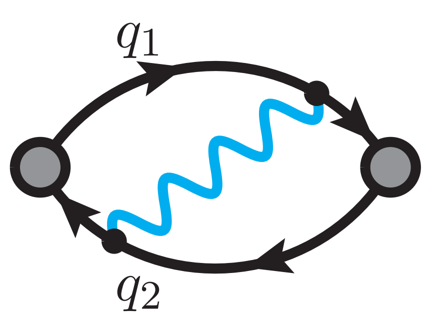
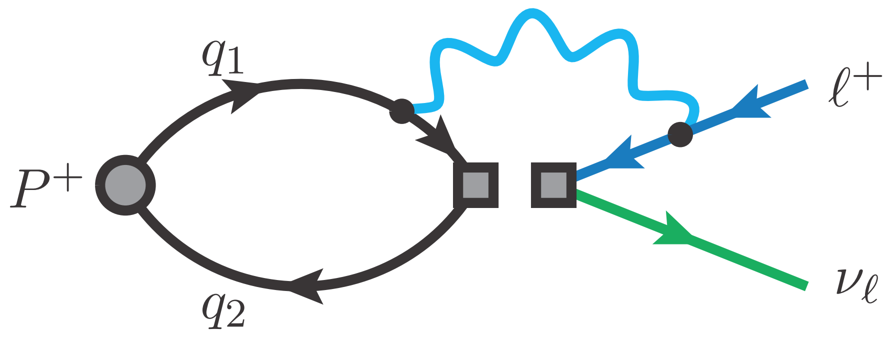
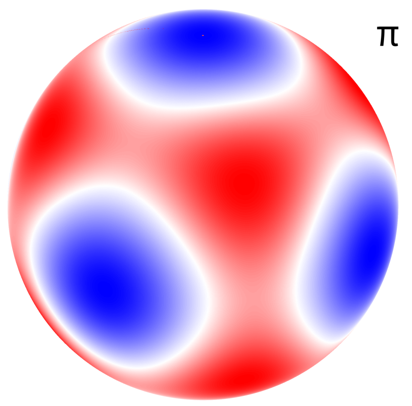
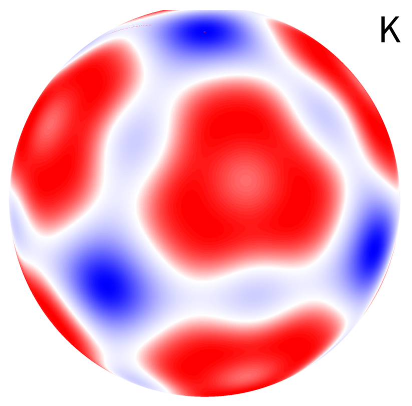
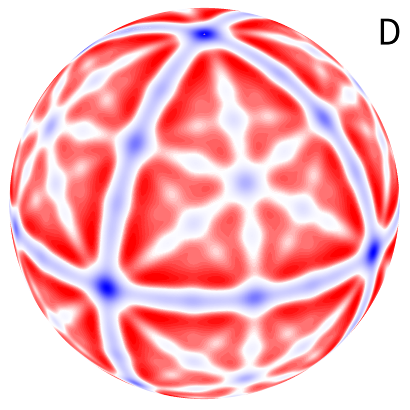
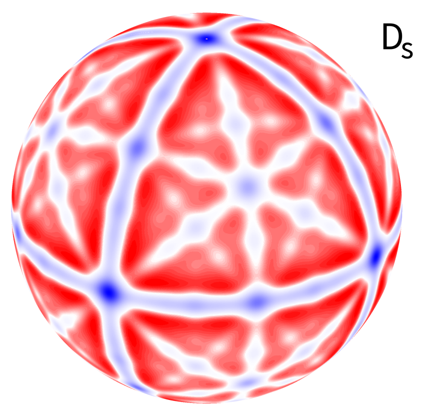












On-shell derivation of QED finite-volume effects

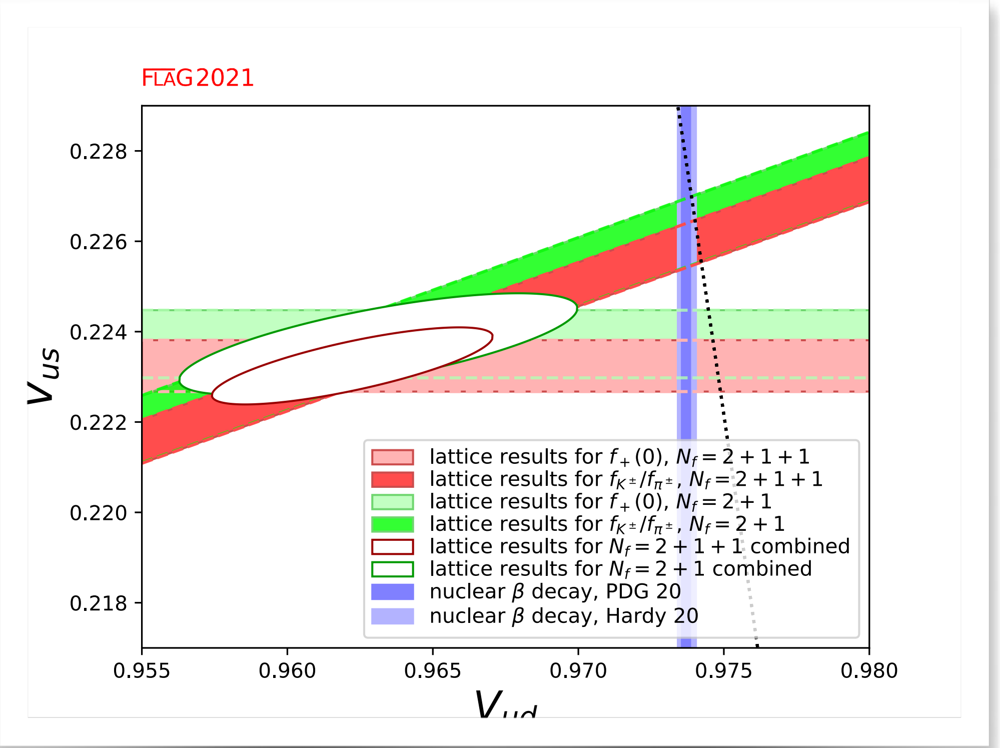
Matteo Di Carlo

2nd August 2024

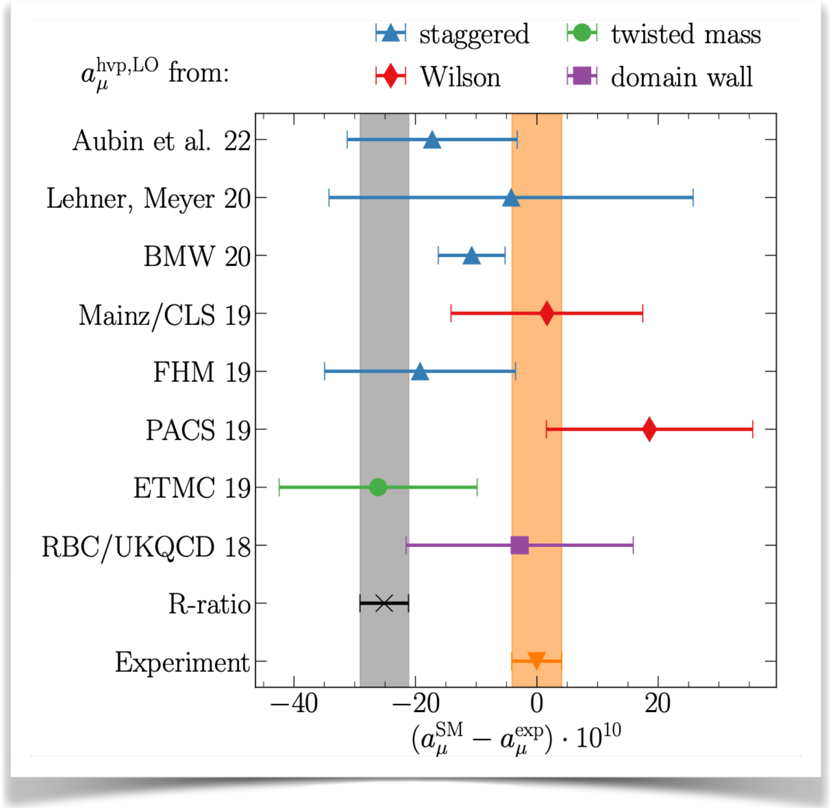
CERN
Funded by the European Union
LATTICE 2024 LIVERPOOL
UNIVERSITY OF LIVERPOOL

Motivations

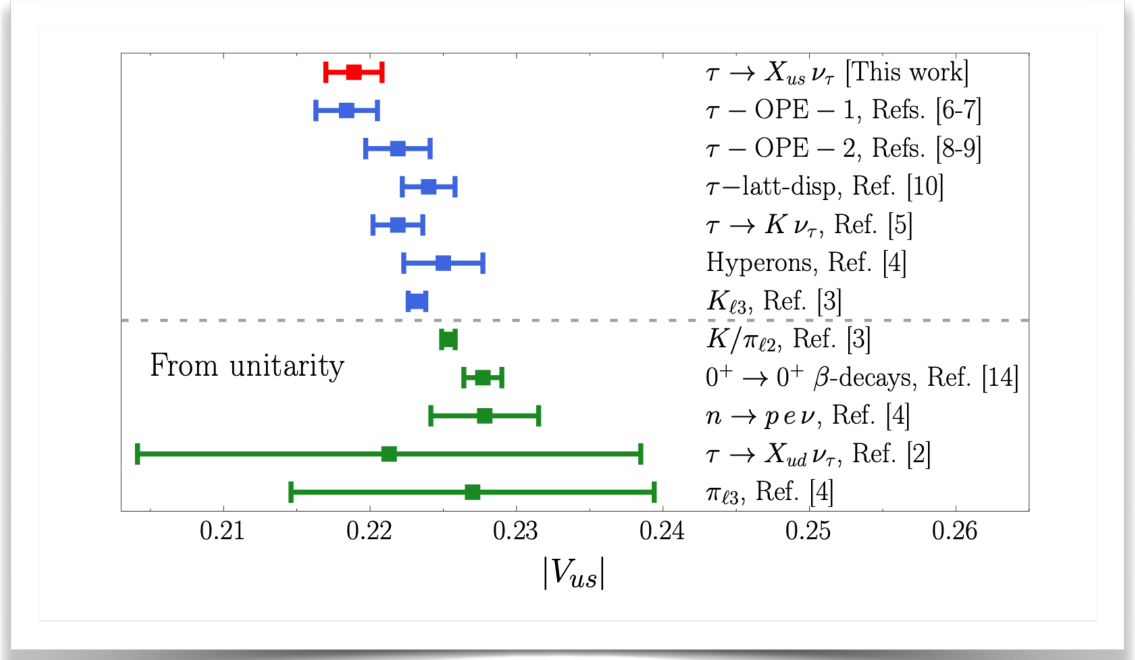
Indirect searches of new physics › high precision › isospin-breaking corrections



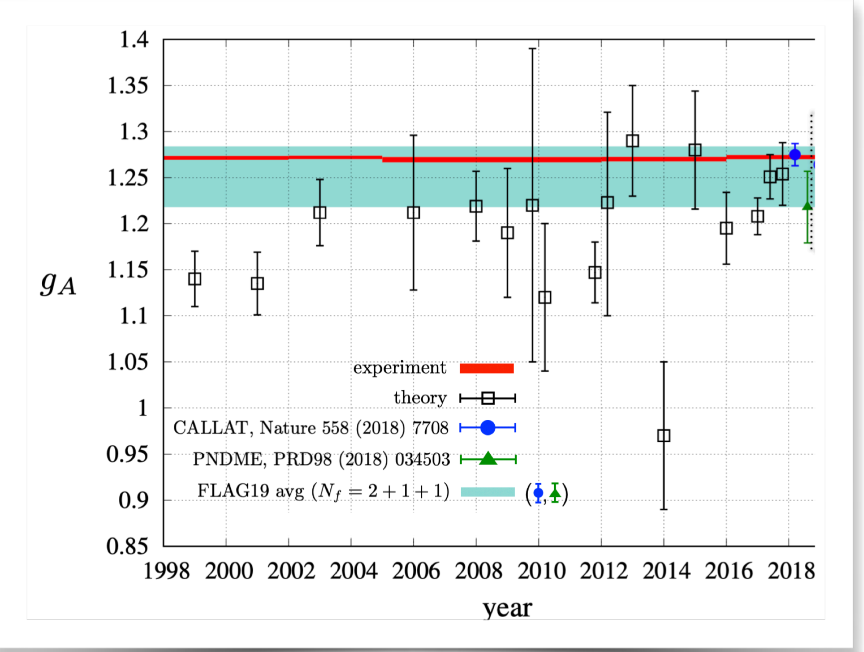
(semi)leptonic decays

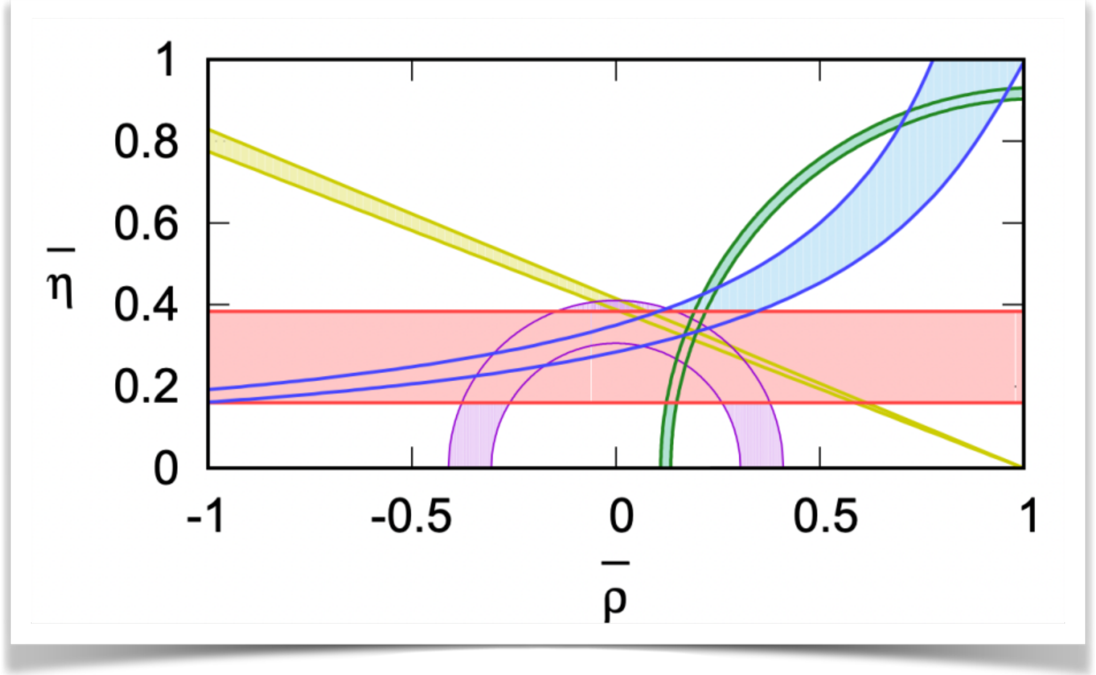


HVP muon g-2



Inclusive τ decays





Nucleon axial charge

CP violation parameters

J.Sitison, Thu 1.08
J.Parrino, Thu 1.08
D.Erb, Thu 1.08
A.Evangelista, Thu 1.08
A.Risch, Thu 1.08
L.Parato, Thu 1.08
C.McNeile, poster

