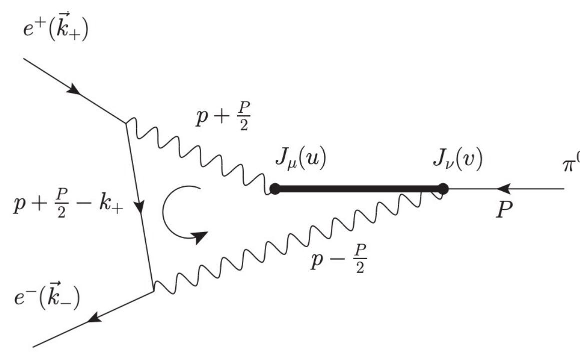

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- This work develops the finite-volume formalism of EW_∞ method if the hadronic intermediate states are one particle or two particles.

How to analyze finite-volume effects?

- FV effects depend on smoothness of the summand $\frac{1}{L^3}\sum_{\boldsymbol{k}} \tilde{f}(\boldsymbol{k})$

$\tilde{f}(\boldsymbol{k})$:
- singular: $O(1/L)$ FV effects
- continuously differentiable up to order N: $O(1/L^{N+1})$ FV effects
- analytic: $O(e^{-mL})$ FV effects

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- What if we do the integral in coordinate space? A simple model:

$$\int_V d^3x A^\infty(\boldsymbol{x}) B^{(L)}(\boldsymbol{x})$$

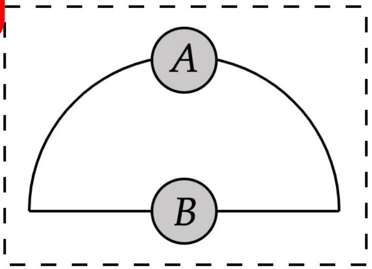
$$A^\infty(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} \tilde{A}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$B^{(L)}(\boldsymbol{x}) = \frac{1}{L^3}\sum_{\boldsymbol{k}'} \tilde{B}(\boldsymbol{k}') e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}$$

$\longrightarrow$

$$\frac{1}{L^3}\sum_{\boldsymbol{k}'} \int \frac{d^3k}{(2\pi)^3} \delta_L(\boldsymbol{k}'-\boldsymbol{k}) \tilde{A}(\boldsymbol{k}) \tilde{B}(\boldsymbol{k}')$$

$$\delta_L(\boldsymbol{q}) = \int_V d^3x e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$$



Integral any two functions with EW_∞ idea

momentum-space analysis

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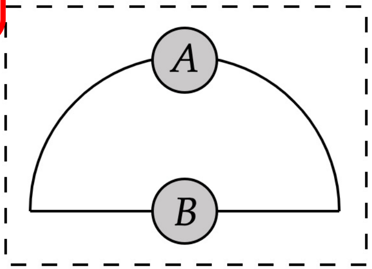
$$B^{(L)}(\boldsymbol{x}) = \frac{1}{L^3}\sum_{\boldsymbol{k}'}\tilde{B}(\boldsymbol{k}')e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \qquad \delta_L(\boldsymbol{q}) = \int_V d^3x e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$$

Integral any two functions with EW_{∞} idea

momentum-space analysis



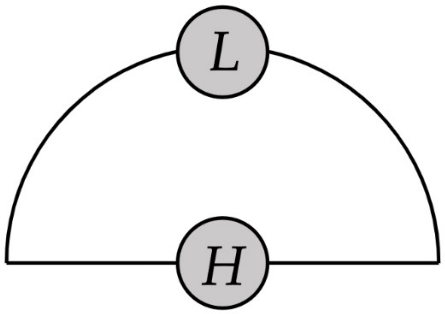
- We prove: FV effects still depend on the smoothness of the summand.
- The momentum-space analysis still works!

Momentum-space analysis of EW_∞ method

- Physical case

$$I^\infty = \int \frac{d^4k}{(2\pi)^4} L^\infty(k) H^\infty(k,p),$$

$$H^\infty(k,p) = \int d^3x \int_{-\infty}^{\infty} dt e^{ik\cdot x} \langle f | T[J_1(t,\boldsymbol{x}) J_2(0)] | i \rangle$$



- EW_∞ method ($t < 0$ time ordering)

$$I^{(LT)} = c_{ME} \int_V d^3x \int_{-t_s}^{0} d\tau L_E^\infty(\tau,\boldsymbol{x}) H_E^{(L)}(\tau,\boldsymbol{x})$$

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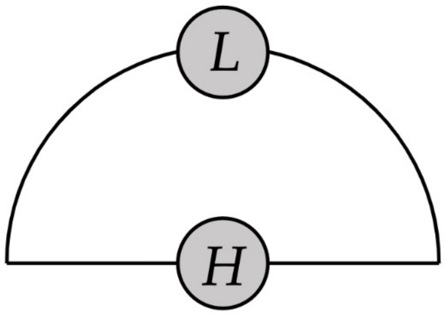

