Four-quark operators with $\Delta F = 2$ in the GIRS scheme

Dr. Marios Costa

Department of Physics
University of Cyprus

Rinnoco Ltd

August 2, 2024

In collaboration with: Temple University (M. Constantinou) and University of Cyprus (H. Herodotou, H. Panagopoulos, G. Spanoudes).

General Outline

Motivations and Challenges

- Scalar and pseudoscalar four-quark operators naturally incorporate weak interaction effects.
- Relevant for calculating CKM matrix elements at high precision.
- Potential discoveries at the Large Hadron Collider (LHC), such as new tetraquarks.
- Phenomenological bag parameters are other important lattice quantities associated with four-quark operators.

Practical difficulties in calculating physical matrix elements of four-quark operators:

• Unwanted mixing encountered under renormalization.

• In GIRS (GFs of gauge-invariant operators), the calculations at any given order in perturbation theory require Feynman diagrams that have more loops than Green's functions (GFs) with external elementary fields.

• When mixing occurs, the calculation of some three-point GIRS GFs may be unavoidable, which are typically more noisy in simulations.

Definitions

We investigate four-quark composite operators of the form:

$$\mathcal{O}_{\Gamma\tilde{\Gamma}}(x) = \bar{\psi}_{f_1}(x)\Gamma\psi_{f_3}(x)\bar{\psi}_{f_2}(x)\tilde{\Gamma}\psi_{f_4}(x),$$

where Γ and $\tilde{\Gamma}$ denote products of Dirac matrices:

$$\Gamma, \tilde{\Gamma} \in \{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}, \gamma_5\sigma_{\mu\nu}\} \equiv \{S, P, V, A, T, \tilde{T}\},$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$; color and spinor indices are implied. In our study, we focus on $\Delta F = 2$ four-quark operators with $\Gamma = \tilde{\Gamma}$ and $\Gamma = \tilde{\Gamma}\gamma_5$ (repeated Lorentz indices are summed over), which are scalar or pseudoscalar quantities under rotational symmetry.

Study of the mixing for four-quark operators

- Mixing among operators with different Dirac matrices is allowed, as dictated by symmetries.

We construct operators with exchanged flavors of their quark fields, which are related to the original operators through the Fierz–Pauli–Kofink identity (the superscript letter F stands for Fierz):

$$
\begin{aligned}
\mathcal{O}_{\Gamma\tilde{\Gamma}} &\equiv (\bar{\psi}_{f_1}\,\Gamma\,\psi_{f_3})(\bar{\psi}_{f_2}\,\tilde{\Gamma}\,\psi_{f_4}) \equiv \sum_x \sum_{a,c} \Big(\bar{\psi}^a_{f_1}(x)\,\Gamma\,\psi^a_{f_3}(x)\Big)\Big(\bar{\psi}^c_{f_2}(x)\,\tilde{\Gamma}\,\psi^c_{f_4}(x)\Big), \\
\mathcal{O}^F_{\Gamma\tilde{\Gamma}} &\equiv (\bar{\psi}_{f_1}\,\Gamma\,\psi_{f_4})(\bar{\psi}_{f_2}\,\tilde{\Gamma}\,\psi_{f_3}) \equiv \sum_x \sum_{a,c} \Big(\bar{\psi}^a_{f_1}(x)\,\Gamma\,\psi^a_{f_4}(x)\Big)\Big(\bar{\psi}^c_{f_2}(x)\,\tilde{\Gamma}\,\psi^c_{f_3}(x)\Big),
\end{aligned}
$$

where Dirac indices are implicit, and color indices are denoted by Latin letters a, c.

Exploitation of key Symmetries

We considered symmetries of the QCD action [1]:

$$\text{Parity}: \quad \begin{cases} \mathcal{P}\psi_f(x) & = \gamma_4 \; \psi_f(x_P) \\ \mathcal{P}\bar{\psi}_f(x) & = \bar{\psi}_f(x_P) \; \gamma_4 \end{cases}$$

$$\text{Charge conjugation}: \quad \begin{cases} \mathcal{C}\psi_f(x) & = -C \; \bar{\psi}_f^T(x) \\ \mathcal{C}\bar{\psi}_f(x) & = \psi_f^T(x) \; C \end{cases}$$

$$\text{Flavor exchange symmetry}: \quad \left\{ \mathcal{S} \equiv (f_3 \leftrightarrow f_4) \right.$$

$$\text{Flavor Switching symmetry 1}: \quad \left\{ \mathcal{S}' \equiv (f_1 \leftrightarrow f_3, f_2 \leftrightarrow f_4) \right.$$

$$\text{Flavor Switching symmetry 2}: \quad \left\{ \mathcal{S}'' \equiv (f_1 \leftrightarrow f_4, f_3 \leftrightarrow f_2) \right.$$

where $x_P = (-\mathbf{x}, t)$, ^{T} means transpose and the matrix C satisfies: $(C\gamma_\mu)^T = C\gamma_\mu$, $C^T = -C$ and $C^\dagger C = 1$.

R. Frezzotti and G. C. Rossi, “Chirally improving Wilson fermions. II. Four-quark operators,” JHEP **10** (2004), 070 [arXiv:hep-lat/0407002 [hep-lat]].