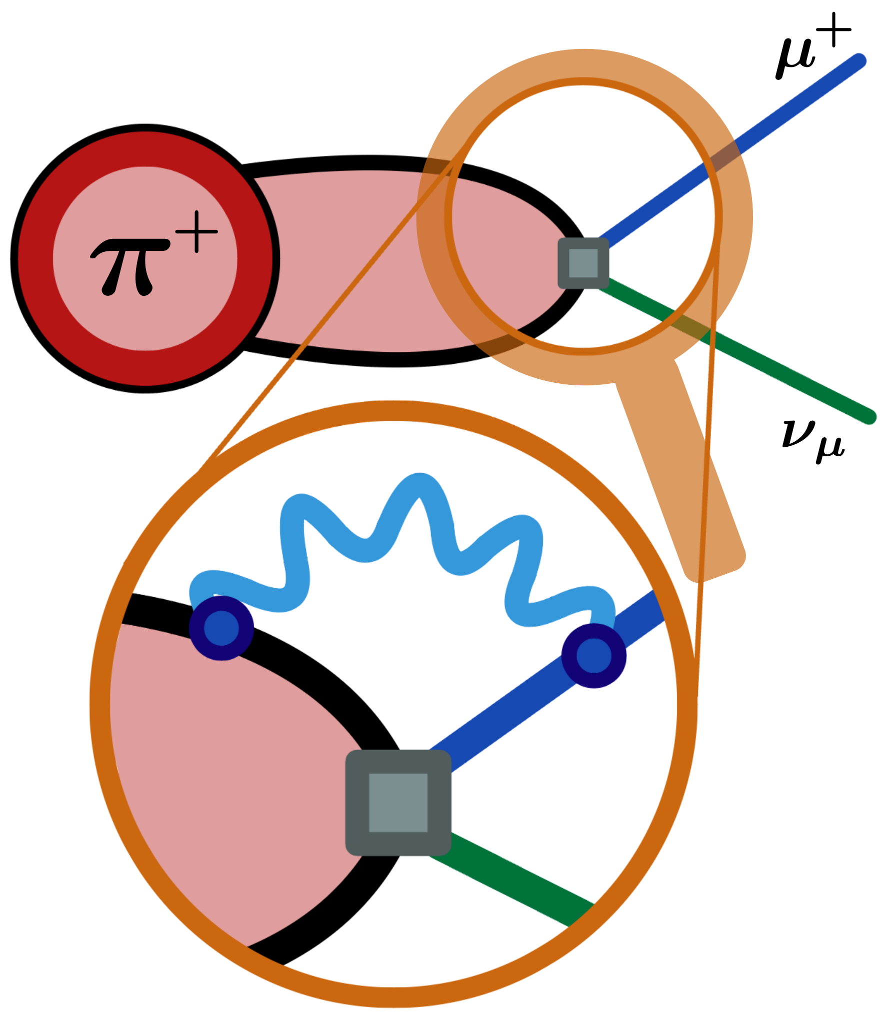
A.Cotellucci, Thu 1.08
R.Hill, Fri 2.08

M.Bruno, Mon 29.07
G.Gagliardi, Tue 30.07

A.Walker-Loud, Thu 1.08

C.Kelly, Thu 1.08

QED and isospin-breaking effects



Current level of precision requires the inclusion of isospin-breaking corrections due to

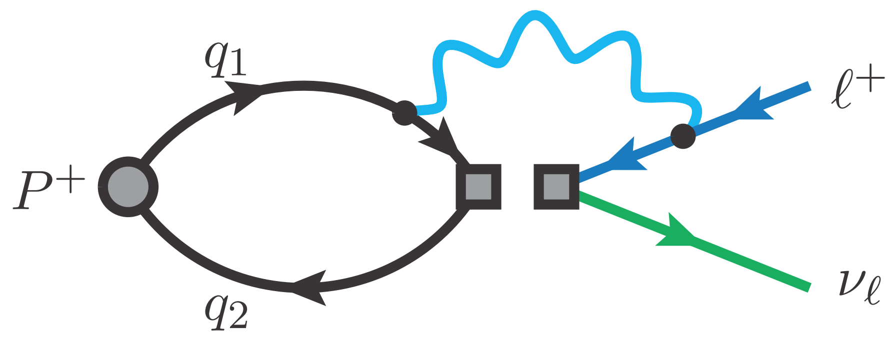
- strong effects $[m_u - m_d]_{\text{QCD}} \neq 0$
- electromagnetic effects $\alpha \neq 0$

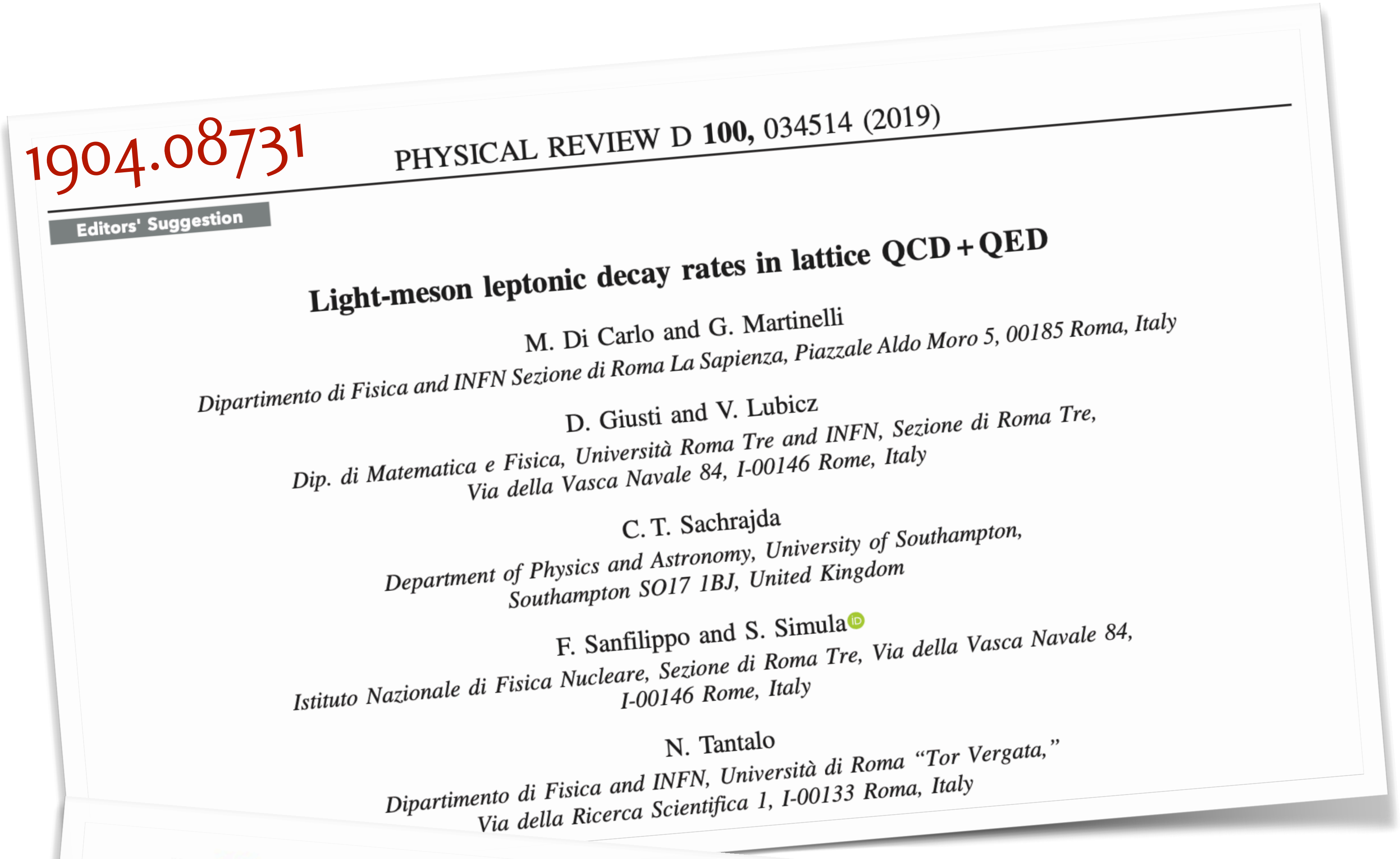
$\sim \mathcal{O}(1\%)$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |V_{us}|^2 \, |f_+^{K\pi}(0)|^2 \, (1 + \delta R^\ell_{K\pi})$$

- results from χPT currently quoted in the PDG — V.Cirigliano & H.Neufeld, PLB **700** (2011)
- but they can be obtained through first-principle **lattice calculations**

Leptonic decays of pseudoscalar mesons




1904.08731

PHYSICAL REVIEW D **100**, 034514 (2019)

Editors' Suggestion

Light-meson leptonic decay rates in lattice QCD+QED

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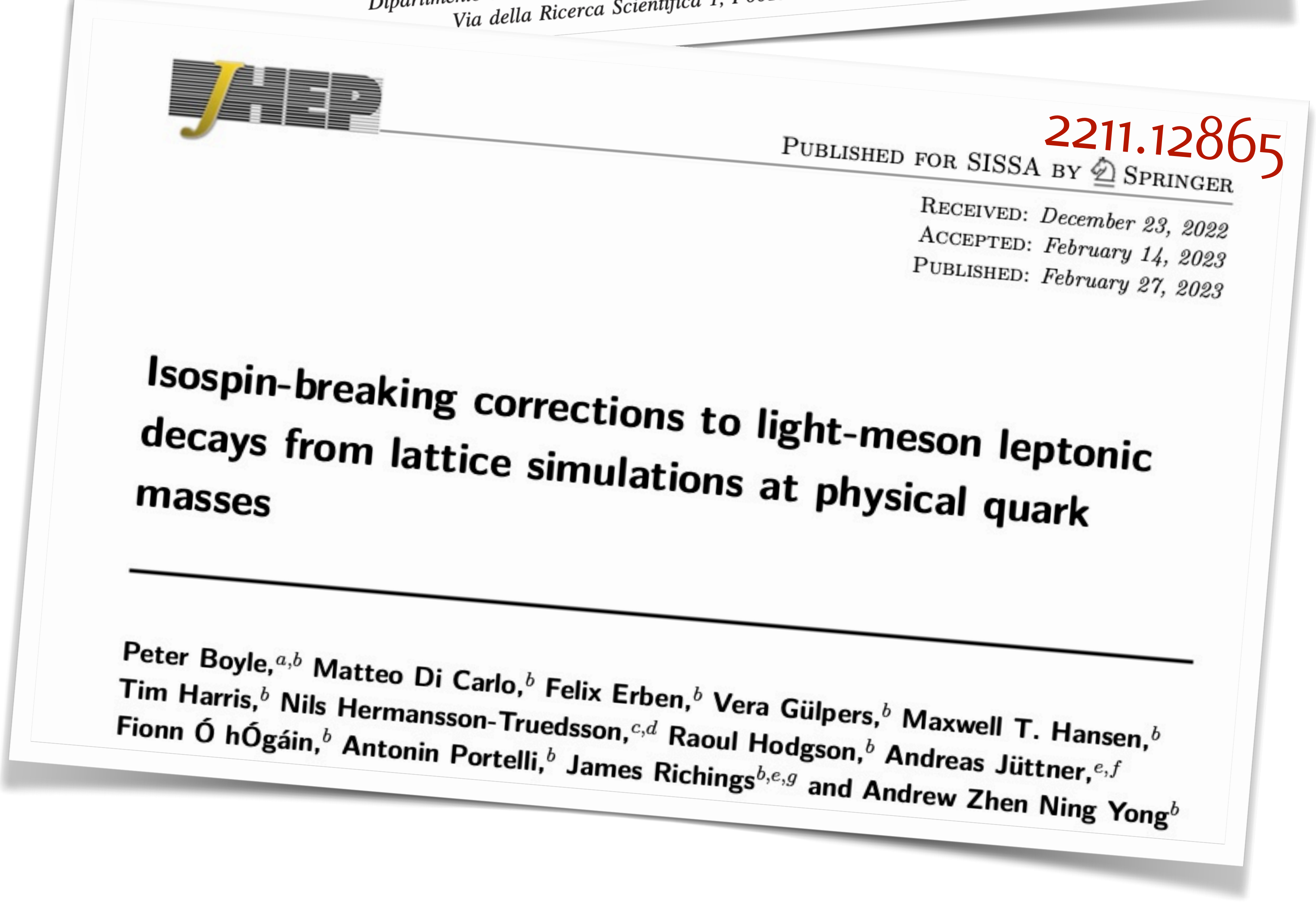
Dipartimento di Fisica and INFN, Università di Roma "Tor Vergata," Via della Ricerca Scientifica 1, I-00133 Roma, Italy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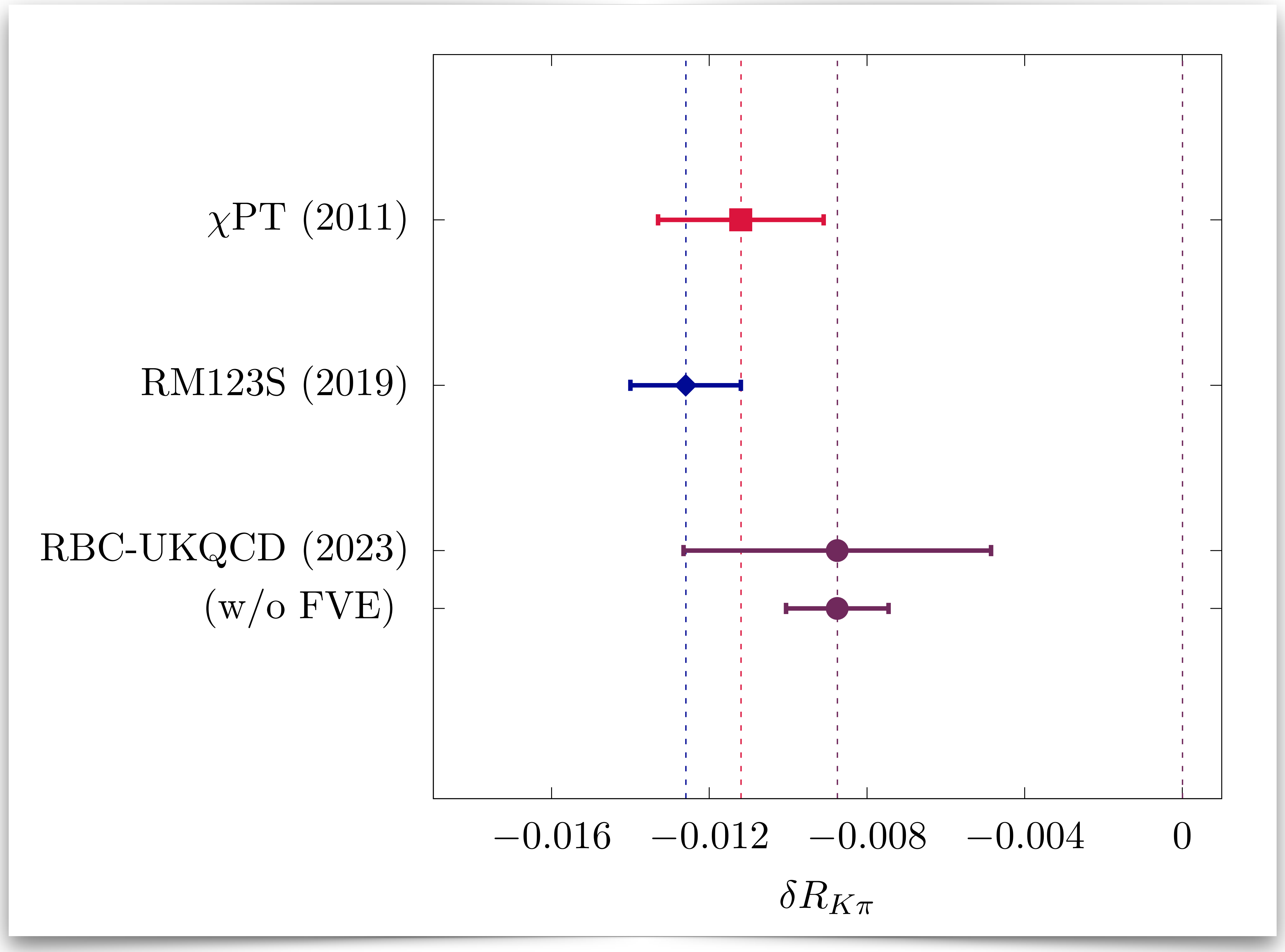
Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses