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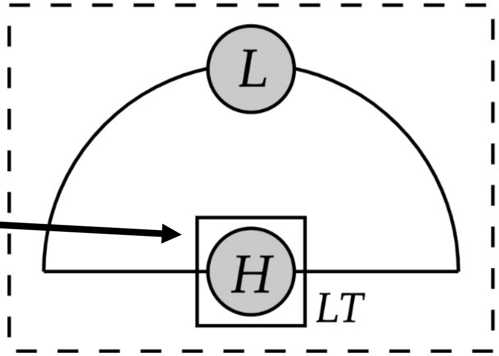
$$I^{(LT)} = c_{ME} \int_V d^3x \int_{-t_s}^{0} d\tau L_E^{\infty}(\tau,\boldsymbol{x}) H_E^{(L)}(\tau,\boldsymbol{x})$$

$$I^{(LT)} = \frac{1}{L^3} \sum_{\boldsymbol{k}' \in \Gamma} \int \frac{d^3k}{(2\pi)^3} \delta_L(\boldsymbol{k}' - \boldsymbol{k}) \int_C \frac{dk^0}{2\pi} L^{\infty}(k) H^{(LT)}(k',p)$$



$I^{(LT)}$

- There are also FV effects in momentum-space hadronic function

$$H^{\infty}_{t<0}(k,p) = i \sum\!\!\!\!\!\!\int_{\alpha} \frac{A_\alpha^{\infty}(-\boldsymbol{k}, E_\alpha)}{m - k^0 - E_\alpha + i\epsilon}$$

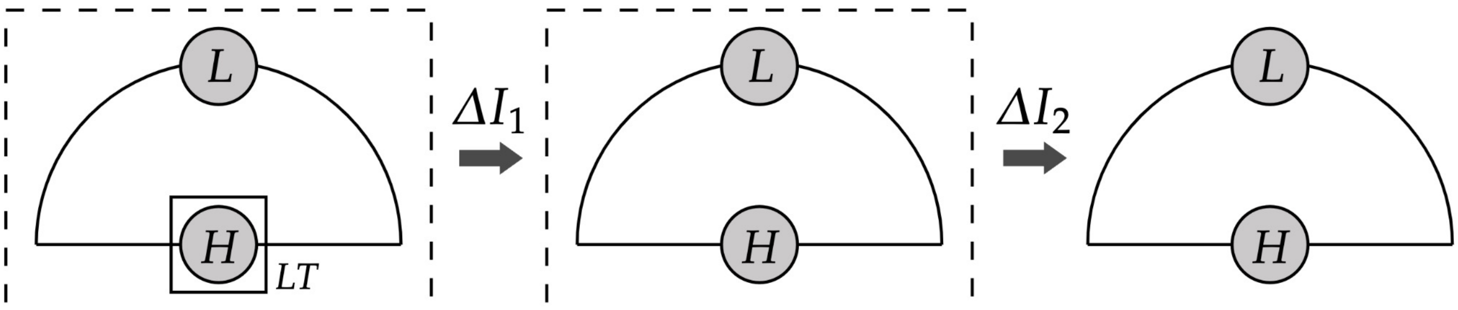
$$H^{(LT)}(k',p) = i \sum_{\alpha_L} \frac{A_\alpha^{(L)}(-\boldsymbol{k}', E_{\alpha_L})}{m - k^0 - E_{\alpha_L}} \left(1 - e^{(m - k^0 - E_{\alpha_L})t_s}\right)$$

Momentum-space analysis of EW_∞ method

- Two corrections

$$H^{(LT)}(k',p). \xrightarrow{\Delta I_1} H^{\infty}(k',p)$$

$$\frac{1}{L^3}\sum_{\boldsymbol{k}'\in\Gamma}\int\frac{d^3k}{(2\pi)^3}\delta_L(\boldsymbol{k}'-\boldsymbol{k}) \xrightarrow{\Delta I_2} \int\frac{d^4k}{(2\pi)^4}$$