Transformation properties of the four-quark operators

	$\mathcal{P}$	$\mathcal{CS}'$	$\mathcal{CS}''$	$\mathcal{CPS}'$	$\mathcal{CPS}''$
$\mathcal{O}_{VV}$	$+$	$+$	$+$	$+$	$+$
$\mathcal{O}_{AA}$	$+$	$+$	$+$	$+$	$+$
$\mathcal{O}_{PP}$	$+$	$+$	$+$	$+$	$+$
$\mathcal{O}_{SS}$	$+$	$+$	$+$	$+$	$+$
$\mathcal{O}_{TT}$	$+$	$+$	$+$	$+$	$+$
$\mathcal{O}_{[VA+AV]}$	$-$	$-$	$-$	$+$	$+$
$\mathcal{O}_{[VA-AV]}$	$-$	$-$	$+$	$+$	$-$
$\mathcal{O}_{[SP-PS]}$	$-$	$+$	$-$	$-$	$+$
$\mathcal{O}_{[SP+PS]}$	$-$	$+$	$+$	$-$	$-$
$\mathcal{O}_{T\tilde{T}}$	$-$	$+$	$+$	$-$	$-$

Table 1: Transformations of the four-quark operators $\mathcal{O}_{\Gamma\tilde{\Gamma}}$ under $\mathcal{P}$, $\mathcal{CS}'$, $\mathcal{CS}''$, $\mathcal{CPS}'$ and $\mathcal{CPS}''$ are noted. The operators $\mathcal{O}_{\tilde{T}T}$ and $\mathcal{O}_{\tilde{T}\tilde{T}}$ are not explicitly shown in the above matrix, as they coincide with $\mathcal{O}_{T\tilde{T}}$ and $\mathcal{O}_{TT}$, respectively. For the Fierz four-quark operators $\mathcal{O}^{F}_{\Gamma\tilde{\Gamma}}$, we must exchange the columns $\mathcal{CS}' \to \mathcal{CS}''$ and $\mathcal{CPS}' \to \mathcal{CPS}''$.

Basis of operators and Green's Functions

The new basis of operators can be further decomposed into smaller independent bases according to symmetries P, S, CPS', CPS''. Following the notation in the literature [1], the 20 operators of Table 1 (including the Fierz operators) are classified into 4 categories:

(a) Parity Conserving ($P = +1$) operators with $S = +1$: $Q_i^{S=+1}$, $(i = 1, 2, \ldots, 5)$,

(b) Parity Conserving ($P = +1$) operators with $S = -1$: $Q_i^{S=-1}$, $(i = 1, 2, \ldots, 5)$,

(c) Parity Violating ($P = -1$) operators with $S = +1$: $\mathcal{Q}_i^{S=+1}$, $(i = 1, 2, \ldots, 5)$,

(d) Parity Violating ($P = -1$) operators with $S = -1$: $\mathcal{Q}_i^{S=-1}$, $(i = 1, 2, \ldots, 5)$,

A. Donini, V. Gimenez, G. Martinelli, M. Talevi and A. Vladikas, Eur. Phys. J. C **10** (1999), 121-142 doi:10.1007/s100529900097 [arXiv:hep-lat/9902030 [hep-lat]].

Mixing of four-quark operators upon Renormalization

The renormalized Parity Conserving (Violating) operators, $\hat{Q}^{S=\pm1}$ ($\hat{\mathcal{Q}}^{S=\pm1}$), are defined via the equations:

$$\hat{Q}_l^{S=\pm1} = Z_{lm}^{S=\pm1} \cdot Q_m^{S=\pm1}, \quad \hat{\mathcal{Q}}_l^{S=\pm1} = \mathcal{Z}_{lm}^{S=\pm1} \cdot \mathcal{Q}_m^{S=\pm1},$$

where $l, m = 1, \ldots, 5$ (a sum over m is implied).

Our focus is on the four-quark operators with $\Delta F = 2$, which do not mix with lower-dimensional operators, which have the same symmetry properties.

$$\mathcal{O}_{\Gamma\tilde{\Gamma}}(x) = \bar{\psi}_{f_1}(x)\Gamma\psi_{f_3}(x)\bar{\psi}_{f_2}(x)\tilde{\Gamma}\psi_{f_4}(x),$$

$$f_1 \notin \{f_3, f_4\}$$

and

$$f_2 \notin \{f_3, f_4\}$$

Basis of operators and Green's Functions

The four-quark operators mix among themselves and are grouped as shown in the curly brackets.

Parity even:

$$\begin{cases} Q_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] + \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right] \\ Q_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] - \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right] \\ Q_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] - \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right] \\ Q_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] + \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right] \\ Q_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{TT} \pm \mathcal{O}^F_{TT}\right] \end{cases}$$

Parity odd:

$$\begin{cases} \mathcal{Q}_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] + \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right] \end{cases}$$

$$\begin{cases} \mathcal{Q}_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] - \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right] \\ \mathcal{Q}_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] - \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right] \end{cases}$$

$$\begin{cases} \mathcal{Q}_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] + \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right] \\ \mathcal{Q}_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{T\tilde{T}} \pm \mathcal{O}^F_{T\tilde{T}}\right] \end{cases}$$

Mixing Matrices

Therefore, the mixing matrices $Z^{S=\pm 1}$ ($\mathcal{Z}^{S=\pm 1}$), which renormalize the Parity Conserving (Violating) operators, take the following form:

$$
\begin{aligned}
Z^{S=\pm 1} &= \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} \end{pmatrix}^{S=\pm 1} , \\
\mathcal{Z}^{S=\pm 1} &= \begin{pmatrix} \mathcal{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22} & \mathcal{Z}_{23} & 0 & 0 \\ 0 & \mathcal{Z}_{32} & \mathcal{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ 0 & 0 & 0 & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}^{S=\pm 1} .
\end{aligned}
$$

Importance of the Conversion Matrices

In order to arrive at the renormalized four-quark operators in the more standard $\overline{\mathrm{MS}}$ scheme (experimental data), the conversion matrices $(C^{S=\pm1})^{\overline{\mathrm{MS}},\mathrm{GIRS}}$ and $(\tilde{C}^{S=\pm1})^{\overline{\mathrm{MS}},\mathrm{GIRS}}$ between GIRS and $\overline{\mathrm{MS}}$ schemes are necessary:

$$
\begin{aligned}
(Z^{S=\pm1})^{\overline{\mathrm{MS}}} &= (C^{S=\pm1})^{\overline{\mathrm{MS}},\mathrm{GIRS}}(Z^{S=\pm1})^{\mathrm{GIRS}}, \\
(\mathcal{Z}^{S=\pm1})^{\overline{\mathrm{MS}}} &= (\tilde{C}^{S=\pm1})^{\overline{\mathrm{MS}},\mathrm{GIRS}}(\mathcal{Z}^{S=\pm1})^{\mathrm{GIRS}}.
\end{aligned}
$$

These conversion matrices can be computed only perturbatively due to the very nature of $\overline{\mathrm{MS}}$. Being regularization-independent, they are evaluated more easily in Dimensional Regularization (DR). In *DR*, one goes to D dimensions and the regulator, ϵ, is defined by $D \equiv 4 - 2\epsilon$.