Peter Boyle,[a,b] Matteo Di Carlo,[b] Felix Erben,[b] Vera Gülpers,[b] Maxwell T. Hansen,[b] Tim Harris,[b] Nils Hermansson-Truedsson,[c,d] Raoul Hodgson,[b] Andreas Jüttner,[e,f] Fionn Ó hÓgáin,[b] Antonin Portelli,[b] James Richings[b,e,g] and Andrew Zhen Ning Yong[b]

- $\delta R_{K\pi} = -0.0112\,(21)$
- $\delta R_{K\pi} = -0.0126\,(14)$
- $\delta R_{K\pi} = -0.0086\,(13)(39)_{\mathrm{vol.}}$



Origin of the large systematic in RBC-UKQCD (2023)

- **Main reason:** calculation performed on a **single volume** ($m_\pi L \simeq 3.9$)
  - no $L \to \infty$ extrapolation

- **Partial knowledge of finite-volume scaling** of virtual decay rate in QED_L

$$\Gamma_0(L) = \Gamma_0^{\rm tree}\left\{1 + 2\,\frac{\alpha}{4\pi}\,Y(L)\right\}$$

V. Lubicz et al., PRD **95** (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

$$Y(L) - Y(\infty) = Y_{\log}(L) + Y_0 + \frac{1}{m_P L}\,Y_1 + \frac{1}{(m_P L)^2}\,Y_2 + \frac{1}{(m_P L)^3}\,Y_3^{\rm pt} + \frac{1}{(m_P L)^3}\,Y_3^{\rm SD} + \mathrm{O}(1/L^4) + \mathrm{O}(\mathrm{e}^{-\alpha L})$$

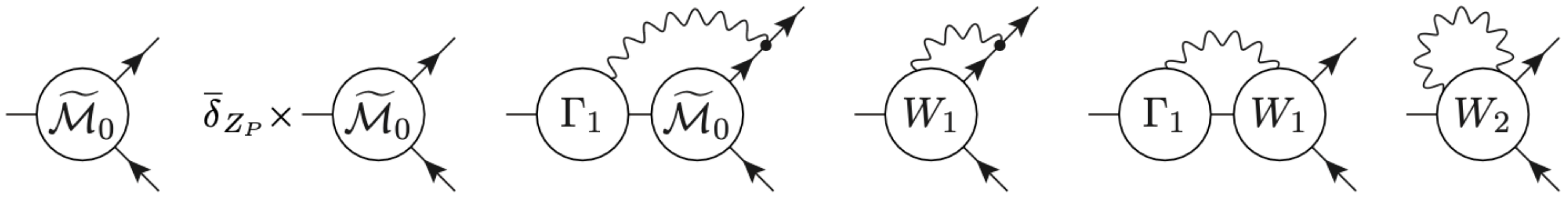
$m_\pi L \approx 3.9$: $Y_{\log}(L) + Y_0 + \frac{1}{m_P L}Y_1$ ≈ -3.96; $\frac{1}{(m_P L)^2}Y_2$ ≈ -2.24; $\frac{1}{(m_P L)^3}Y_3^{\rm pt}$ ≈ 3.37; $\frac{1}{(m_P L)^3}Y_3^{\rm SD}$: currently **unknown**

QED finite-volume effects to leptonic decays

- RM123+Soton — V. Lubicz et al., PRD 95 (2017)
  - first **proof of universality** of $\log(L)$ and $1/L$ terms
  - first calculation of universal FV effects via Poisson summation formula

- N.Tantalo, Lattice 2016 — N.Tantalo et al., [1612.00199v2]
  - proof of **universality** via **effective Lagrangian** (composite particle & soft photons)
  - derivation of FV effects for pointlike mesons up to $1/L^3$

- MDC, M.Hansen, N.Hermansson-Truedsson, A.Portelli — MDC et al., PRD 105 (2022)
  - use of **skeleton expansion** to derive FV effects, including structure dependence up to $1/L^2$



On-shell derivation of QED finite-volume effects

The goal:

- derive **all-orders expression** for finite-volume effects
- study its **asymptotic behaviour**
- put more stringent **bounds** on the unsubtracted higher order terms

›› We use an on-shell derivation of QED FV effects, based on **spectral analysis** of correlators:

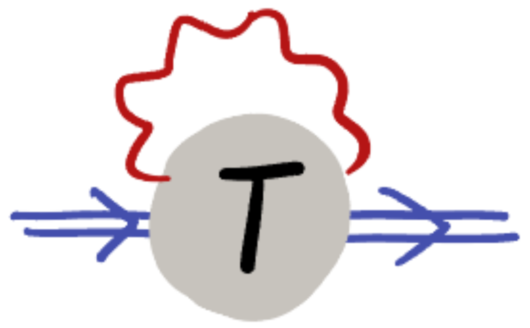
1. Define quantity of interest in terms of infinite-volume Minkowski **correlation function**
2. Study **spectral decomposition** of all time orderings, perform **Wick rotation** and **integrate** over photon energies k_0
3. **Repeat at finite** L and take $L \to \infty$ expansion of **sum-integral differences**

QED finite-volume effects

Hadron masses

Approach used for hadron masses by RC* collaboration in B.Lucini et al., JHEP 1602 (2016)

1. Cottingham formula:



$$m_P = m_P^{(0)} + \frac{\mathrm{i}\, e^2}{4m_P} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{T_{\mu\nu}(k_0, \mathbf{k}) g^{\mu\nu}}{k_0^2 - \mathbf{k}^2 + \mathrm{i}\epsilon}$$

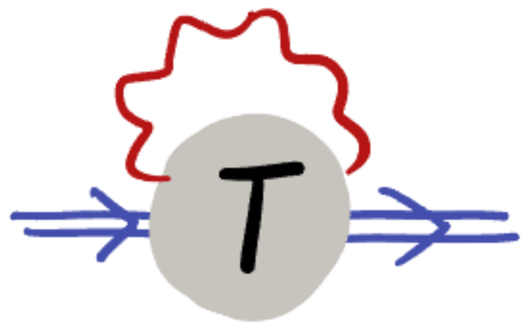
$$T_\mu{}^\mu(k_0, \mathbf{k}) = \mathrm{i} \int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}kx} \left\langle P(\mathbf{0}) | \mathrm{T}\{J_\mu(x) J^\mu(0)\} | P(\mathbf{0}) \right\rangle_{\mathrm{c}}$$

QED finite-volume effects

Hadron masses

Approach used for hadron masses by RC* collaboration in B.Lucini et al., JHEP 1602 (2016)

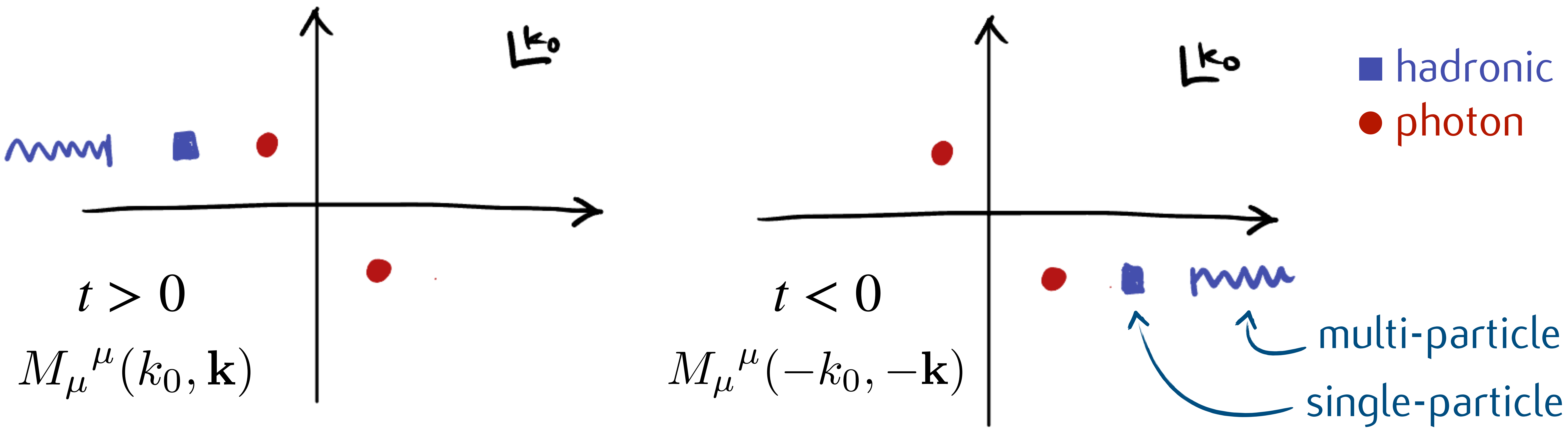
1. Cottingham formula:



$$m_P = m_P^{(0)} + \frac{\mathrm{i}\,e^2}{4m_P}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{T_{\mu\nu}(k_0,\mathbf{k})g^{\mu\nu}}{k_0^2-\mathbf{k}^2+\mathrm{i}\epsilon}$$

$$T_\mu{}^\mu(k_0,\mathbf{k}) = \mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}kx}\,\langle P(\mathbf{0})|\mathrm{T}\{J_\mu(x)J^\mu(0)\}|P(\mathbf{0})\rangle_\mathrm{c}$$

2.



QED finite-volume effects

Hadron masses

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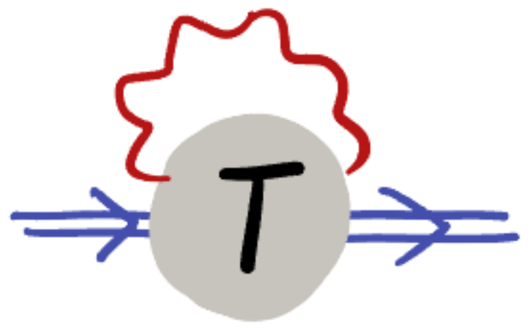
1. Cottingham formula:



$$m_P = m_P^{(0)} + \frac{\mathrm{i}\,e^2}{4m_P}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\frac{T_{\mu\nu}(k_0,\mathbf{k})g^{\mu\nu}}{k_0^2-\mathbf{k}^2+\mathrm{i}\epsilon}$$

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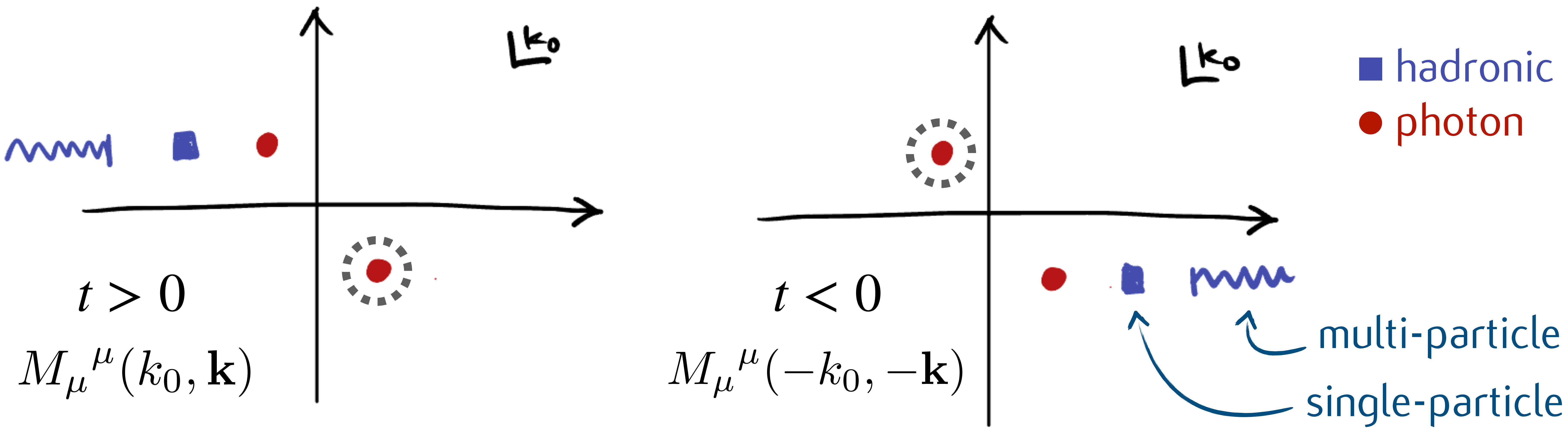
2.