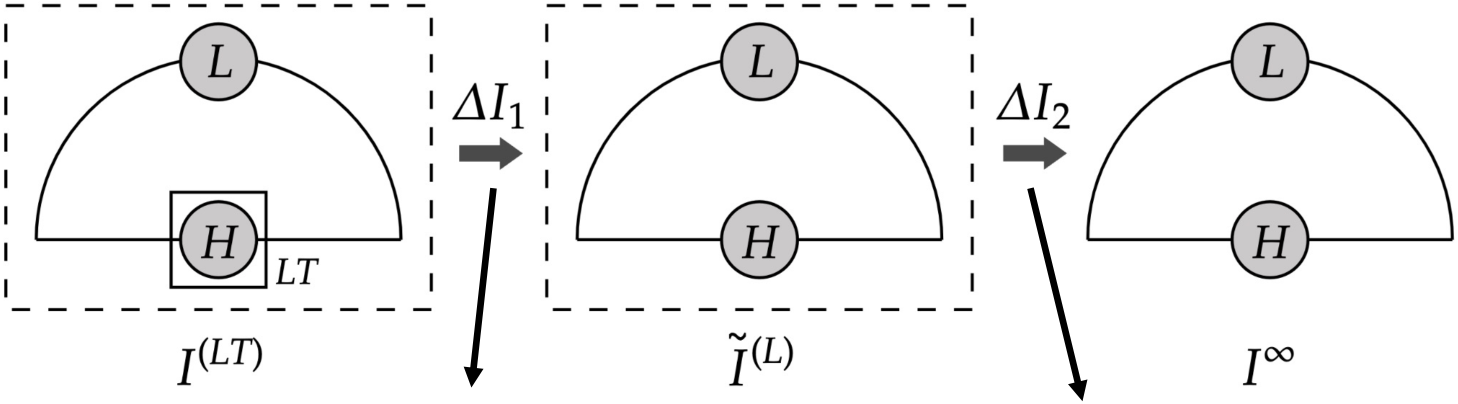


$I^{(LT)}$ $\tilde{I}^{(L)}$ I^{∞}

Momentum-space analysis of EW_∞ method

- Two corrections

$$\begin{cases} H^{(LT)}(k',p). \xrightarrow{\Delta I_1} H^{\infty}(k',p) \\ \frac{1}{L^3}\sum_{\boldsymbol{k}'\in\Gamma}\int\frac{d^3k}{(2\pi)^3}\delta_L(\boldsymbol{k}'-\boldsymbol{k}) \xrightarrow{\Delta I_2} \int\frac{d^4k}{(2\pi)^4} \end{cases}$$



What we know

e.g., for two-particles intermediate states

PRD 101 (2020) 1, 014509

arxiv:1911.04036

What's new

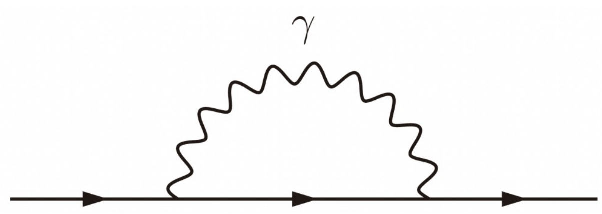
FV effects due to coordinate space integral in EW_∞ method

↓

Analysis the singularity of the summand

One-particle case

- Example: QED self-energy



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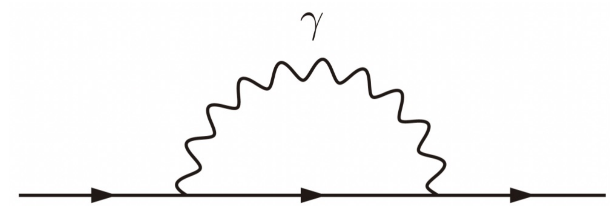

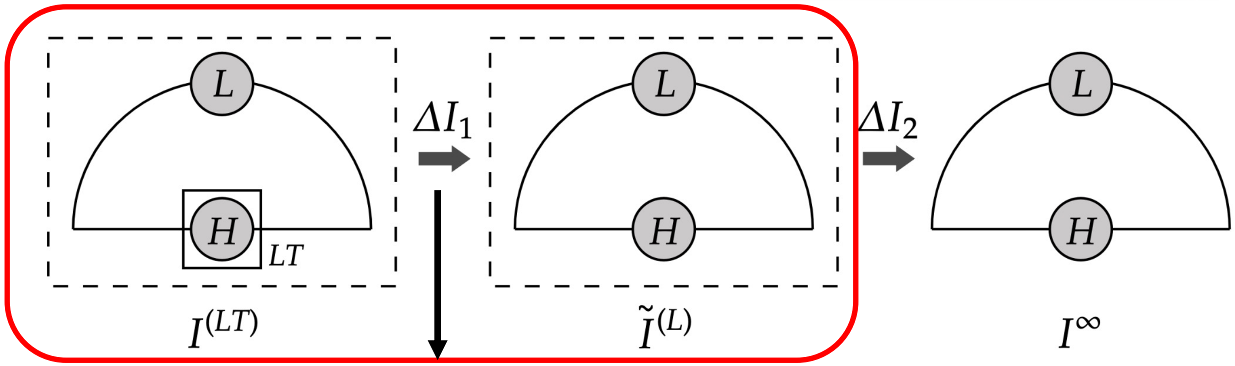

ΔI_1 is dominated by temporal truncation effects

$$H^{(LT)}\left(k',p\right)=i\frac{A_N\left(-\boldsymbol{k}'\right)}{2E_N\left(\boldsymbol{k}'\right)\left(m_n-k^0-E_N\left(\boldsymbol{k}'\right)\right)}\left(1-e^{\left(m_n-k^0-E_N\left(\boldsymbol{k}'\right)\right)t_s}\right)$$

One-particle case

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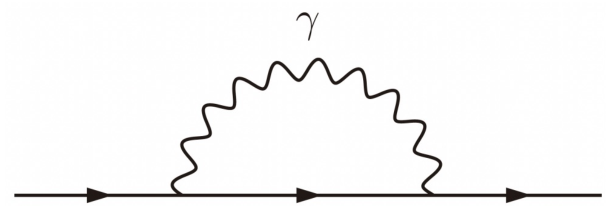