Renormalization procedure in the GIRS scheme

- The GIRS scheme is employed for extracting the conversion matrices $(C^{S=\pm 1})^{\overline{\rm MS},{\rm GIRS}}$ and $(\tilde{C}^{S=\pm 1})^{\overline{\rm MS},{\rm GIRS}}$ between GIRS and $\overline{\rm MS}$ schemes; we calculate the one-loop expressions for these conversion matrices for various GIRS variants.

In the case of a multiplicatively renormalizable operator, $\mathcal{O}$, a typical condition in GIRS has the following form:

$$(Z_{\mathcal{O}}^{\rm GIRS})^2 \langle \mathcal{O}(x)\, \mathcal{O}^\dagger(y)\rangle\Big|_{x-y=\bar{z}} = \langle \mathcal{O}(x)\, \mathcal{O}^\dagger(y)\rangle^{\rm tree}\Big|_{x-y=\bar{z}},$$

where $\bar{z}$ is a nonzero renormalization 4-vector scale.

- $\langle \mathcal{O}(x)\, \mathcal{O}^\dagger(y)\rangle$ is gauge independent.
- When operator mixing occurs, one needs to consider a set of conditions involving more than one Green's functions of two or more gauge-invariant operators.

Determination of mixing matrices – Green's functions

The determination of the 5×5 mixing matrices requires the calculation of: **(i)** two-point Green's functions with two four-quark operators:

$$G^{\rm 2pt}_{\mathcal{O}_{\Gamma\tilde\Gamma};\,\mathcal{O}_{\Gamma'\tilde\Gamma'}}(z) \;\equiv\; \langle \mathcal{O}_{\Gamma\tilde\Gamma}(x)\,\mathcal{O}^\dagger_{\Gamma'\tilde\Gamma'}(y)\rangle, \qquad z \equiv x-y,\; x \neq y.$$

(ii) three-point Green's functions with one four-quark operator and two quark bilinear operators ($\mathcal{O}_\Gamma(x) = \bar\psi_{f_1}(x)\Gamma\psi_{f_2}(x)$):

$$G^{\rm 3pt}_{\mathcal{O}_{\Gamma'};\,\mathcal{O}_{\Gamma\tilde\Gamma};\,\mathcal{O}_{\Gamma''}}(z,z') \;\equiv\; \langle \mathcal{O}_{\Gamma'}(x)\,\mathcal{O}_{\Gamma\tilde\Gamma}(y)\mathcal{O}_{\Gamma''}(w)\rangle, \qquad z \equiv x-y,\; z' \equiv y-w,\; x \neq y \neq w \neq x.$$

Two-point Green's functions with one four-quark operator and one bilinear operator are not considered since they vanish when $\Delta F = 2$.

Green's functions – Corresponding Feynman diagrams

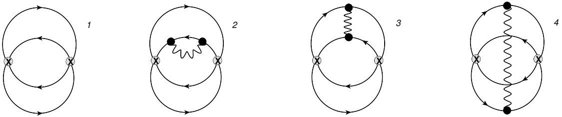

Figure 2: Feynman diagrams contributing to $\langle \mathcal{O}_{\Gamma\tilde{\Gamma}}(x)\, \mathcal{O}^{\dagger}_{\Gamma'\tilde{\Gamma}'}(y)\rangle$, to order $\mathcal{O}(g^0)$ (diagram 1) and $\mathcal{O}(g^2)$ (the remaining diagrams). Wavy (solid) lines represent gluons (quarks). Diagrams 2 and 4 have also mirror variants.

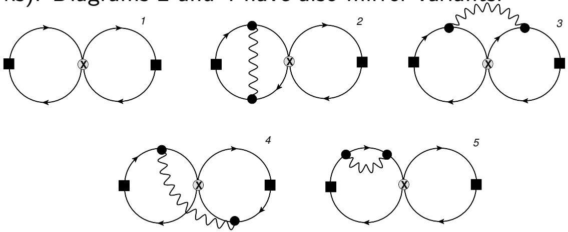

Figure 3: Feynman diagrams contributing to $\langle \mathcal{O}_{\Gamma'}(x)\, \mathcal{O}_{\Gamma\tilde{\Gamma}}(0)\mathcal{O}_{\Gamma''}(y)\rangle$, to order $\mathcal{O}(g^0)$ (diagram 1) and $\mathcal{O}(g^2)$ (the remaining diagrams). Wavy (solid) lines represent gluons (quarks). A circled cross denotes the insertion of the four-quark operator, and the solid squares denote the quark bilinear operators. Diagrams 2-5 have also mirror variants.

Renormalization Conditions for Parity Conserving operators (Q_i)

In the case of the Parity Conserving operators (Q_i) we need to calculate the 25 elements of the mixing matrix for both $S=+1$ and $S=-1$:
From relevant 2-pt GFs, we get 15 conditions, and another 10 conditions that will be extracted from the relevant 3-pt GFs.

$$[\tilde{G}^{\rm 2pt}_{Q_i^{S=\pm1};Q_j^{S=\pm1}}(t)]^{\rm GIRS} \equiv \sum_{k,l=1}^{5} (Z_{ik}^{S\pm1})^{\rm GIRS} (Z_{jl}^{S\pm1})^{\rm GIRS}\ \tilde{G}^{\rm 2pt}_{Q_k^{S=\pm1};Q_l^{S=\pm1}}(t) = [\tilde{G}^{\rm 2pt}_{Q_i^{S=\pm1};Q_j^{S=\pm1}}(t)]^{\rm tree},$$

where i, j run from 1 to 5 and $i \leq j$; $z_4 := t$ is the GIRS renormalization scale.