QED finite-volume effects

Hadron masses

3. $$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\left[\frac{1}{L^3}\sum_{\mathbf{k}\in\Pi_{\boldsymbol{\theta}}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right] \frac{{M_\mu}^{\mu}(-|\mathbf{k}|,\mathbf{k})}{|\mathbf{k}|}$$

$${M_\mu}^{\mu}(-|\mathbf{k}|,\mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$

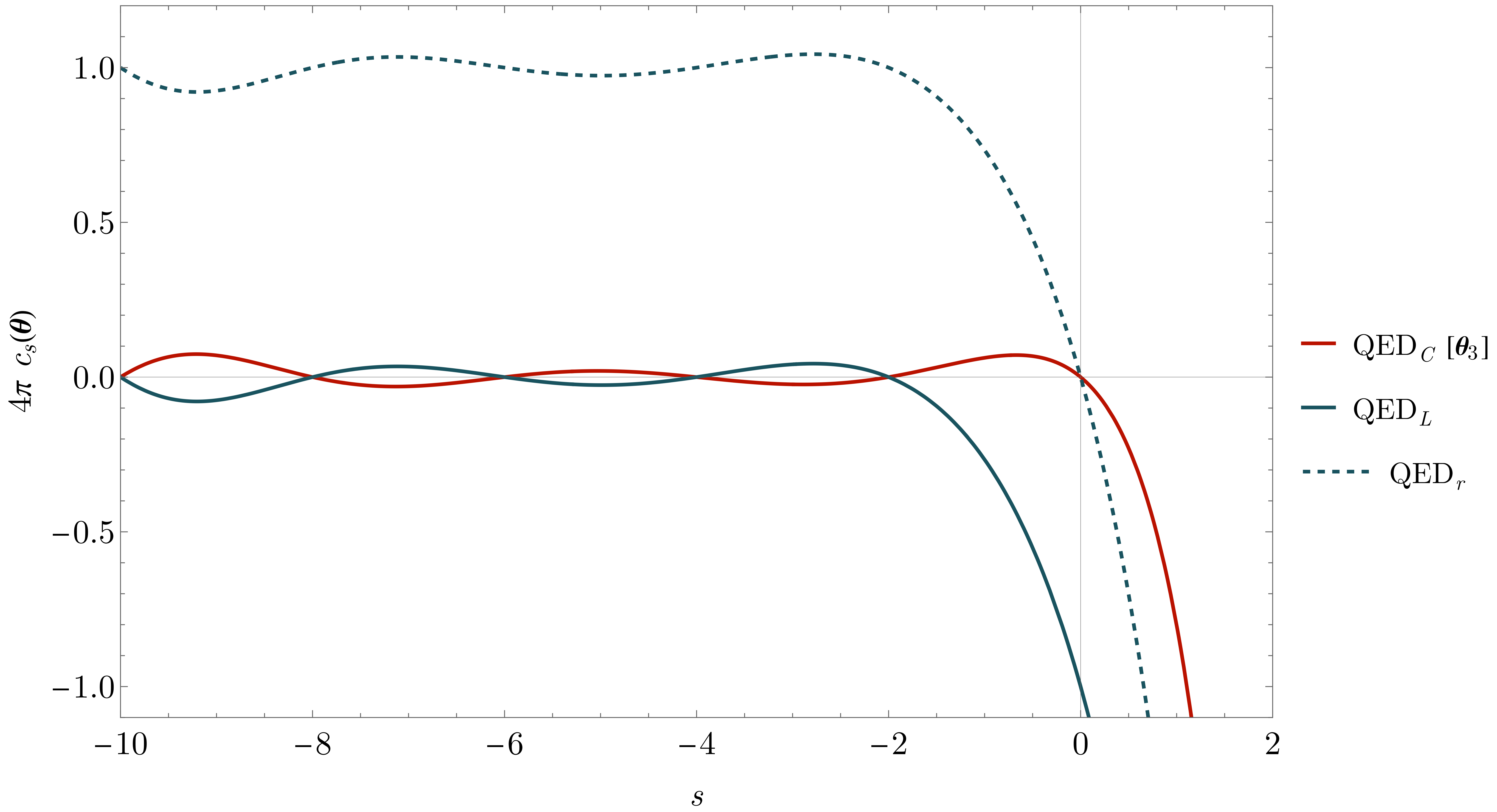
$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s}$$

universal terms fixed by Ward identities — structure + multi-particle dependence

QED finite-volume effects

Hadron masses

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$



QED finite-volume effects

V. Lubicz et al., PRD **95** (2017)

Leptonic decays

Finite-volume calculation more tricky due to appearance of **infrared divergences** & dependence on **external lepton momentum** p_ℓ

$$\Gamma_P = \mathcal{K}_P\, f_P^2 (1 + e^2\, \delta R_P^{\rm virt} + e^2\, \delta R_P^{\rm real}) \qquad \delta R_P^{\rm virt}(L) = \frac{Y(L)}{8\pi^2}$$

$$Y(L) = \lim_{\varepsilon\to 0} Y_\varepsilon(L) \equiv \lim_{\varepsilon\to 0} \left\{ Y_\varepsilon(\infty) + \Delta Y_\varepsilon(L) \right\} = Y^{\rm SD}(\infty) + \lim_{\varepsilon\to 0} \left\{ Y_\varepsilon^{\rm uni}(\infty) + \Delta Y_\varepsilon(L) \right\}$$

target of our lattice calculation ($Y^{\rm SD}(\infty)$)

finite-volume effects $\Delta Y(L)$:
- point-like decay rate with massive photon ($Y_\varepsilon^{\rm uni}(\infty)$)
- sum-integral differences at finite photon mass ($\Delta Y_\varepsilon(L)$)

QED finite-volume effects

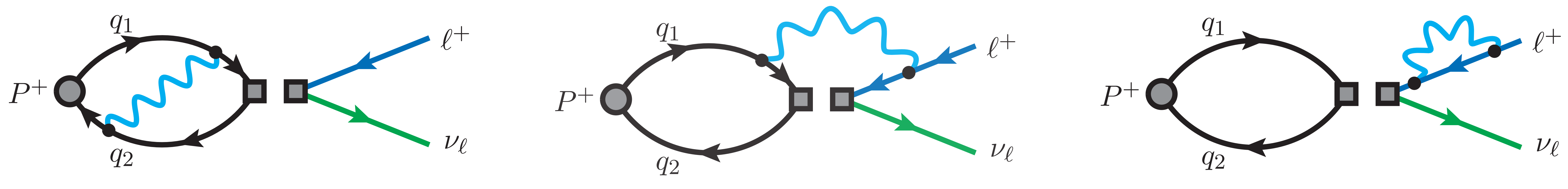
Leptonic decays

1. We define a reduction formula:

$$\delta R_P = \lim_{\epsilon\to 0} \frac{1}{e^2}\left[\frac{\epsilon\, 2m_P\, C_{\mathrm{w}}(m_P)}{|\mathcal{M}_P^{\mathrm{tree}}|^2} - 1\right]_{e=0}$$

$$C_{\mathrm{w}}(p_0) = \int \mathrm{d}^4 z\, \mathrm{e}^{\mathrm{i}(p-p_\ell-p_\nu)\cdot z}\, \langle \ell(\mathbf{p}_\ell)\bar{\nu}_\ell(\mathbf{p}_\nu)|\mathrm{T}\{\mathcal{L}_{\mathrm{w}}(z)\mathcal{L}_{\mathrm{w}}(0)\}|\ell(\mathbf{p}_\ell)\bar{\nu}_\ell(\mathbf{p}_\nu)\rangle_{\mathrm{QCD+QED}}$$

The expansion of $C_{\mathrm{w}}(m_P)$ around $e = 0$ generates 3 kinds of contributions:



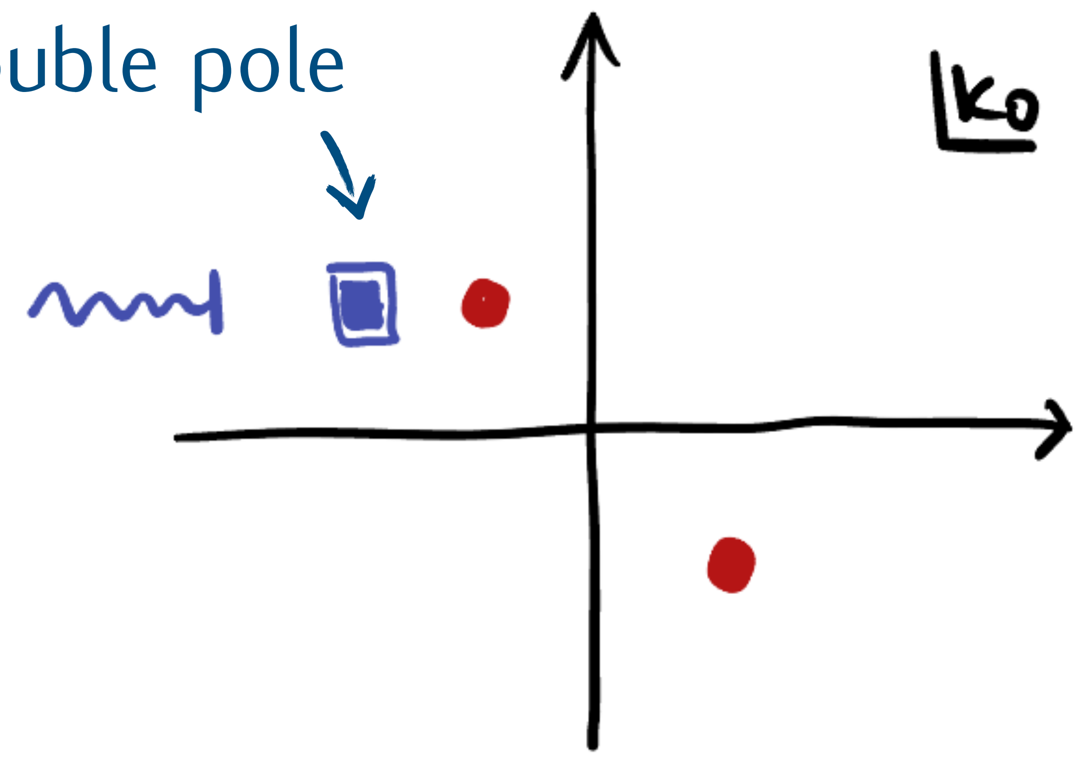
factorizable non-factorizable leptonic

QED finite-volume effects

Leptonic decays

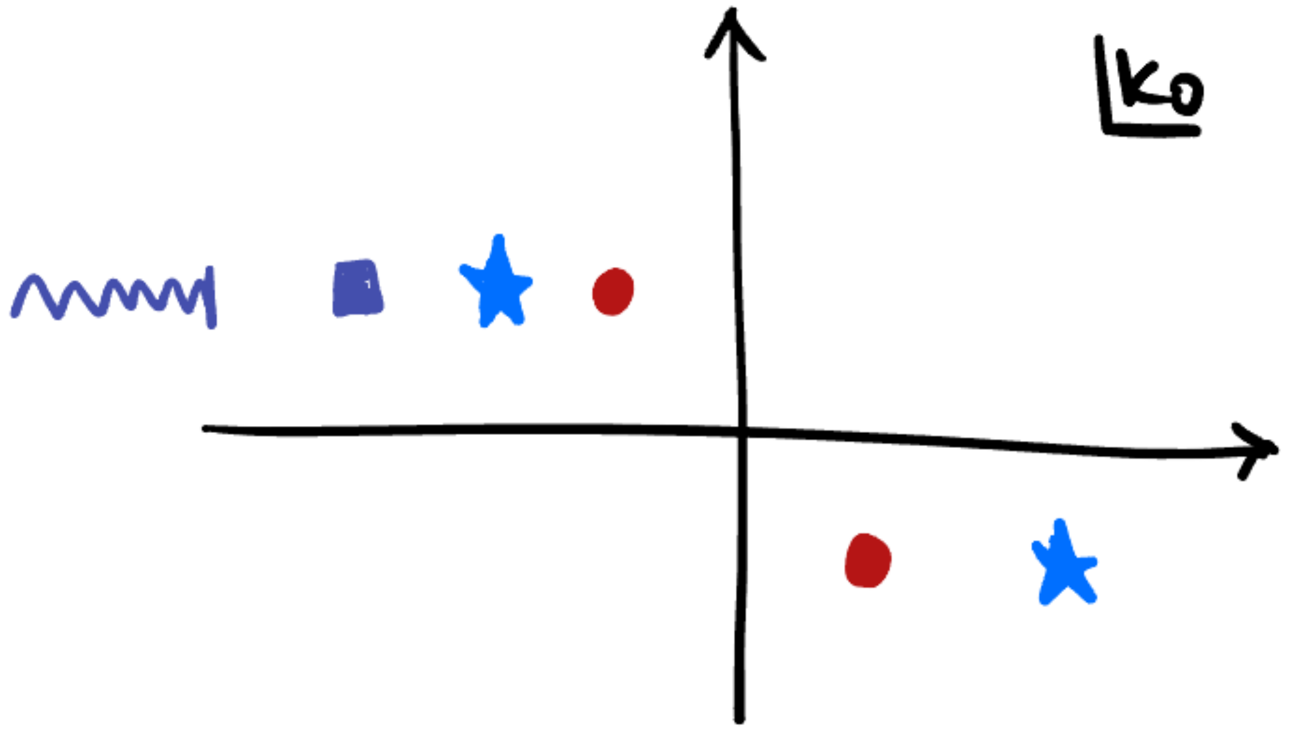
2. 6 time orderings for factorizable corrections + 2 for the non-factorizable yield

$$\delta R_P^{\mathrm{f}} = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{g_{\mu\nu} W_{\mathrm{f}}^{\mu\nu}(k_0, \mathbf{k})}{k_0^2 - \mathbf{k}^2 + \mathrm{i}\epsilon}$$