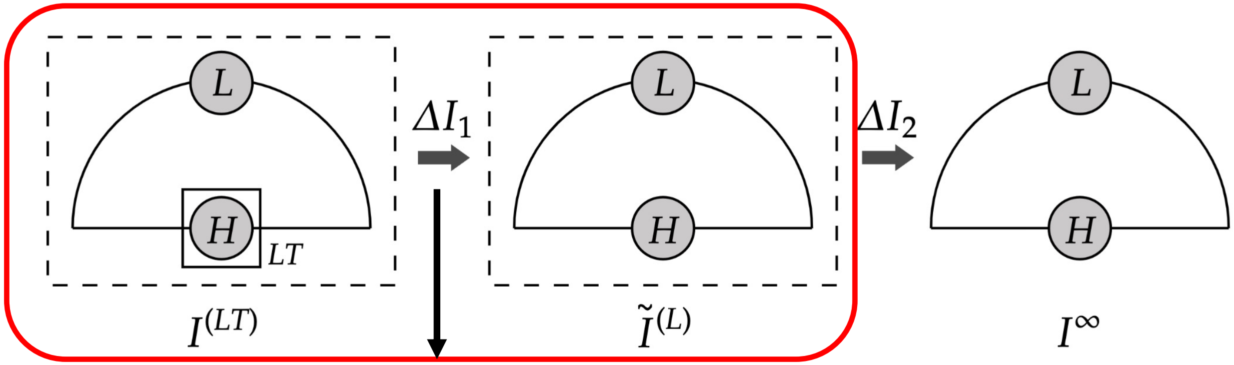

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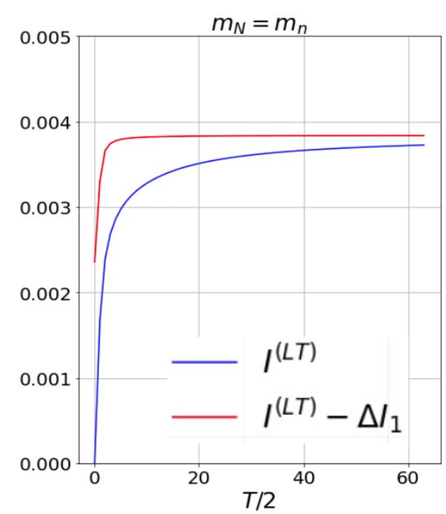
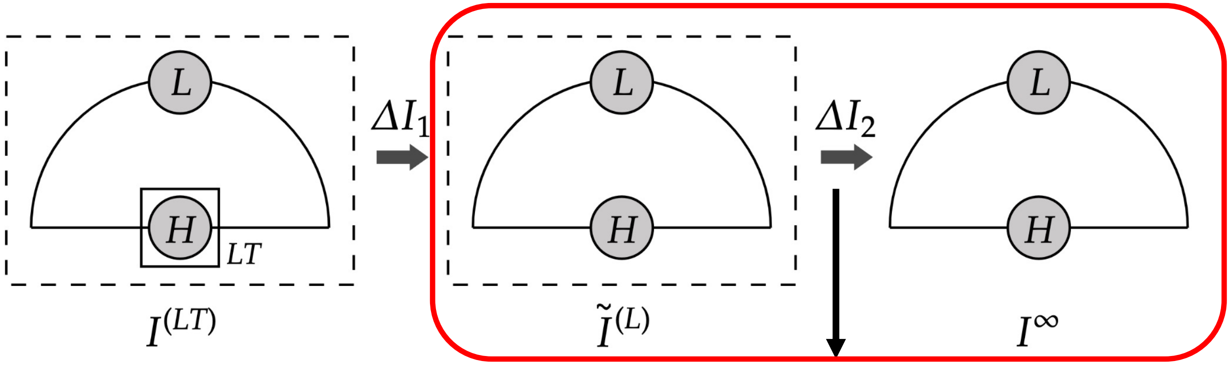
Infinite-volume reconstruction (IVR)

ΔI_1 can be reconstructed by ground-state dominance from lattice data at $t = t_s$.



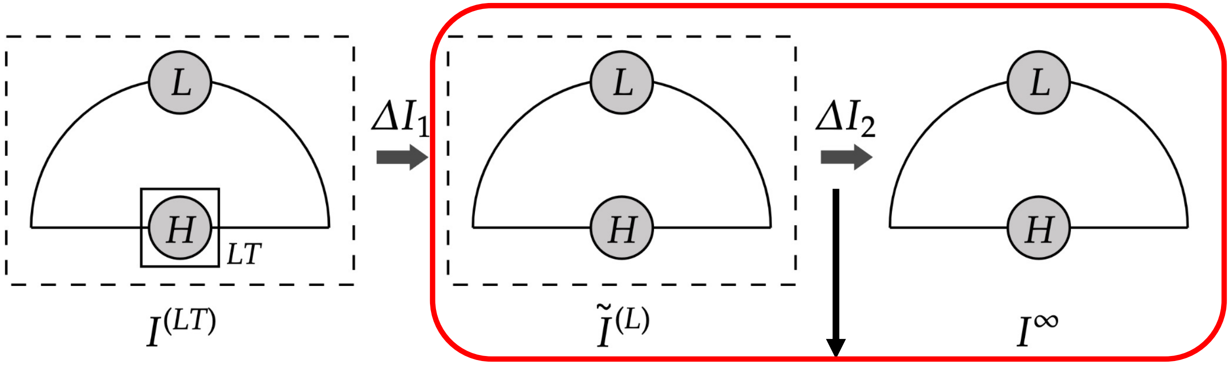
One-particle case



$$\tilde{I}^{(L)} = \frac{1}{L^3} \sum_{\boldsymbol{k}' \in \Gamma} \hat{I}(\boldsymbol{k}')$$