We have a variety of options for selecting the remaining conditions involving three-point Green's functions:

$$[\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;Q_i^{S=\pm1};\mathcal{O}_\Gamma}(t,\,t')]^{\rm GIRS} \equiv (Z_{\mathcal{O}_\Gamma}^{\rm GIRS})^2 \sum_{k=1}^{5} (Z_{ik}^{S\pm1})^{\rm GIRS}\ \tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;Q_k^{S=\pm1};\mathcal{O}_\Gamma}(t,\,t') = [\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;Q_i^{S=\pm1};\mathcal{O}_\Gamma}(t,\,t')]^{\rm tree},$$

where $i \in [1,5]$, $\Gamma \in \{\mathbb{1},\, \gamma_5,\, \gamma_\mu,\, \gamma_\mu\gamma_5,\, \sigma_{\mu\nu}\}$, and $z_4 := t$, $z_4' := t'$ are GIRS renormalization scales. In this case, the two bilinears must be the same in order to obtain a nonzero Green's function. $Z^{\rm GIRS}_{\mathcal{O}_\Gamma}$ is the renormalization factor of the bilinear operator $\mathcal{O}_\Gamma$.

Renormalization Conditions for Parity Violating operators ($\mathcal{Q}_i$)

In the case of the Parity Violating operators ($\mathcal{Q}_i$), the 5×5 mixing matrix is block diagonal for both $S = +1$ and $S = -1$, we need to calculate only 9 elements (7 conditions for 2-pt GFs and 2 conditions for 3-pt GFs).

$$[\tilde{G}^{\rm 2pt}_{\mathcal{Q}_1^{S=\pm1};\mathcal{Q}_1^{S=\pm1}}(t)]^{\rm GIRS} \equiv [(\mathcal{Z}_{11}^{S\pm1})^{\rm GIRS}]^2\, \tilde{G}^{\rm 2pt}_{\mathcal{Q}_1^{S=\pm1};\mathcal{Q}_1^{S=\pm1}}(t) = [\tilde{G}^{\rm 2pt}_{\mathcal{Q}_1^{S=\pm1};\mathcal{Q}_1^{S=\pm1}}(t)]^{\rm tree},$$

$$[\tilde{G}^{\rm 2pt}_{\mathcal{Q}_i^{S=\pm1};\mathcal{Q}_j^{S=\pm1}}(t)]^{\rm GIRS} \equiv \sum_{k,l=2}^{3} (\mathcal{Z}_{ik}^{S\pm1})^{\rm GIRS} (\mathcal{Z}_{jl}^{S\pm1})^{\rm GIRS}\, \tilde{G}^{\rm 2pt}_{\mathcal{Q}_k^{S=\pm1};\mathcal{Q}_l^{S=\pm1}}(t) = [\tilde{G}^{\rm 2pt}_{\mathcal{Q}_i^{S=\pm1};\mathcal{Q}_j^{S=\pm1}}(t)]^{\rm tree},$$

$$[\tilde{G}^{\rm 2pt}_{\mathcal{Q}_i^{S=\pm1};\mathcal{Q}_j^{S=\pm1}}(t)]^{\rm GIRS} \equiv \sum_{k,l=4}^{5} (\mathcal{Z}_{ik}^{S\pm1})^{\rm GIRS} (\mathcal{Z}_{jl}^{S\pm1})^{\rm GIRS}\, \tilde{G}^{\rm 2pt}_{\mathcal{Q}_k^{S=\pm1};\mathcal{Q}_l^{S=\pm1}}(t) = [\tilde{G}^{\rm 2pt}_{\mathcal{Q}_i^{S=\pm1};\mathcal{Q}_j^{S=\pm1}}(t)]^{\rm tree}.$$

Note that in the above equations $i \leq j$. The two conditions that include three-point functions can be:

$$\begin{aligned}
[\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_i^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t')]^{\rm GIRS} &\equiv Z^{\rm GIRS}_{\mathcal{O}_\Gamma}\, Z^{\rm GIRS}_{\mathcal{O}_{\Gamma\gamma_5}} \sum_{k=2}^{3} (\mathcal{Z}_{ik}^{S\pm1})^{\rm GIRS}\, \tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_k^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t') \\
&= [\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_i^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t')]^{\rm tree},
\end{aligned}$$

$$\begin{aligned}
[\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_i^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t')]^{\rm GIRS} &\equiv Z^{\rm GIRS}_{\mathcal{O}_\Gamma}\, Z^{\rm GIRS}_{\mathcal{O}_{\Gamma\gamma_5}} \sum_{k=4}^{5} (\mathcal{Z}_{ik}^{S\pm1})^{\rm GIRS}\, \tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_k^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t') \\
&= [\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_i^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t')]^{\rm tree}.
\end{aligned}$$

Results for tree level Green's functions

We present our results for the bare tree-level two-point Green's function of two four-quark operators with arbitrary Dirac matrices and arbitrary flavors (f_i, f_i', $i = 1, 2, 3, 4$) carried by the quark fields.

$$\langle\left(\bar\psi_{f_1}(x)\Gamma\psi_{f_3}(x)\bar\psi_{f_2}(x)\Gamma\psi_{f_4}(x)\right)\left(\bar\psi_{f_1'}(y)\Gamma'\psi_{f_3'}(y)\bar\psi_{f_2'}(y)\Gamma'\psi_{f_4'}(y)\right)\rangle^{\rm tree} = \frac{N_c\ \Gamma(2-\epsilon)^4}{16\ \pi^{8-4\epsilon}\ (z^2)^{8-4\epsilon}} \times$$
$$\Big\{\delta_{f_1 f_3'}\delta_{f_2 f_4'}\ [N_c\ \delta_{f_3 f_1'}\delta_{f_4 f_2'}\ {\rm tr}(\Gamma\not{z}\Gamma'\not{z})\ {\rm tr}(\Gamma\not{z}\Gamma'\not{z}) - \delta_{f_3 f_2'}\delta_{f_4 f_1'}\ {\rm tr}(\Gamma\not{z}\Gamma'\not{z}\Gamma\not{z}\Gamma'\not{z})]$$
$$+\delta_{f_1 f_4'}\delta_{f_2 f_3'}\ [N_c\ \delta_{f_3 f_2'}\delta_{f_4 f_1'}\ {\rm tr}(\Gamma\not{z}\Gamma'\not{z})\ {\rm tr}(\Gamma\not{z}\Gamma'\not{z}) - \delta_{f_3 f_1'}\delta_{f_4 f_2'}\ {\rm tr}(\Gamma\not{z}\Gamma'\not{z}\Gamma\not{z}\Gamma'\not{z})]\Big\},$$

where N_c is the number of colors.