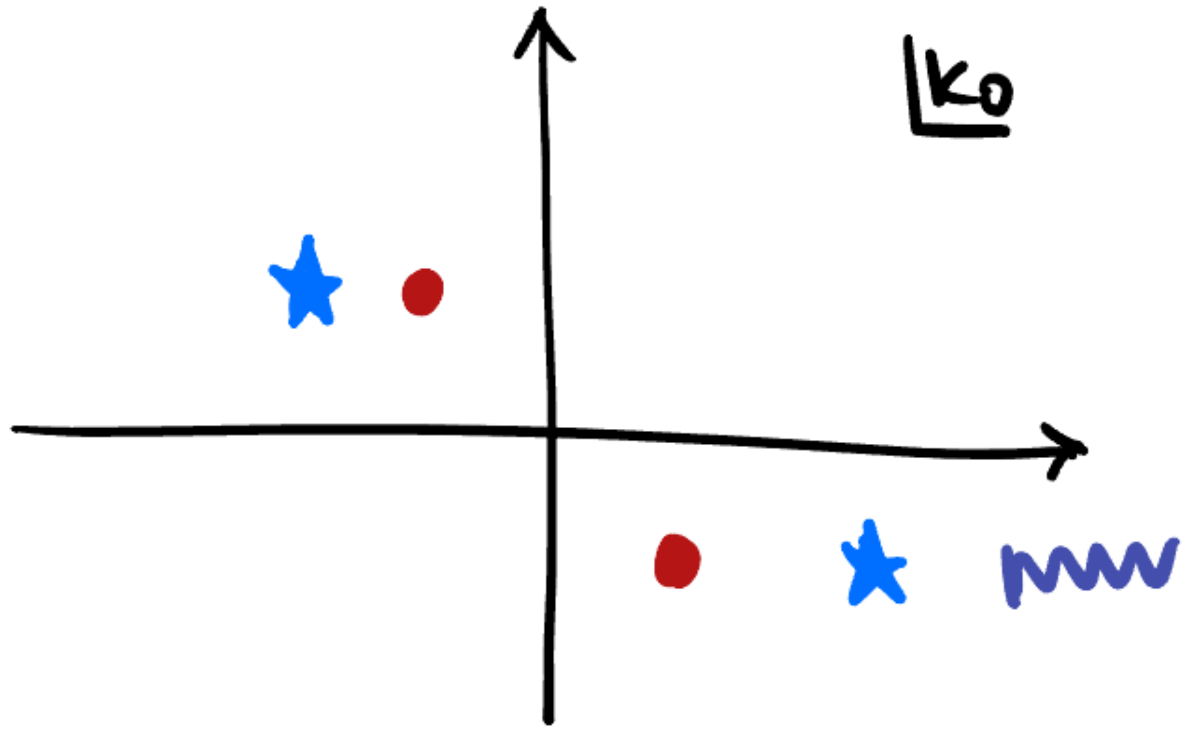


$W_{\mathrm{f}}^{\mu\nu}(k_0, \mathbf{k})$

$$\delta R_P^{\mathrm{nf}} = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{g_{\mu\nu} \left[W_{\mathrm{nf},1}^{\mu\nu}(k_0, \mathbf{k}) + W_{\mathrm{nf},2}^{\mu\nu}(k_0, \mathbf{k}) \right]}{k_0^2 - \mathbf{k}^2 + \mathrm{i}\epsilon}$$



$W_{\mathrm{nf},1}^{\mu\nu}(k_0, \mathbf{k})$



$W_{\mathrm{nf},2}^{\mu\nu}(k_0, \mathbf{k})$

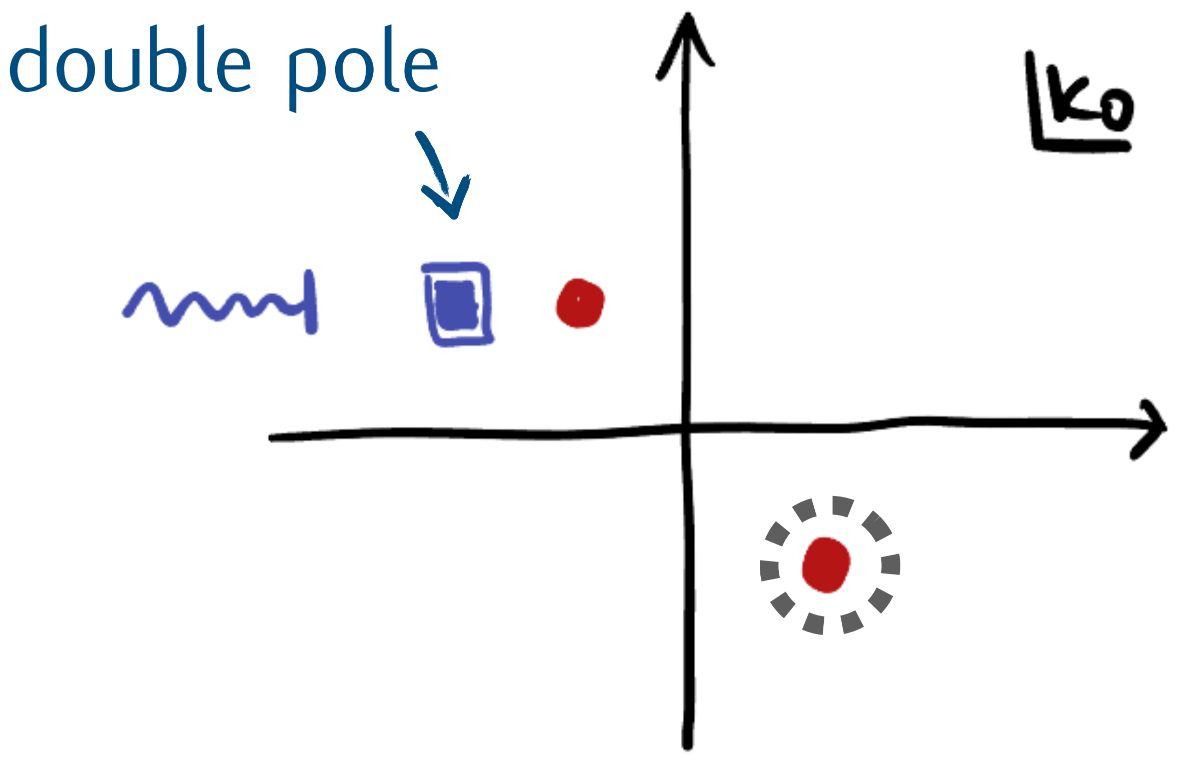
■ hadronic ● photon ★ lepton

QED finite-volume effects

Leptonic decays

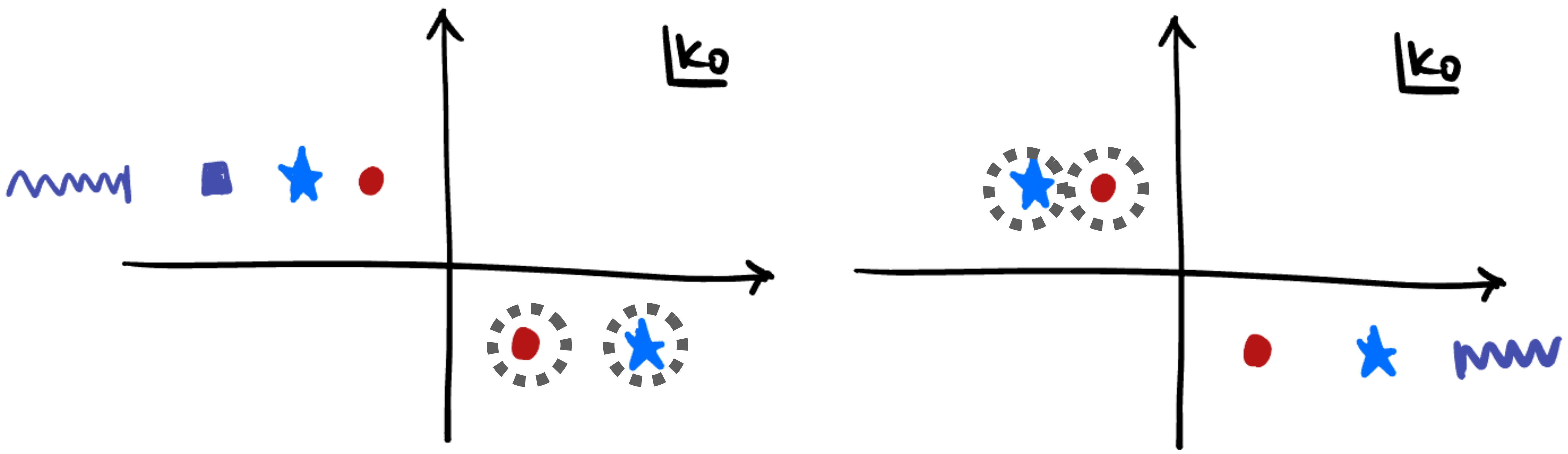
2. 6 time orderings for factorizable corrections + 2 for the non-factorizable yield

$$\delta R_P^{\mathrm{f}} = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{g_{\mu\nu} W_{\mathrm{f}}^{\mu\nu}(k_0, \mathbf{k})}{k_0^2 - \mathbf{k}^2 + \mathrm{i}\epsilon}$$



$W_{\mathrm{f}}^{\mu\nu}(k_0, \mathbf{k})$

$$\delta R_P^{\mathrm{nf}} = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{g_{\mu\nu} \left[W_{\mathrm{nf},1}^{\mu\nu}(k_0, \mathbf{k}) + W_{\mathrm{nf},2}^{\mu\nu}(k_0, \mathbf{k})\right]}{k_0^2 - \mathbf{k}^2 + \mathrm{i}\epsilon}$$



$W_{\mathrm{nf},1}^{\mu\nu}(k_0, \mathbf{k})$ $W_{\mathrm{nf},2}^{\mu\nu}(k_0, \mathbf{k})$

■ hadronic ● photon ★ lepton

QED finite-volume effects

Leptonic decays

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]

3. Asymptotic $L \to \infty$ expansion of the sum-integral difference (after the k_0 integration) yields

$$\Delta Y(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$
$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$
$$+ \frac{1}{(m_P L)^2}\left[-\frac{\boldsymbol{F_A(0)}}{f_P}\,\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$
$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, \boldsymbol{C_\ell^{(1)}} + c_0(\mathbf{v}_\ell)\, \boldsymbol{C_\ell^{(2)}}\right]$$
$$+ \cdots$$

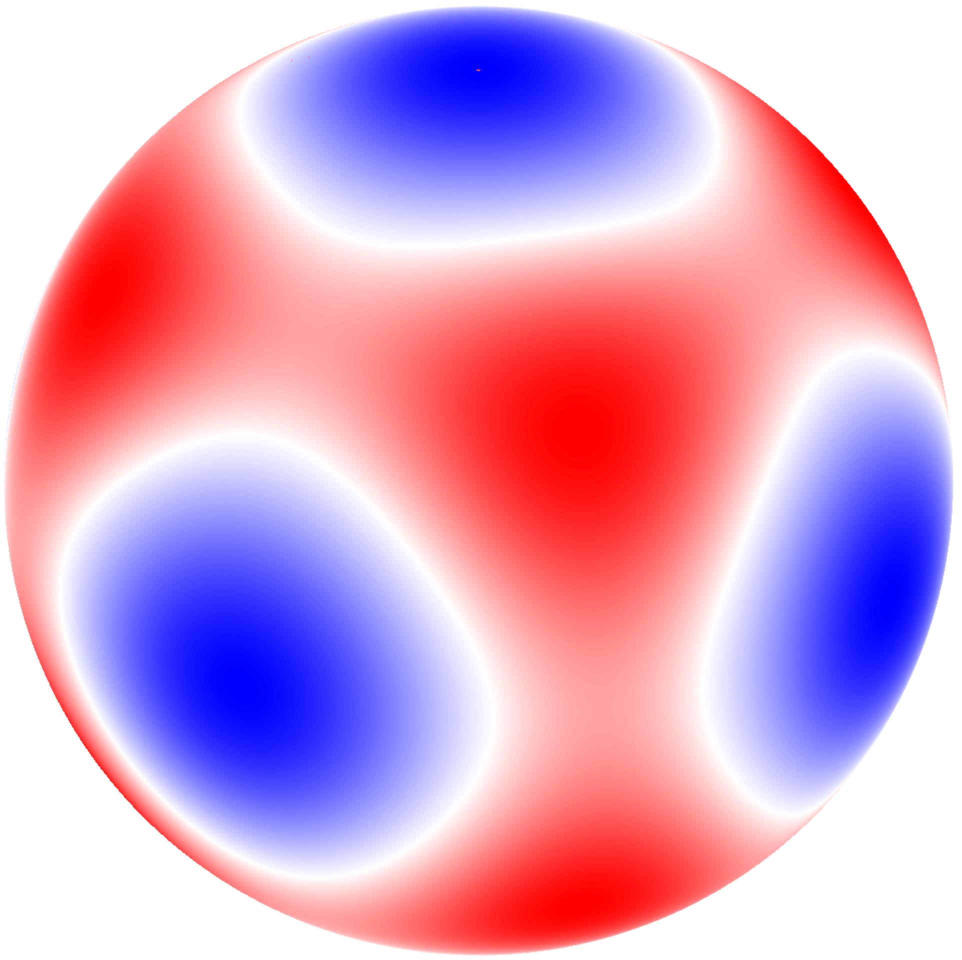
›› here shown only up to $1/L^3$ just for convenience — in agreement with other published results