Analysis the singularity of the summand

One-particle case



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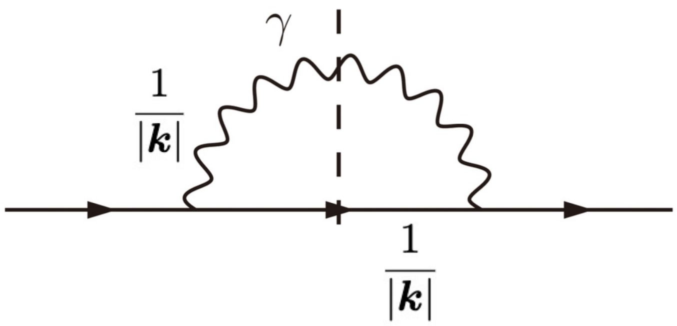

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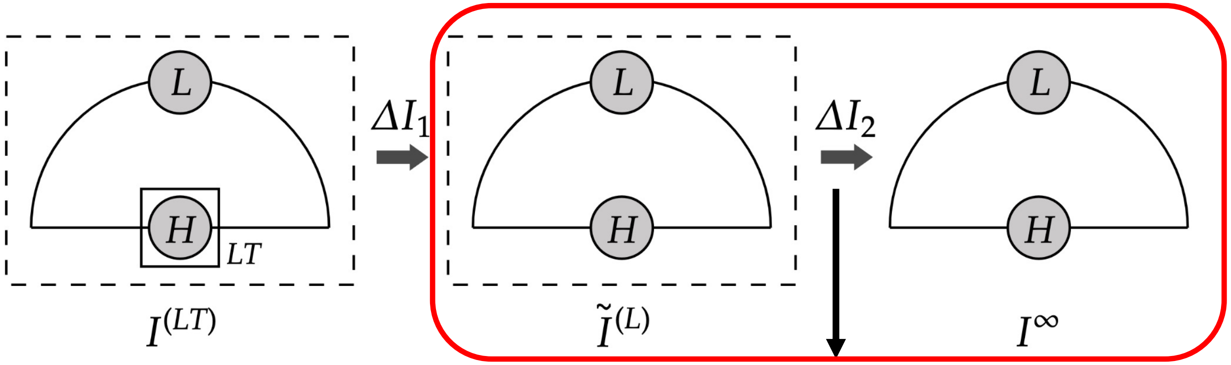

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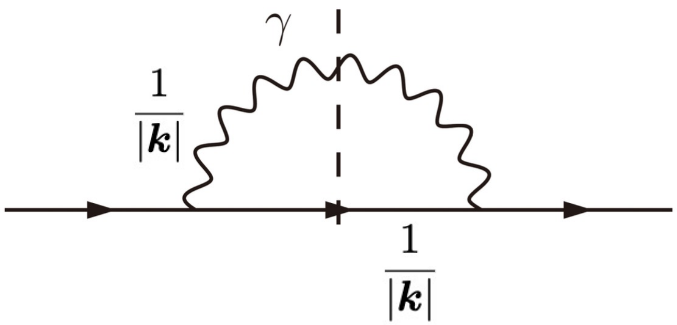
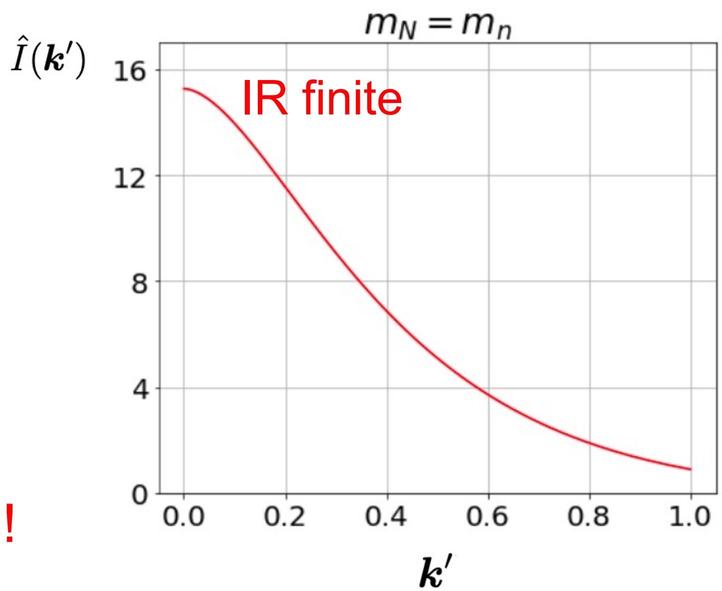