The tree-level three-point Green's function of one four-quark and two quark bilinear operators for arbitrary Dirac matrices and flavors is given below to all orders in ϵ and in terms of the D-vectors $z \equiv x - y$ and $z' \equiv y - w$, which connect the four-quark operator with the left and right bilinear operators, respectively:

$$\langle\left(\bar\psi_{f_1'}(x)\Gamma'\psi_{f_2'}(x)\right)\left(\bar\psi_{f_1}(y)\Gamma\psi_{f_3}(y)\bar\psi_{f_2}(y)\Gamma\psi_{f_4}(y)\right)\left(\bar\psi_{f_1''}(w)\Gamma''\psi_{f_2''}(w)\right)\rangle^{\rm tree} = \frac{N_c\ \Gamma(2-\epsilon)^4}{16\ \pi^{8-4\epsilon}\ (z^2)^{4-2\epsilon}\ (z'^2)^{4-2\epsilon}} \times$$
$$\Big\{\delta_{f_3 f_1'}\delta_{f_4 f_1''}[N_c\ \delta_{f_1 f_2'}\delta_{f_2 f_2''}\ {\rm tr}(\Gamma'\not{z}\Gamma\not{z})\ {\rm tr}(\Gamma\not{z}'\Gamma''\not{z}') - \delta_{f_2 f_2'}\delta_{f_1 f_2''}\ {\rm tr}(\Gamma'\not{z}\Gamma\not{z}'\Gamma''\not{z}'\Gamma\not{z})]$$
$$+\delta_{f_4 f_1'}\delta_{f_3 f_1''}[N_c\ \delta_{f_2 f_2'}\delta_{f_1 f_2''}\ {\rm tr}(\Gamma'\not{z}\Gamma\not{z})\ {\rm tr}(\Gamma\not{z}'\Gamma''\not{z}') - \delta_{f_1 f_2'}\delta_{f_2 f_2''}\ {\rm tr}(\Gamma'\not{z}\Gamma\not{z}'\Gamma''\not{z}'\Gamma\not{z})]\Big\}.$$

Results - examples for $\overline{\mathrm{MS}}$ Green's functions

As an example, we provide one two-point and one three-point Green's function renormalized in $\overline{\mathrm{MS}}$; they depend on the scales z and/or z' corresponding to the separations between the operators that are present in each Green's function, as well as on the $\overline{\mathrm{MS}}$ renormalization scale $\bar{\mu}$ appearing in the renormalization of the coupling constant in D dimensions: $g_R = \mu^{-\epsilon} Z_g^{-1} g_B$ [g_B (g_R) is the bare (renormalized) coupling constant, $\mu = \bar{\mu}\sqrt{e^{\gamma_E}/4\pi}$].

$$
\begin{aligned}
[G^{2\mathrm{pt}}_{Q_1^{S=\pm1};Q_1^{S=\pm1}}(z)]^{\overline{\mathrm{MS}}} = {} & \frac{4N_c}{\pi^8 (z^2)^6} \left(\delta_{f_1 f_4'}\,\delta_{f_2 f_3'} \pm \delta_{f_1 f_3'}\,\delta_{f_2 f_4'}\right)\left(\delta_{f_3 f_2'}\,\delta_{f_4 f_1'} \pm \delta_{f_3 f_1'}\,\delta_{f_4 f_2'}\right) \times \\
& \left\{ \pm 1 + N_c + 2\,\frac{g^2_{\overline{\mathrm{MS}}}\,C_F}{16\pi^2}\left[\pm 6 + 7N_c \mp 6\left(\ln\left(\bar{\mu}^2 z^2\right) + 2\gamma_E - 2\ln(2)\right)\right] + \mathcal{O}(g^4_{\overline{\mathrm{MS}}})\right\},
\end{aligned}
$$

$$
\begin{aligned}
[G^{3\mathrm{pt}}_{V_\mu;Q_1^{S=\pm1};V_\mu}(z,z')]^{\overline{\mathrm{MS}}} = {} & \frac{N_c}{\pi^8 (z^2)^3 (z'^2)^3} \left(\delta_{f_1' f_4}\,\delta_{f_1'' f_3} \pm \delta_{f_1' f_3}\,\delta_{f_1'' f_4}\right)\left(\delta_{f_2' f_2}\,\delta_{f_2'' f_1} \pm \delta_{f_2' f_1}\,\delta_{f_2'' f_2}\right) \times \\
& \left\{ \frac{N_c \pm 1}{2}\left[1 - 2\frac{z_\mu}{z^2} - 2\frac{z'_\mu}{z'^2} + 4\frac{(z\cdot z')z_\mu z'_\mu}{z^2 z'^2}\right] \pm \frac{g^2_{\overline{\mathrm{MS}}}\,C_F}{16\pi^2}\left[1 - 2\frac{(z_\mu + z'_\mu)^2}{(z+z')^2}\right] \right. \\
& \pm \frac{g^2_{\overline{\mathrm{MS}}}\,C_F}{16\pi^2}\left[1 - 2\frac{z_\mu}{z^2} - 2\frac{z'_\mu}{z'^2} + 4\frac{(z\cdot z')z_\mu z'_\mu}{z^2 z'^2}\right] \times \\
& \left. \left[2 - 3\left(\ln\left(\frac{\bar{\mu}^2 z^2 z'^2}{(z+z')^2}\right) + 2\gamma_E - 2\ln(2) \mp N_c\right)\right] + \mathcal{O}(g^4_{\overline{\mathrm{MS}}})\right\}.
\end{aligned}
$$