Velocity-dependent coefficients

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int \mathrm{d}^3\mathbf{n}\right) \frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- **Collinear** divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

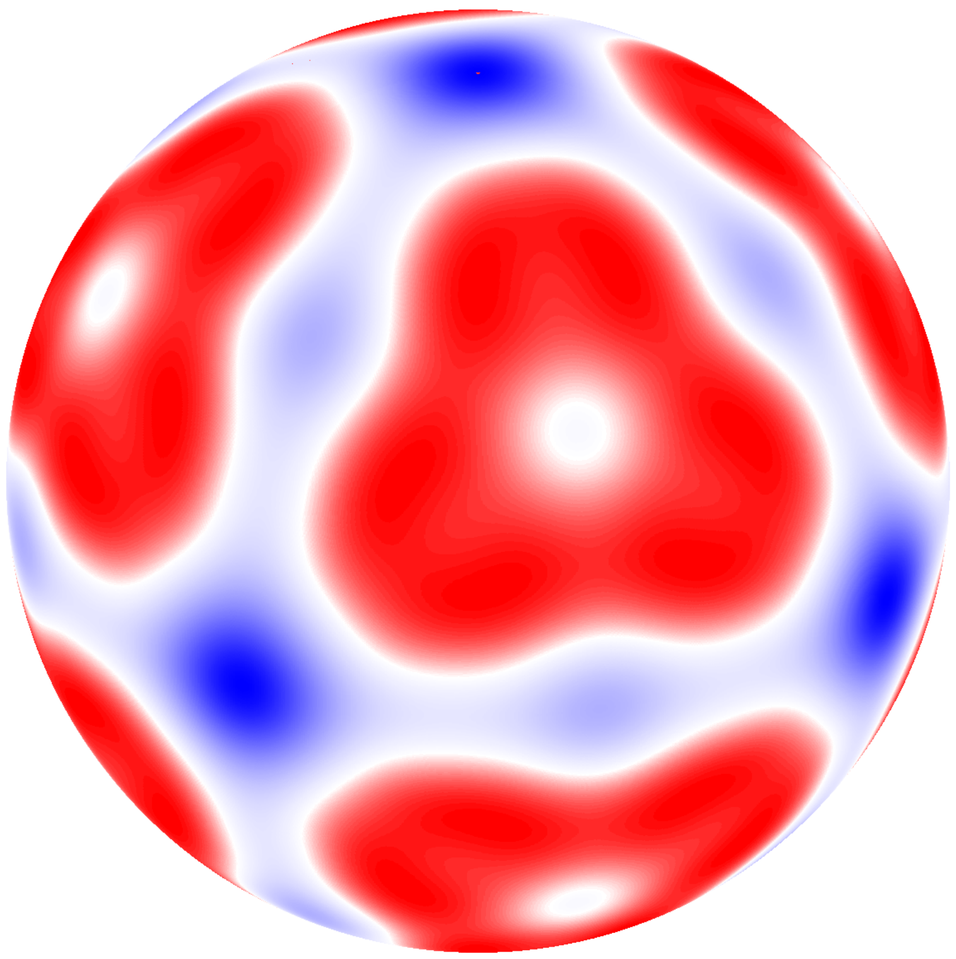
|v| = 0.40



$\max \bar{c}_0(\mathbf{v}) = 0.0171$

$\min \bar{c}_0(\mathbf{v}) = -0.0114$

|v| = 0.95



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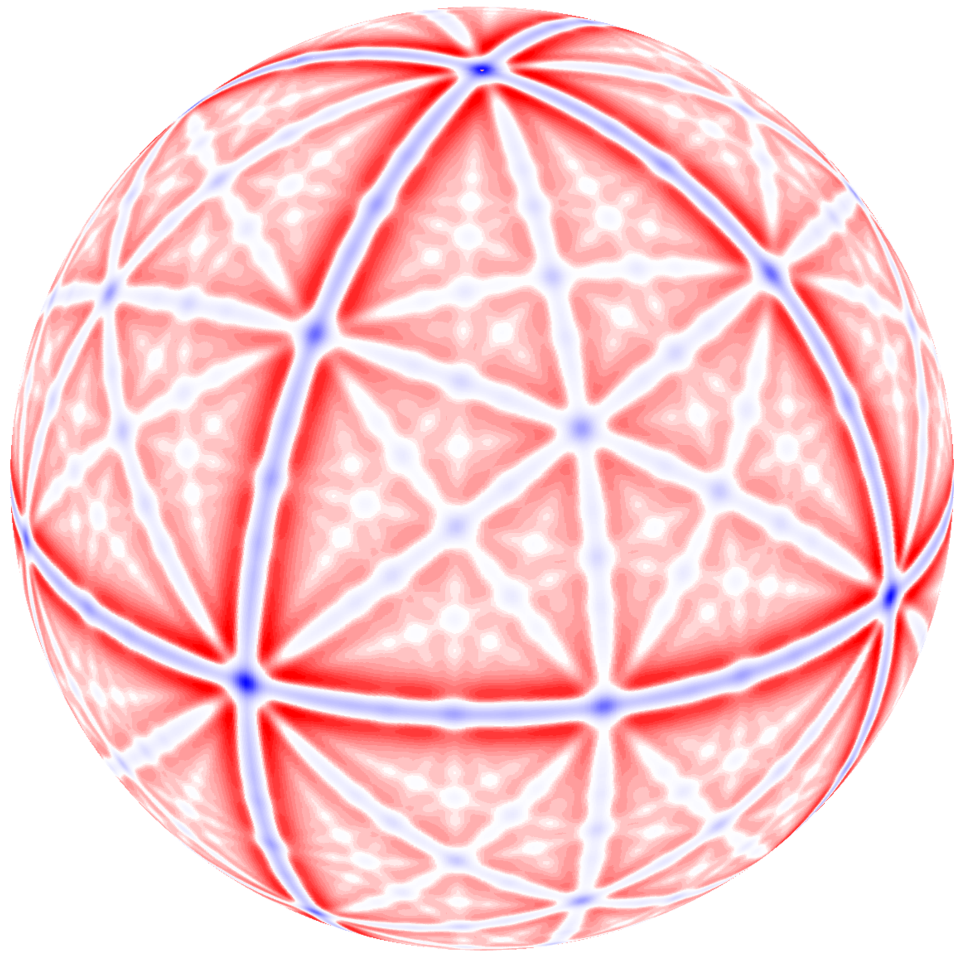

$\max \bar{c}_0(\mathbf{v}) = 9002.2317$

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Ongoing numerical studies in $\mathrm{QED_L}$ and $\mathrm{QED_r}$ of strategies to tame such effects

Conclusions

- Work in progress to fully **understand finite-volume scaling** of leptonic decay rates $P \to \ell\bar{\nu}$
- **On-shell approach** allows one to derive **all-order formulas** for FV effects
  - › Understand asymptotic behaviour of the $1/L$ series and put bounds on neglected higher orders
- **Velocity-dependent coefficients** $c_s(\mathbf{v}_\ell)$ can be very large:
  - › numerical studies to tame these effects are ongoing in $\mathrm{QED_L}$ and $\mathrm{QED_r}$
- I look with interest at the work on EW_∞ discussed previously by X.Tuo for extension to $K \to \pi\ell\bar{\nu}$

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Thank you

and to A.Patella and M.Hansen, N.Hermansson-Truedsson & A.Portelli
for useful discussions and work on topics discussed in the talk

This work has received funding from the European Union's Horizon Europe research and innovation programme under the Marie Sklodowska-Curie grant agreement No 101108006

Backup slides

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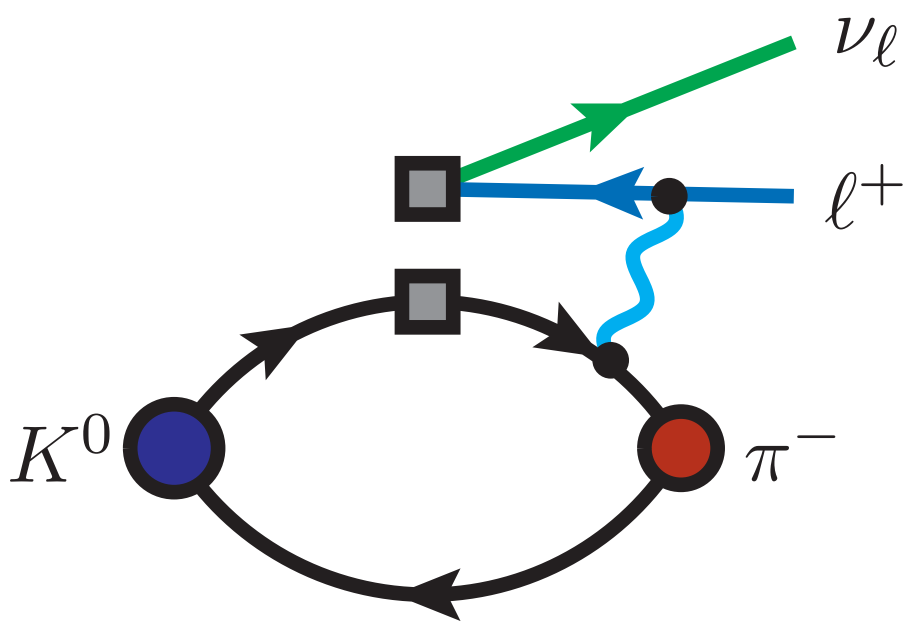
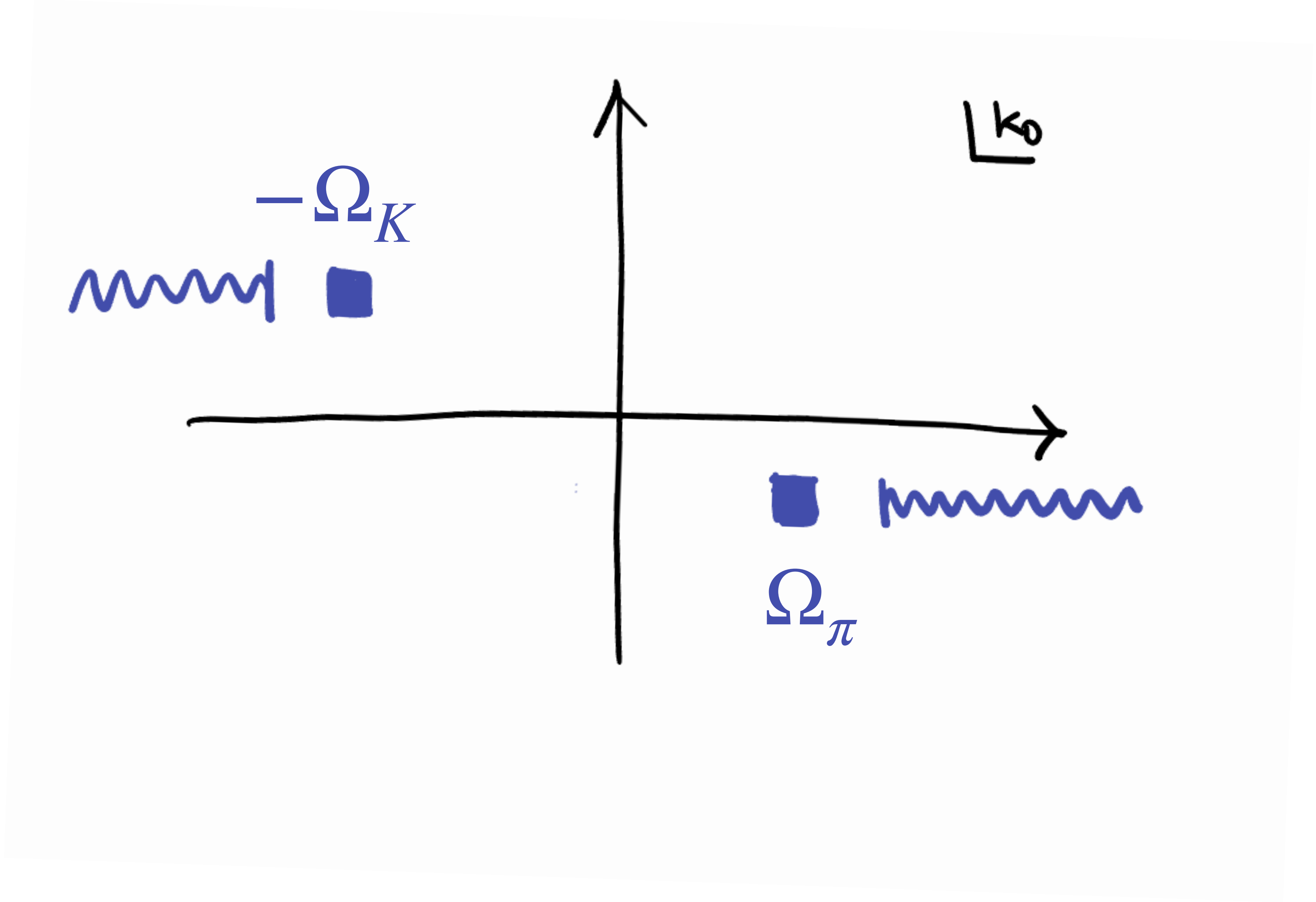
Beyond leptonic decays?