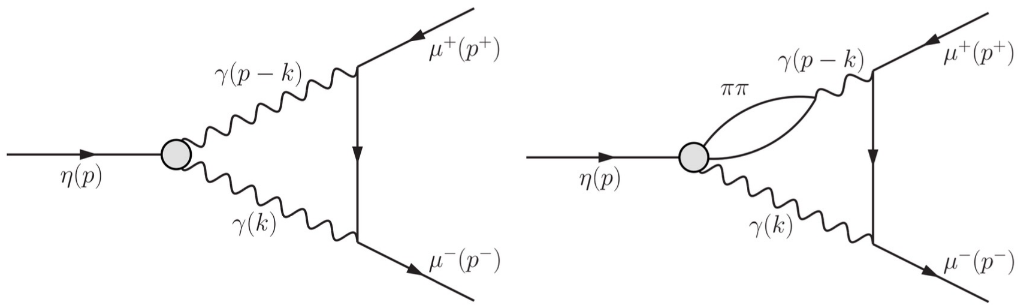


- Even if photon has IR singularity, the summand is still IR finite!

→ IVR has only $O(e^{-mL})$ FV effects!

Two-particle case

- Example:



[1] R. A. Briceno, Z. Davoudi, M. T. Hansen, M. R. Schindler, and A. Baroni, Phys. Rev. D101, 014509 (2020)

Two-particle case

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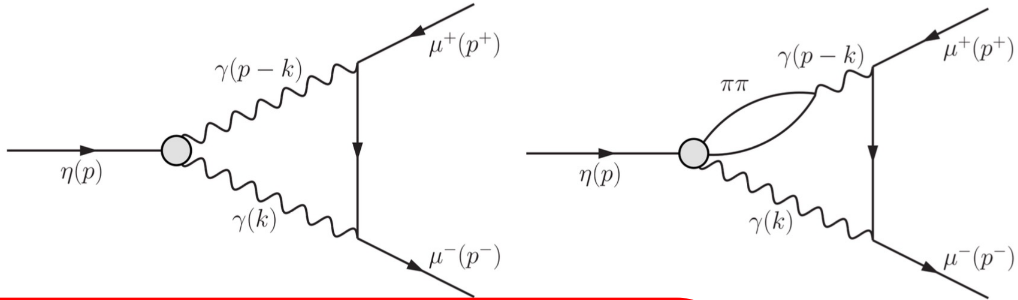

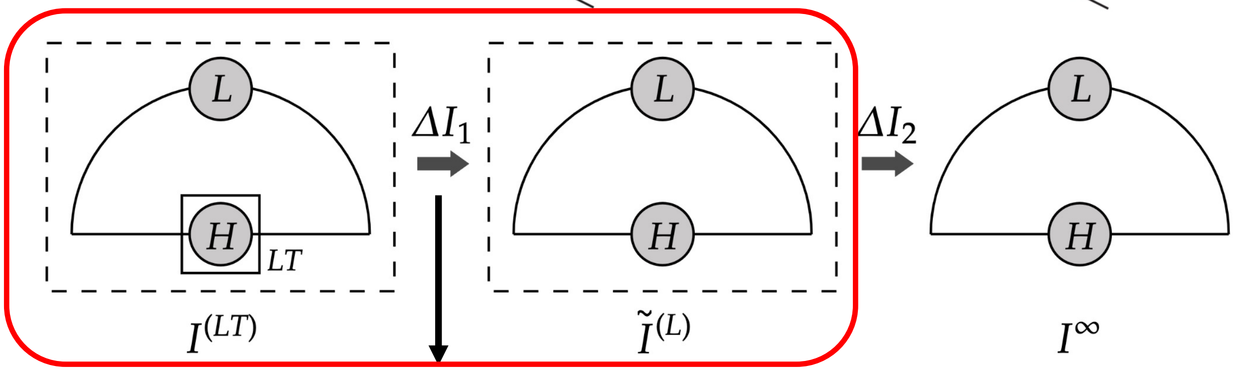

$\Delta H = H^{(LT)} - H^{\infty}$ has been studied by Ref. [1]

ΔI_1 { exponential growing term; $O(1/L)$ FV effects }

(Inputs: $\pi\pi$ phase shift, matrix element of vertex)

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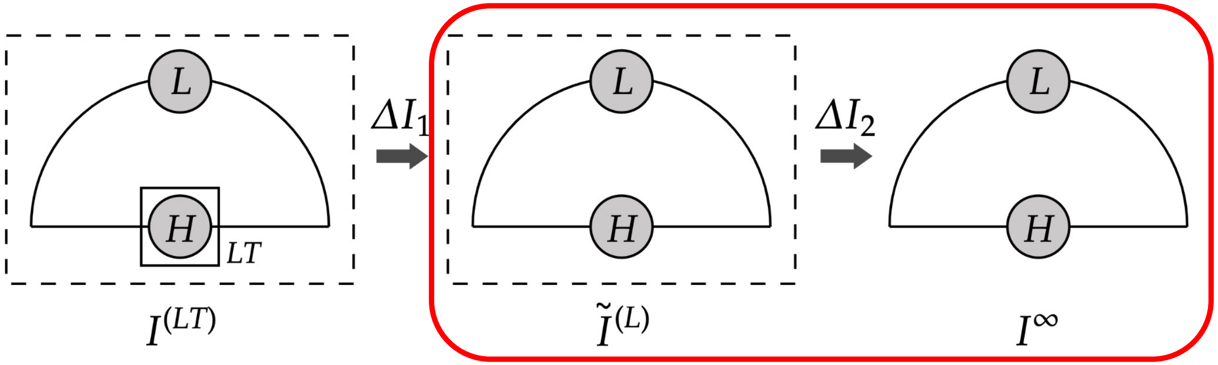
Two-particle case