Results – $\overline{\mathrm{MS}}$-renormalized Green's functions

The $\overline{\mathrm{MS}}$-renormalized two-point and three-point Green's functions after integration over timeslices are shown below. These are relevant for the extraction of the conversion matrices. They are written in a compact form for all four-quark and quark bilinear operators:

$$[\tilde{G}^{2\mathrm{pt}}_{Q_i^{S=\pm1};Q_j^{S=\pm1}}(t)]^{\overline{\mathrm{MS}}} = \frac{N_c}{\pi^6\,|t|^9}\left(\delta_{f_1 f_4'}\,\delta_{f_2 f_3'} \pm \delta_{f_1 f_3'}\,\delta_{f_2 f_4'}\right)\left(\delta_{f_3 f_2'}\,\delta_{f_4 f_1'} \pm \delta_{f_3 f_1'}\,\delta_{f_4 f_2'}\right)\Big\{\left(a^{\pm}_{ij;0} + a^{\pm}_{ij;1} N_c\right)$$
$$+ \frac{g^2_{\overline{\mathrm{MS}}}\,C_F}{16\pi^2}\left[\left(b^{\pm}_{ij;0} + b^{\pm}_{ij;1} N_c\right) + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\left(c^{\pm}_{ij;0} + c^{\pm}_{ij;1} N_c\right)\right] + \mathcal{O}(g^4_{\overline{\mathrm{MS}}})\Big\},$$

$$[\tilde{G}^{2\mathrm{pt}}_{\mathcal{Q}_i^{S=\pm1};\mathcal{Q}_j^{S=\pm1}}(t)]^{\overline{\mathrm{MS}}} = \frac{N_c}{\pi^6\,|t|^9}\left(\delta_{f_1 f_4'}\,\delta_{f_2 f_3'} \pm \delta_{f_1 f_3'}\,\delta_{f_2 f_4'}\right)\left(\delta_{f_3 f_2'}\,\delta_{f_4 f_1'} \pm (-1)^{\delta_{i2}+\delta_{i3}}\,\delta_{f_3 f_1'}\,\delta_{f_4 f_2'}\right)\Big\{\left(\tilde{a}^{\pm}_{ij;0} + \tilde{a}^{\pm}_{ij;1} N_c\right)$$
$$+ \frac{g^2_{\overline{\mathrm{MS}}}\,C_F}{16\pi^2}\left[\left(\tilde{b}^{\pm}_{ij;0} + \tilde{b}^{\pm}_{ij;1} N_c\right) + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\left(\tilde{c}^{\pm}_{ij;0} + \tilde{c}^{\pm}_{ij;1} N_c\right)\right] + \mathcal{O}(g^4_{\overline{\mathrm{MS}}})\Big\},$$

$$[\tilde{G}^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;Q_i^{S=\pm1};\mathcal{O}_\Gamma}(t,t)]^{\overline{\mathrm{MS}}} = \frac{N_c}{\pi^4 t^6}\left(\delta_{f_1' f_4}\,\delta_{f_1'' f_3} \pm \delta_{f_1' f_3}\,\delta_{f_1'' f_4}\right)\left(\delta_{f_2' f_2}\,\delta_{f_2'' f_1} \pm \delta_{f_2' f_1}\,\delta_{f_2'' f_2}\right)\Big\{\left(d^{\pm}_{i\Gamma;0} + d^{\pm}_{i\Gamma;1} N_c\right)$$
$$+ \frac{g^2_{\overline{\mathrm{MS}}}\,C_F}{16\pi^2}\left[\left(e^{\pm}_{i\Gamma;0} + e^{\pm}_{i\Gamma;1} N_c\right) + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\left(f^{\pm}_{i\Gamma;0} + f^{\pm}_{i\Gamma;1} N_c\right)\right] + \mathcal{O}(g^4_{\overline{\mathrm{MS}}})\Big\},$$

$$[\tilde{G}^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}_i^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t)]^{\overline{\mathrm{MS}}} = \frac{N_c}{\pi^4 t^6}\left(\delta_{f_1' f_4}\,\delta_{f_1'' f_3} \pm \delta_{f_1' f_3}\,\delta_{f_1'' f_4}\right)\left(\delta_{f_2' f_2}\,\delta_{f_2'' f_1} \pm (-1)^{\delta_{i2}+\delta_{i3}}\,\delta_{f_2' f_1}\,\delta_{f_2'' f_2}\right)\Big\{\left(\tilde{d}^{\pm}_{i\Gamma;0} + \tilde{d}^{\pm}_{i\Gamma;1} N_c\right)$$
$$+ \frac{g^2_{\overline{\mathrm{MS}}}\,C_F}{16\pi^2}\left[\left(\tilde{e}^{\pm}_{i\Gamma;0} + \tilde{e}^{\pm}_{i\Gamma;1} N_c\right) + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\left(\tilde{f}^{\pm}_{i\Gamma;0} + \tilde{f}^{\pm}_{i\Gamma;1} N_c\right)\right] + \mathcal{O}(g^4_{\overline{\mathrm{MS}}})\Big\},$$

where $a^{\pm}_{ij;k}$, $b^{\pm}_{ij;k}$, $c^{\pm}_{ij;k}$, $\tilde{a}^{\pm}_{ij;k}$, $\tilde{b}^{\pm}_{ij;k}$, $\tilde{c}^{\pm}_{ij;k}$, $d^{\pm}_{i\Gamma;k}$, $e^{\pm}_{i\Gamma;k}$, $f^{\pm}_{i\Gamma;k}$, $\tilde{d}^{\pm}_{i\Gamma;k}$, $\tilde{e}^{\pm}_{i\Gamma;k}$, $\tilde{f}^{\pm}_{i\Gamma;k}$ are numerical coefficients [https://arxiv.org/pdf/2406.08065].