- The on-shell approach strongly relies on the study of the **analytical properties** of a given amplitude $W(k_0, \mathbf{k})$
- This makes it potentially suitable for more complicated processes like $K \to \pi \ell \bar{\nu}$





$$\Omega_K = \sqrt{m_K^2 + \mathbf{k}^2} - m_K$$

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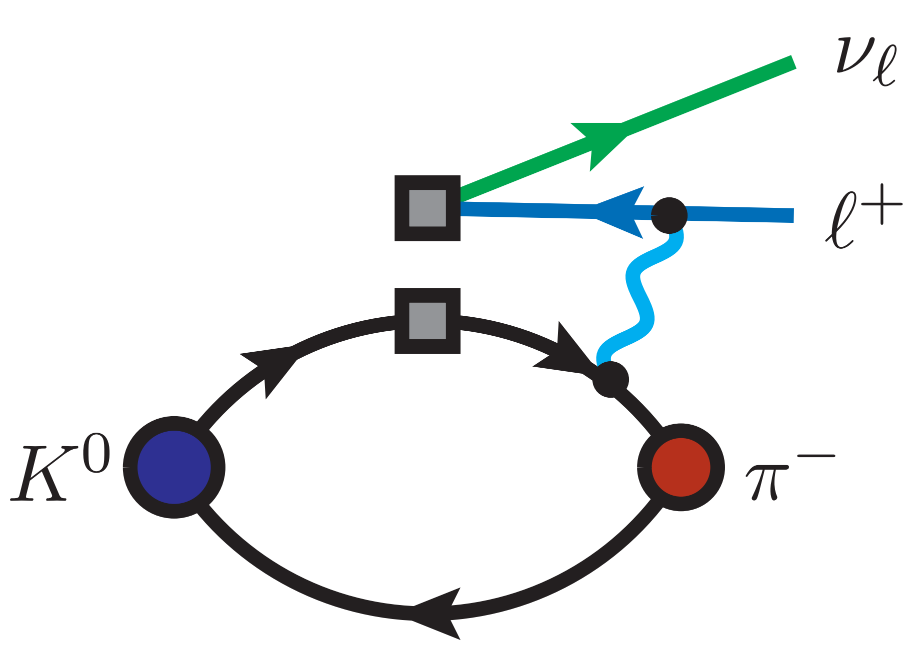

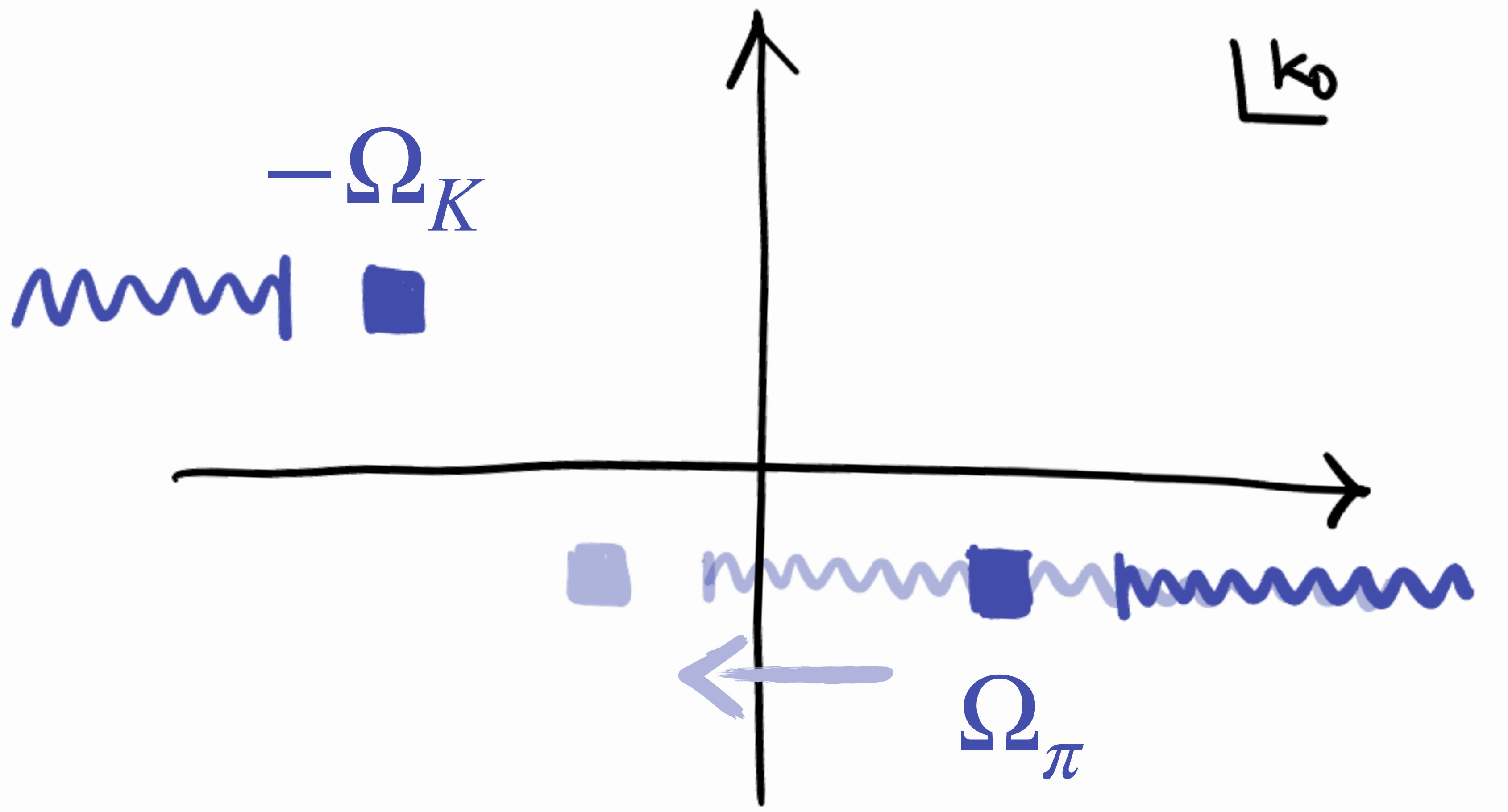

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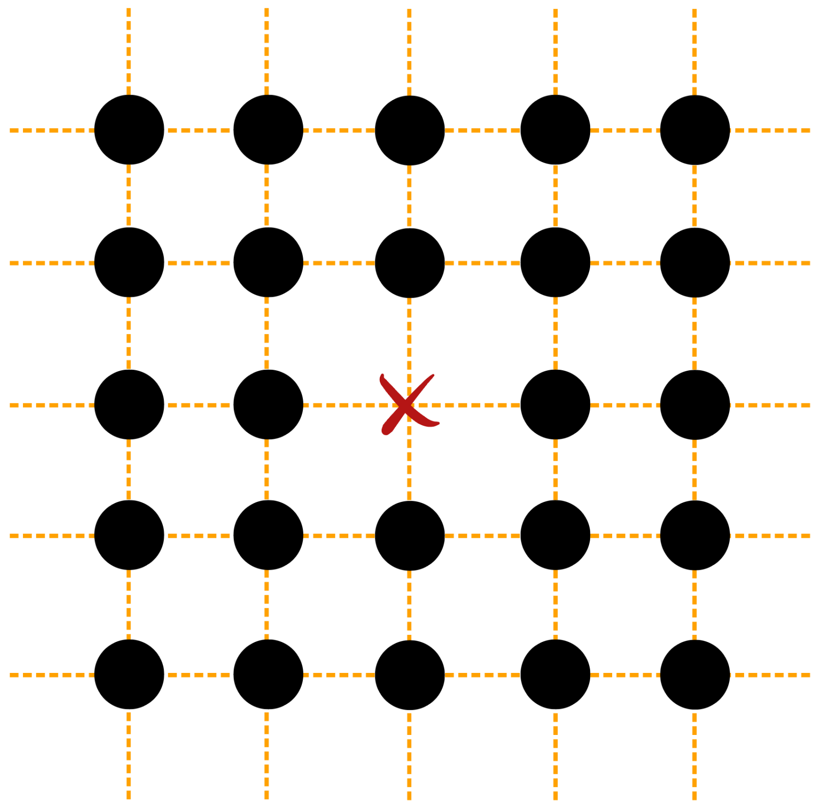
For certain kinematical configurations Wick rotation is not possible due to **lighter internal states**.
More work to be done to study extension. Perhaps in the direction of X.Tuo & X.Feng [2407.16930]

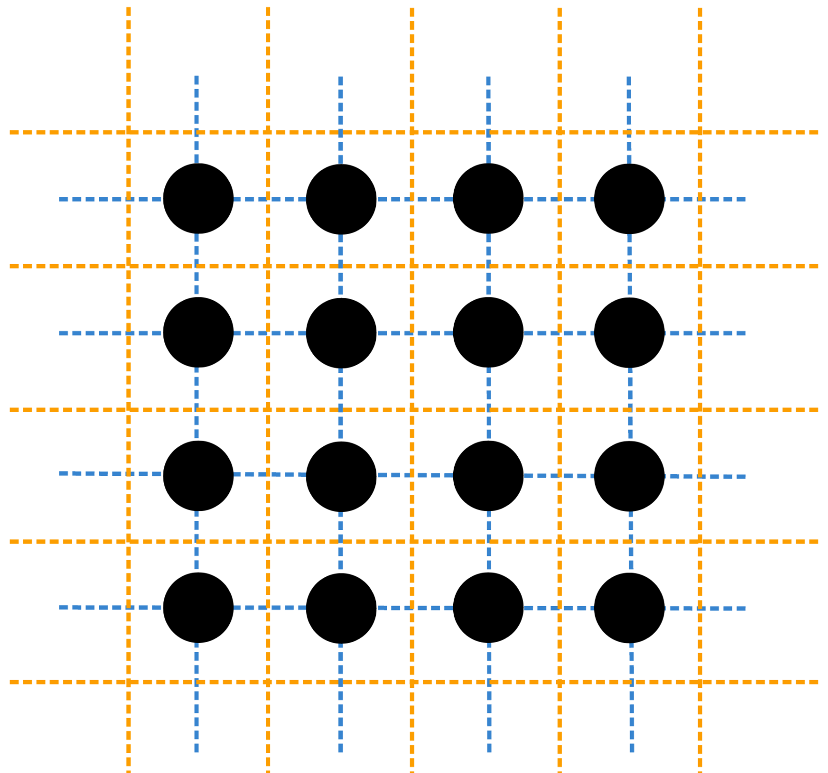
Lattice QED formulations

$\mathrm{QED_L}$



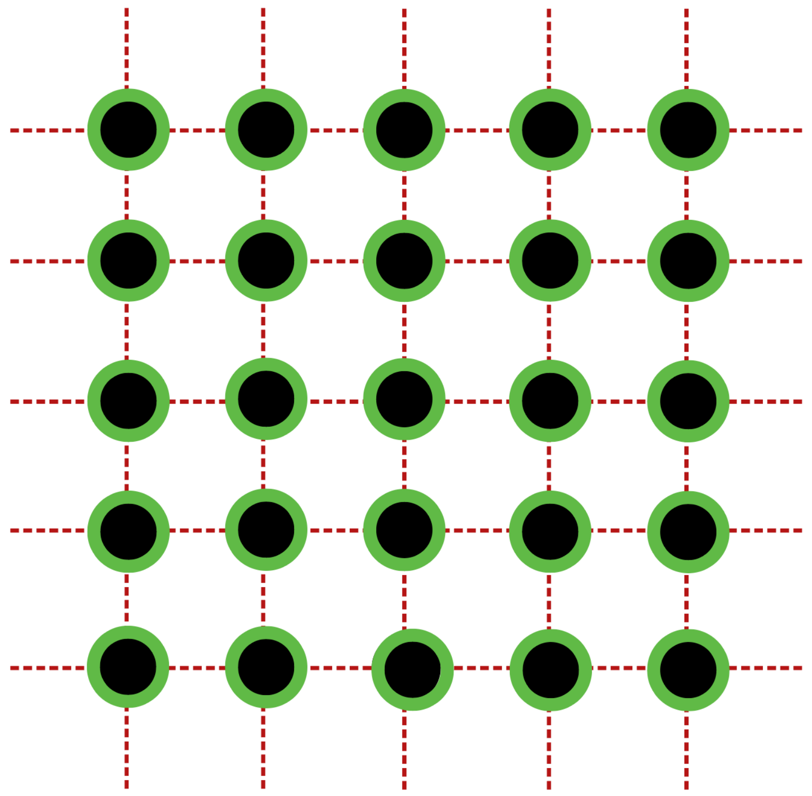
$\Omega_3 = 2\pi\mathbb{Z}^3/L$

$\mathrm{QED_{C^*}}$



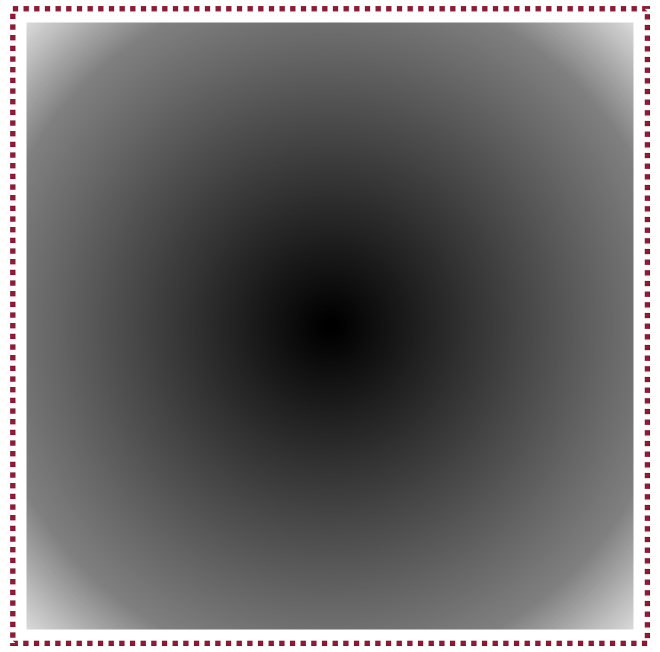
$\Omega_3 = 2\pi\mathbb{Z}^3/L \quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