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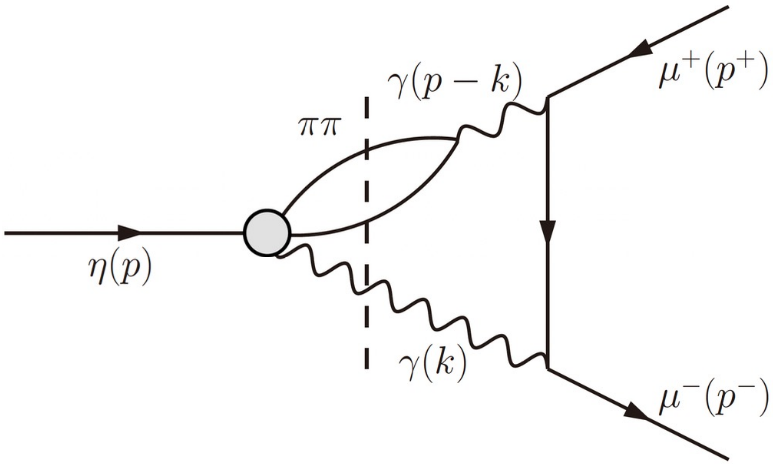
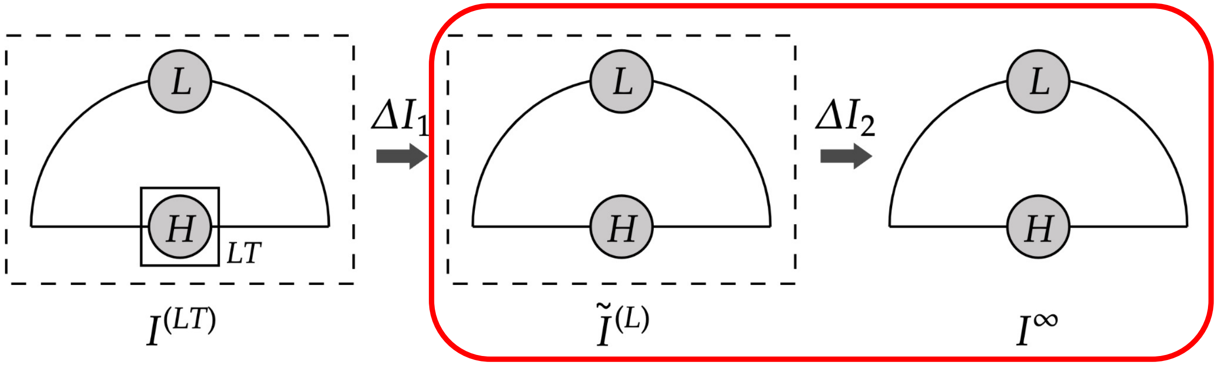
- Similar as one-particle case, the summand is still IR finite



Two-particle case

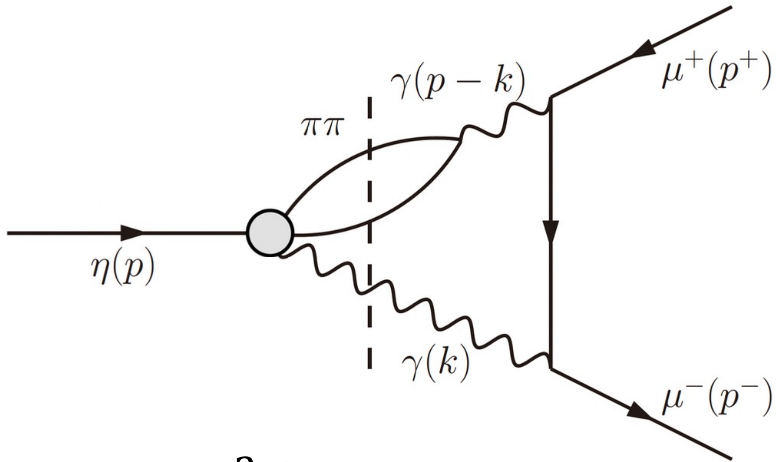


$$\tilde{I}^{(L)} = \frac{1}{L^3} \sum_{\boldsymbol{k}' \in \Gamma} \hat{I}(\boldsymbol{k}')$$

$$\hat{I}\left(\boldsymbol{k}'\right) \sim \int \frac{d^3 k}{(2\pi)^3} \delta_L\left(\boldsymbol{k}' - \boldsymbol{k}\right) \frac{\boldsymbol{k} \cdot \boldsymbol{k}'}{2|\boldsymbol{k}| \cdot |\boldsymbol{k}| \cdot m_\eta} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{2E_\pi(\boldsymbol{q}) 2E_\pi\left(\boldsymbol{k}' + \boldsymbol{q}\right)|\boldsymbol{k}|} \sim \int d^3 k \frac{1}{|\boldsymbol{k}|^2}$$