Results – one-loop conversion matrices

The one-loop conversion matrices between different variants of GIRS and the $\overline{\rm MS}$ scheme are extracted from our results by rewriting the GIRS conditions in terms of the conversion matrices, as follows:

$$[\tilde{G}^{\rm 2pt}_{Q_i^{S=\pm1};Q_j^{S=\pm1}}(t)]^{\overline{\rm MS}} = \sum_{k,l=1}^{5} (C_{ik}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}} \; (C_{jl}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}} \; [\tilde{G}^{\rm 2pt}_{Q_k^{S=\pm1};Q_l^{S=\pm1}}(t)]^{\rm tree},$$

$$[\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;Q_i^{S=\pm1};\mathcal{O}_\Gamma}(t,t)]^{\overline{\rm MS}} = (C_{\mathcal{O}_\Gamma}^{\overline{\rm MS},{\rm GIRS}})^2 \; \sum_{k=1}^{5} (C_{ik}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}} \; [\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;Q_k^{S=\pm1};\mathcal{O}_\Gamma}(t,t)]^{\rm tree},$$

$$[\tilde{G}^{\rm 2pt}_{\mathcal{Q}_i^{S=\pm1};\mathcal{Q}_j^{S=\pm1}}(t)]^{\overline{\rm MS}} = \sum_{k,l=1}^{5} (\tilde{C}_{ik}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}} \; (\tilde{C}_{jl}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}} \; [\tilde{G}^{\rm 2pt}_{\mathcal{Q}_k^{S=\pm1};\mathcal{Q}_l^{S=\pm1}}(t)]^{\rm tree},$$

$$[\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_i^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t)]^{\overline{\rm MS}} = (C_{\mathcal{O}_\Gamma}^{\overline{\rm MS},{\rm GIRS}}) \; (C_{\mathcal{O}_{\Gamma\gamma_5}}^{\overline{\rm MS},{\rm GIRS}}) \; \sum_{k=1}^{5} (\tilde{C}_{ik}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}} \times [\tilde{G}^{\rm 3pt}_{\mathcal{O}_\Gamma;\mathcal{Q}_k^{S=\pm1};\mathcal{O}_{\Gamma\gamma_5}}(t,t)]^{\rm tree},$$

where $C_{\mathcal{O}_\Gamma}^{\overline{\rm MS},{\rm GIRS}}$ is the conversion factor of the quark bilinear operator $\mathcal{O}_\Gamma$; they have been calculated to one loop in previous works. Note that the conversion matrix $(\tilde{C}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}}$ has the block diagonal form of $\mathcal{Z}^{S=\pm1}$.

Results – one-loop conversion matrices for parity conserving operators

From the options that give the smallest sum of squares of the off-diagonal coefficients (smallest mixing contributions), we choose one to present below. We avoid including tensor operators in the selected set of conditions, which are typically more noisy in simulations. Also, we prefer to have more scalar or pseudoscalar operators which are computationally cheaper compared to other bilinear operators. The selected set of conditions includes the following 10 renormalized three-point functions:

$$\tilde{G}^{\rm 3pt}_{S;Q_1^{S=\pm1};S}(t,t),\ \tilde{G}^{\rm 3pt}_{P;Q_1^{S=\pm1};P}(t,t),\ \tilde{G}^{\rm 3pt}_{V_i;Q_1^{S=\pm1};V_i}(t,t),\ \tilde{G}^{\rm 3pt}_{S;Q_2^{S=\pm1};S}(t,t),\ \tilde{G}^{\rm 3pt}_{P;Q_2^{S=\pm1};P}(t,t),$$

$$\tilde{G}^{\rm 3pt}_{S;Q_3^{S=\pm1};S}(t,t),\ \tilde{G}^{\rm 3pt}_{S;Q_5^{S=\pm1};S}(t,t),\ \tilde{G}^{\rm 3pt}_{P;Q_5^{S=\pm1};P}(t,t),\ \tilde{G}^{\rm 3pt}_{V_i;Q_5^{S=\pm1};V_i}(t,t),\ \tilde{G}^{\rm 3pt}_{A_i;Q_5^{S=\pm1};A_i}(t,t),$$

and the solution reads:

$$(C^{S\pm1}_{ij})^{\overline{\rm MS},{\rm GIRS}} = \delta_{ij} + \frac{g^2_{\overline{\rm MS}}}{16\pi^2}\sum_{k=-1}^{+1}\left[g^{\pm}_{ij;k} + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\ h^{\pm}_{ij;k}\right] N_c^k + \mathcal{O}(g^4_{\overline{\rm MS}}),$$

where $g^{\pm}_{ij;k}$, $h^{\pm}_{ij;k}$ are numerical coefficients.

Results – one-loop conversion matrices for parity conserving operators

i	j	$g^{\pm}_{ij;-1}$	$g^{\pm}_{ij;0}$	$g^{\pm}_{ij;+1}$	$h^{\pm}_{ij;-1}$	$h^{\pm}_{ij;0}$	$h^{\pm}_{ij;+1}$
1	1	$-869/140$	$\pm 379/140$	$7/2$	3	∓ 3	0
1	2	2	$\mp(723/280 - 6\ln(2))$	-2	0	0	0
1	3	$-723/140 + 12\ln(2)$	0	0	0	0	0
1	4	-4	± 4	0	0	0	0
1	5	-2	± 2	0	0	0	0
2	1	$397/280 + 6\ln(2)$	$\pm(163/280 - 6\ln(2))$	-2	0	0	0
2	2	$-9/2$	± 2	$7/2$	-3	0	0
2	3	4	∓ 2	0	0	∓ 6	0
2	4	4	± 8	0	0	0	0
2	5	-2	0	0	0	0	0
3	1	-1	± 1	0	0	0	0
3	2	1	$\pm 99/280$	0	0	0	0
3	3	$-38/35$	± 2	$251/140$	-3	0	3
3	4	4	$\pm 239/280$	$-321/140$	0	0	0
3	5	0	$\mp 239/560$	0	0	0	0
4	1	-1	± 1	0	0	0	0
4	2	1	$\mp 239/280$	0	0	0	0
4	3	4	± 2	$-799/140$	0	0	0
4	4	$-307/112 + 3\ln(2)$	$\pm 169/140$	$251/140$	-3	∓ 3	3
4	5	$-269/480 + 1/2\ \ln(2)$	$\pm(869/1680 - \ln(2))$	0	1	$\mp 1/2$	0
5	1	-6	± 6	0	0	0	0
5	2	-6	0	0	0	0	0
5	3	0	∓ 12	0	0	0	0
5	4	$-269/40 + 6\ln(2)$	$\mp(29/140 - 12\ln(2))$	0	12	± 6	0
5	5	$-1229/240 - 3\ln(2)$	$\pm 309/140$	$1709/420$	1	∓ 3	-1