- Similar as one-particle case, the summand is still IR finite

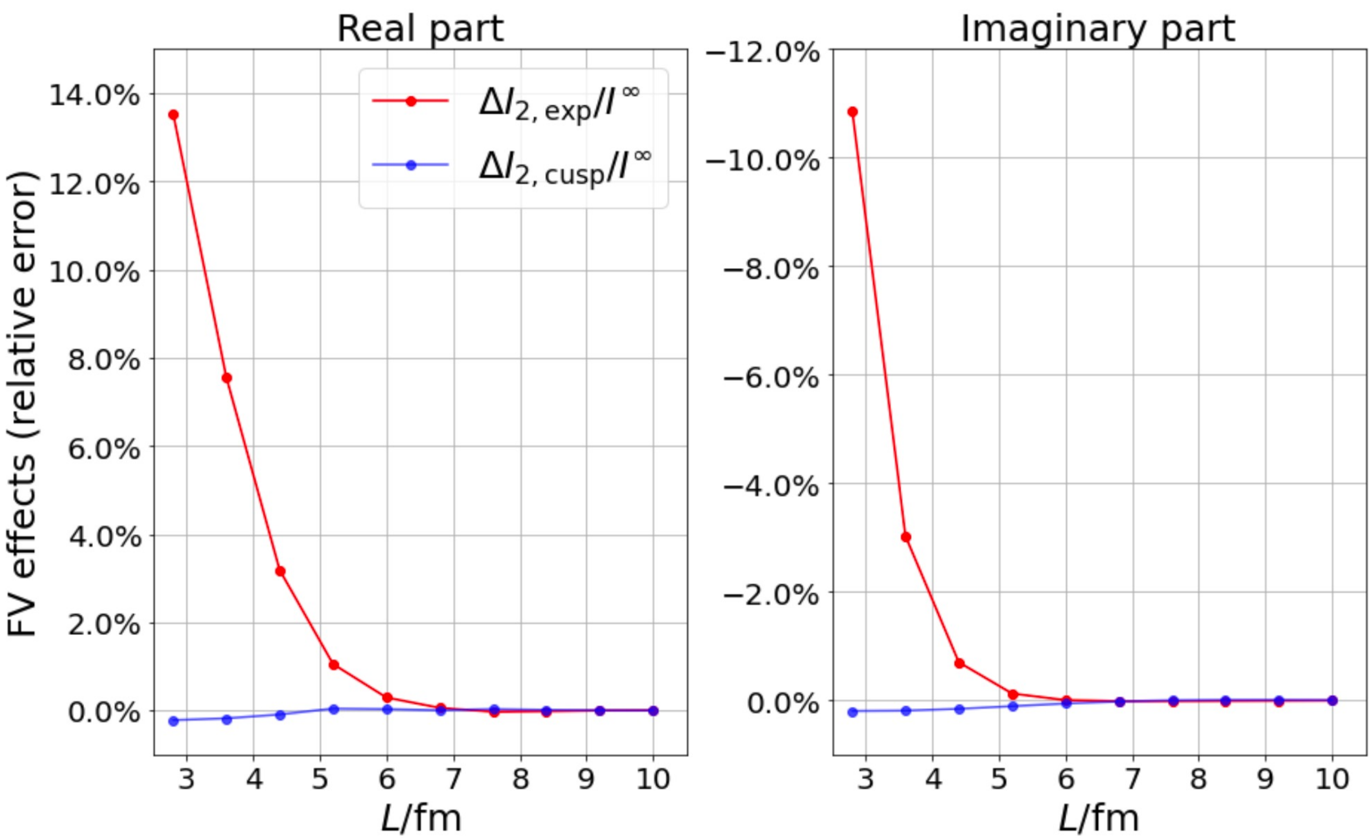
- Cusp effects: $\hat{I}(\boldsymbol{k}')$ has non-smooth points at threshold
$$|\boldsymbol{k}'| = \sqrt{\left(m_\eta - k^0\right)^2 - 4m_\pi^2},$$
and is differentiable up to order $N = l$ (angular momentum).



⟶ $O(1/L^{l+1})$ FV effects. (P wave: $O(1/L^2)$)

Numerical test of ΔI_2

- Numerical test of ΔI_2 in $\eta \to \mu^+\mu^-$: one $\pi\pi$ loop (ignore rescattering effects), GS model



Conclusion

- This work develops the finite-volume formalism of EW_∞ method in

  1.One-particle case: Why in IVR method $\delta_{IVR} \sim O(e^{-mL})$?

  2.Two-particle case:

ΔI_1
- exponential growing term
- $O(1/L)$ FV effects

ΔI_2
- cusp effects: $O(1/L^{l+1})$ FV effects
- $O(e^{-\Lambda L})$ FV effects → From ρ resonance region

(exponential growing term, $O(1/L)$ FV effects, cusp effects:) From low-lying $\pi\pi$
We derive correction formulas

Low-lying $\pi\pi$ state in P wave is suppressed due to rho resonance.

En-Hung Chao, Norman Christ, (2024), arxiv:2406.07447