Table 2: Numerical values of the coefficients $g^{\pm}_{ij;k}$, $h^{\pm}_{ij;k}$ appearing in

$$(C^{S\pm1}_{ij})^{\overline{\mathrm{MS}},\mathrm{GIRS}} = \delta_{ij} + \frac{g^2_{\overline{\mathrm{MS}}}}{16\pi^2} \sum_{k=-1}^{+1} \left[g^{\pm}_{ij;k} + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\ h^{\pm}_{ij;k} \right] N_c^k .$$

Results – one-loop conversion matrices for parity violating operators

The option that gives the smallest sum of squares of the off-diagonal coefficients include the following renormalized three-point functions:

$$\tilde{G}^{\rm 3pt}_{S;\mathcal{Q}_2^{S=\pm1};P}(t,t), \qquad \tilde{G}^{\rm 3pt}_{S;\mathcal{Q}_5^{S=\pm1};P}(t,t),$$

and the solution reads:

$$(\tilde{C}^{S\pm1}_{ij})^{\overline{\rm MS},{\rm GIRS}} = \delta_{ij} + \frac{g^2_{\overline{\rm MS}}}{16\pi^2} \sum_{k=-1}^{+1} \left[\tilde{g}^{\pm}_{ij;k} + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\ \tilde{h}^{\pm}_{ij;k}\right] N_c^k + \mathcal{O}(g^4_{\overline{\rm MS}}),$$

where the coefficients $\tilde{g}^{\pm}_{ij;k}$, $\tilde{h}^{\pm}_{ij;k}$, are numerical coefficients.

Results – one-loop conversion matrices for parity violating operators

i	j	$\tilde{g}^{\pm}_{ij;-1}$	$\tilde{g}^{\pm}_{ij;0}$	$\tilde{g}^{\pm}_{ij;+1}$	$\tilde{h}^{\pm}_{ij;-1}$	$\tilde{h}^{\pm}_{ij;0}$	$\tilde{h}^{\pm}_{ij;+1}$
1	1	$-869/140$	$\pm 379/140$	$7/2$	3	∓ 3	0
1	2	0	0	0	0	0	0
1	3	0	0	0	0	0	0
1	4	0	0	0	0	0	0
1	5	0	0	0	0	0	0
2	1	0	0	0	0	0	0
2	2	$-9/2$	0	$7/2$	-3	0	0
2	3	0	∓ 2	0	0	∓ 6	0
2	4	0	0	0	0	0	0
2	5	0	0	0	0	0	0
3	1	0	0	0	0	0	0
3	2	0	$\pm 99/280$	0	0	0	0
3	3	$-38/35$	0	$251/140$	-3	0	3
3	4	0	0	0	0	0	0
3	5	0	0	0	0	0	0
4	1	0	0	0	0	0	0
4	2	0	0	0	0	0	0
4	3	0	0	0	0	0	0
4	4	$-307/112 + 3\ln(2)$	$\pm 169/140$	$251/140$	-3	∓ 3	3
4	5	$-269/480 + 1/2\ \ln(2)$	$\pm(869/1680 - \ln(2))$	0	1	$\mp 1/2$	0
5	1	0	0	0	0	0	0
5	2	0	0	0	0	0	0
5	3	0	0	0	0	0	0
5	4	$-269/40 + 6\ln(2)$	$\mp(29/140 - 12\ln(2))$	0	12	± 6	0
5	5	$-1229/240 - 3\ln(2)$	$\pm 309/140$	$1709/420$	1	∓ 3	-1

Table 3: Numerical values of the coefficients $\tilde{g}^{\pm}_{ij;k}$, $\tilde{h}^{\pm}_{ij;k}$ appearing in $(\tilde{C}^{S\pm1}_{ij})^{\overline{\mathrm{MS}},\mathrm{GIRS}} = \delta_{ij} + \frac{g^2_{\overline{\mathrm{MS}}}}{16\pi^2} \sum_{k=-1}^{+1} \left[\tilde{g}^{\pm}_{ij;k} + \left(\ln\left(\bar{\mu}^2 t^2\right) + 2\gamma_E\right)\ \tilde{h}^{\pm}_{ij;k}\right] N_c^k$.

Summary – Future plans

- Perturbative study of the renormalization of four-quark operators involved in $\Delta F = 2$ processes by using a gauge-invariant renormalization scheme (GIRS).
- One-loop perturbative calculation of two-point Green's functions involving products of two four-quark operators, as well as three-point Green's functions with one four-quark and two bilinear operators in DR.
- Mixing matrices: $(Z^{S\pm1})^{\rm GIRS}$ and $(\mathcal{Z}^{S\pm1})^{\rm GIRS}$.
- Conversion matrices from GIRS to $\overline{\rm MS}$: $(C^{S\pm1})^{\overline{\rm MS},{\rm GIRS}}$ and $(\tilde{C}^{S\pm1})^{\overline{\rm MS},{\rm GIRS}}$.
- Future plans: Investigation of four-quark operators with $\Delta F = 1$ and $\Delta F = 0$.

Acknowledgements

Thank you for your attention!

Co-funded by the European Union

Republic of Cyprus

RESEARCH & INNOVATION FOUNDATION

The project (EXCELLENCE/0421/0025) is implemented under the programme of social cohesion "THALIA 2021-2027" co-funded by the European Union through the Research and Innovation Foundation (RIF). The results are generated within the FEDILA software (project: CONCEPT/0823/0052), which is also implemented under the same "THALIA 2021-2027" programme, co-funded by the European Union through RIF.