$\mathrm{QED_m}$



$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

$\mathrm{QED_\infty}$



$\Omega_4 = \mathbb{R}^4$

finite-volume photon

non-local

local

∞-volume photon

power-like finite-volume effects

exponential finite-volume effects

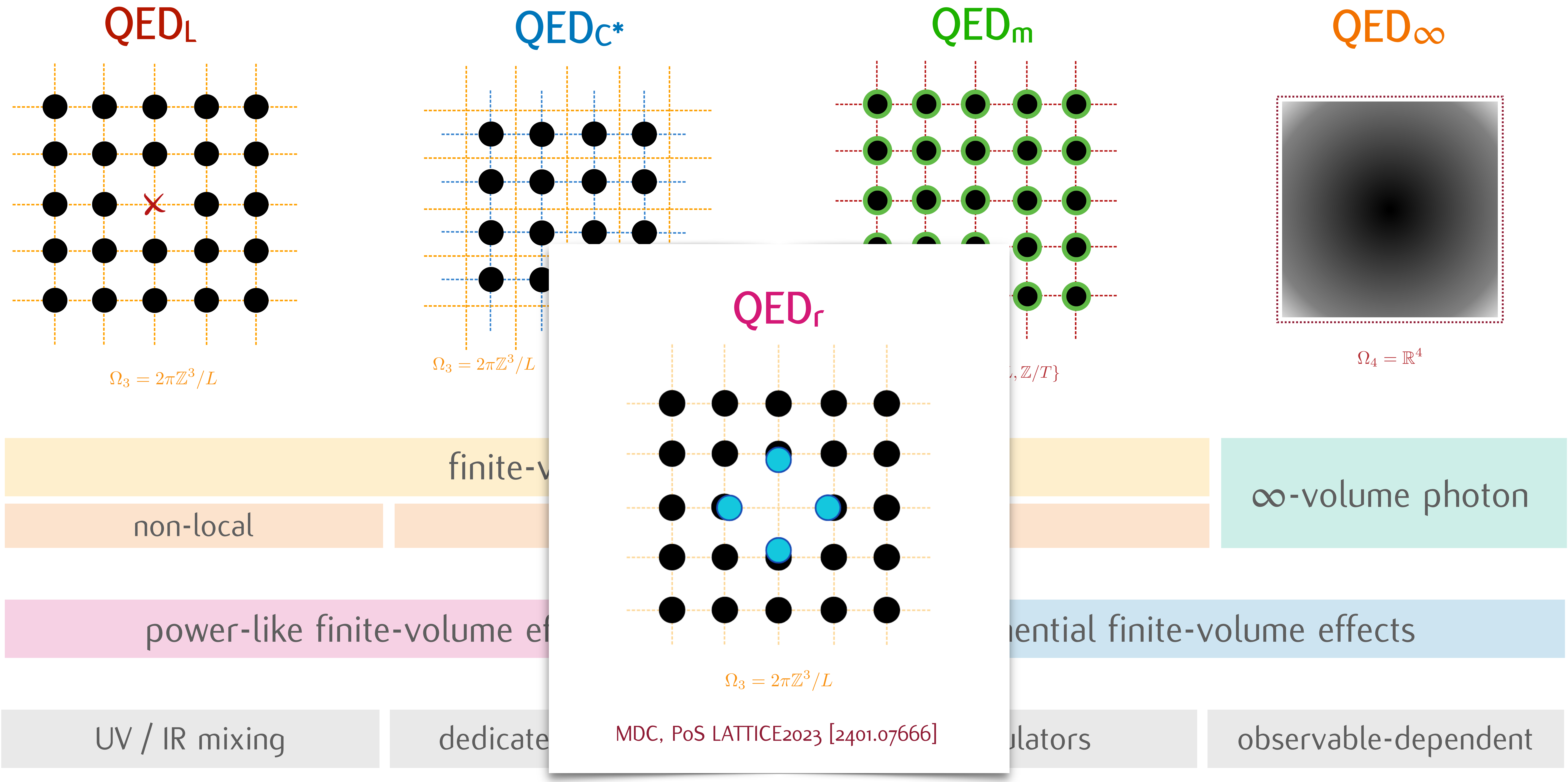
UV / IR mixing

dedicated ensembles

two IR regulators

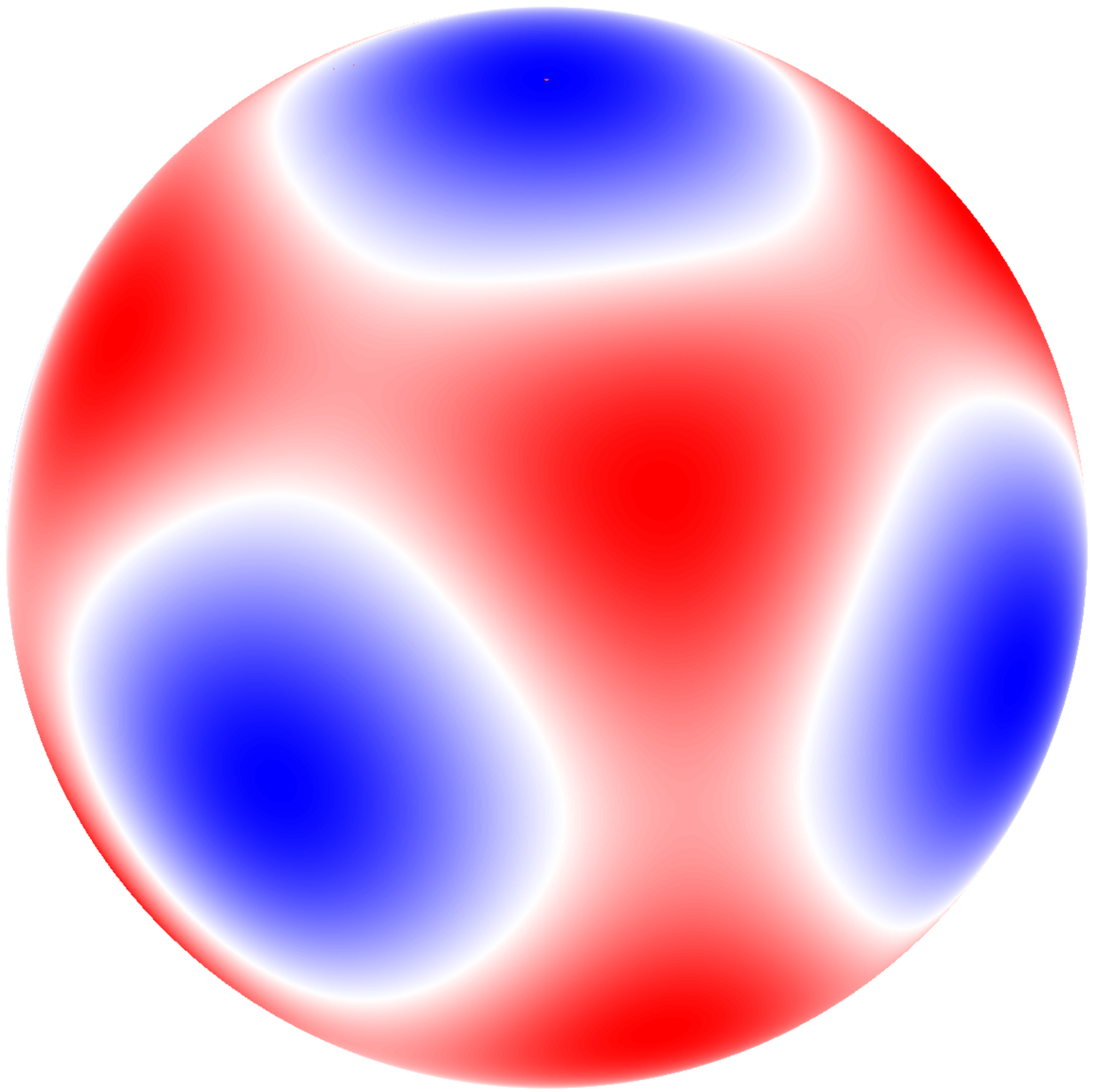
observable-dependent

Lattice QED formulations



Velocity-dependent coefficients in QED$_r$

$|\mathbf{v}| = 0.40$



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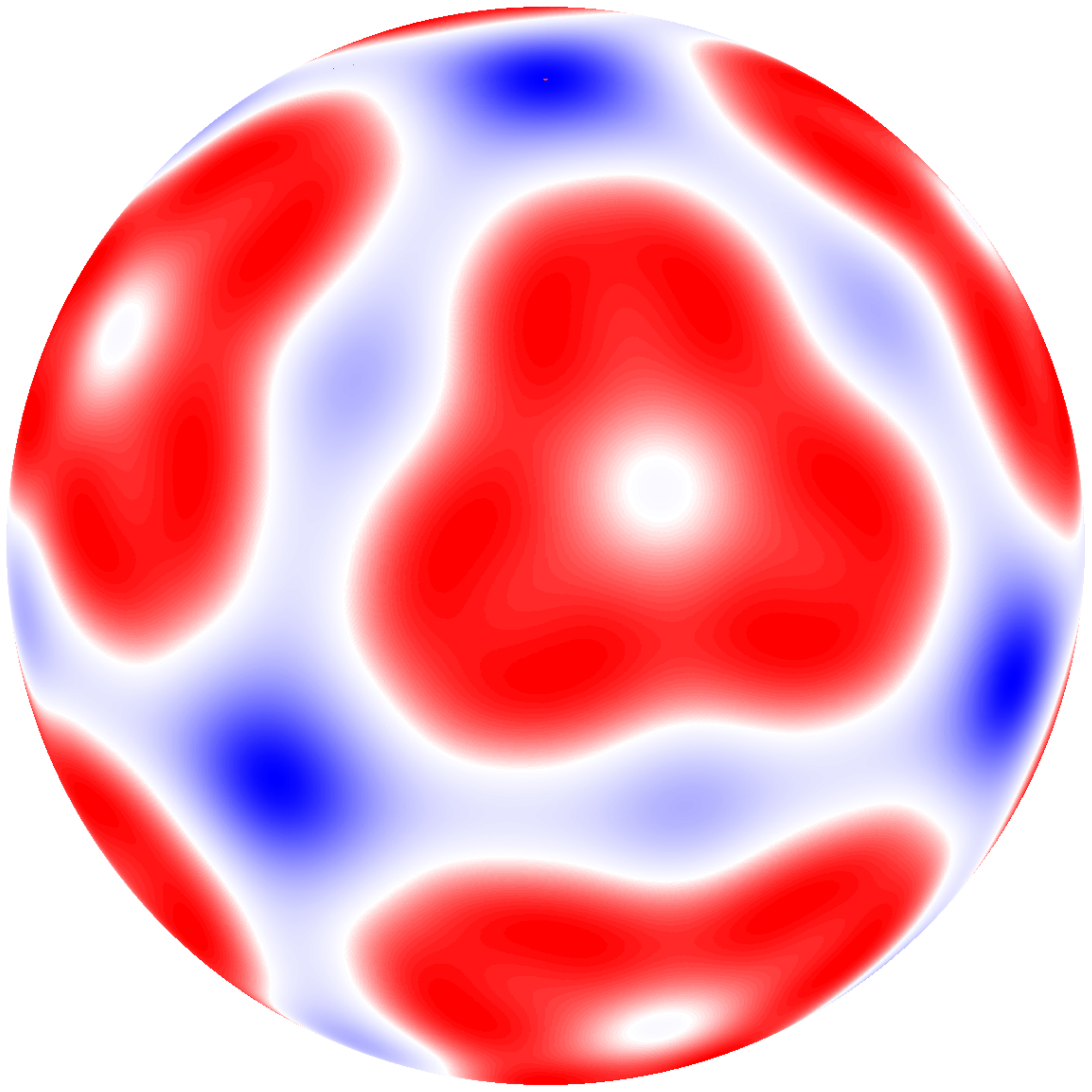

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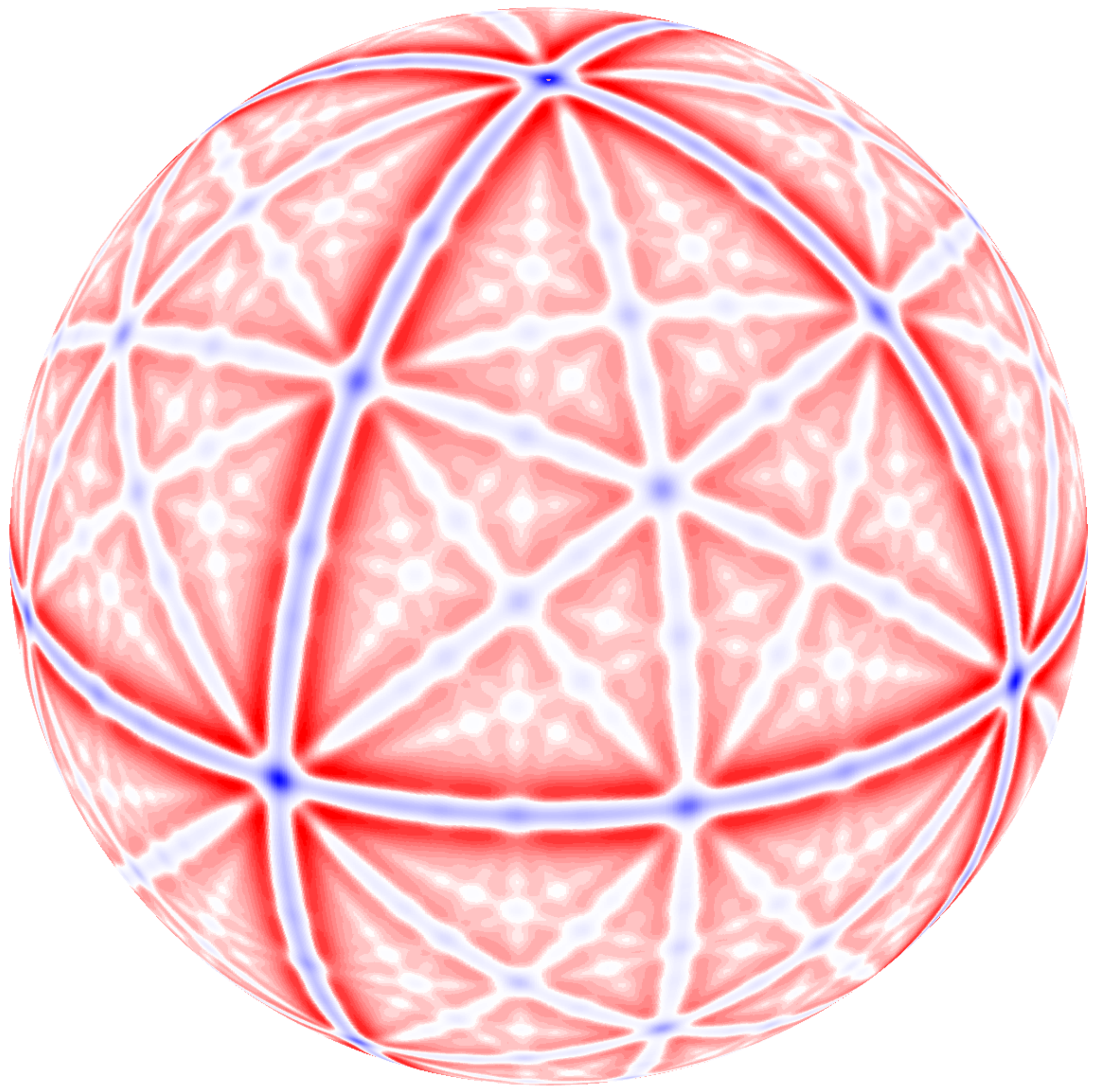

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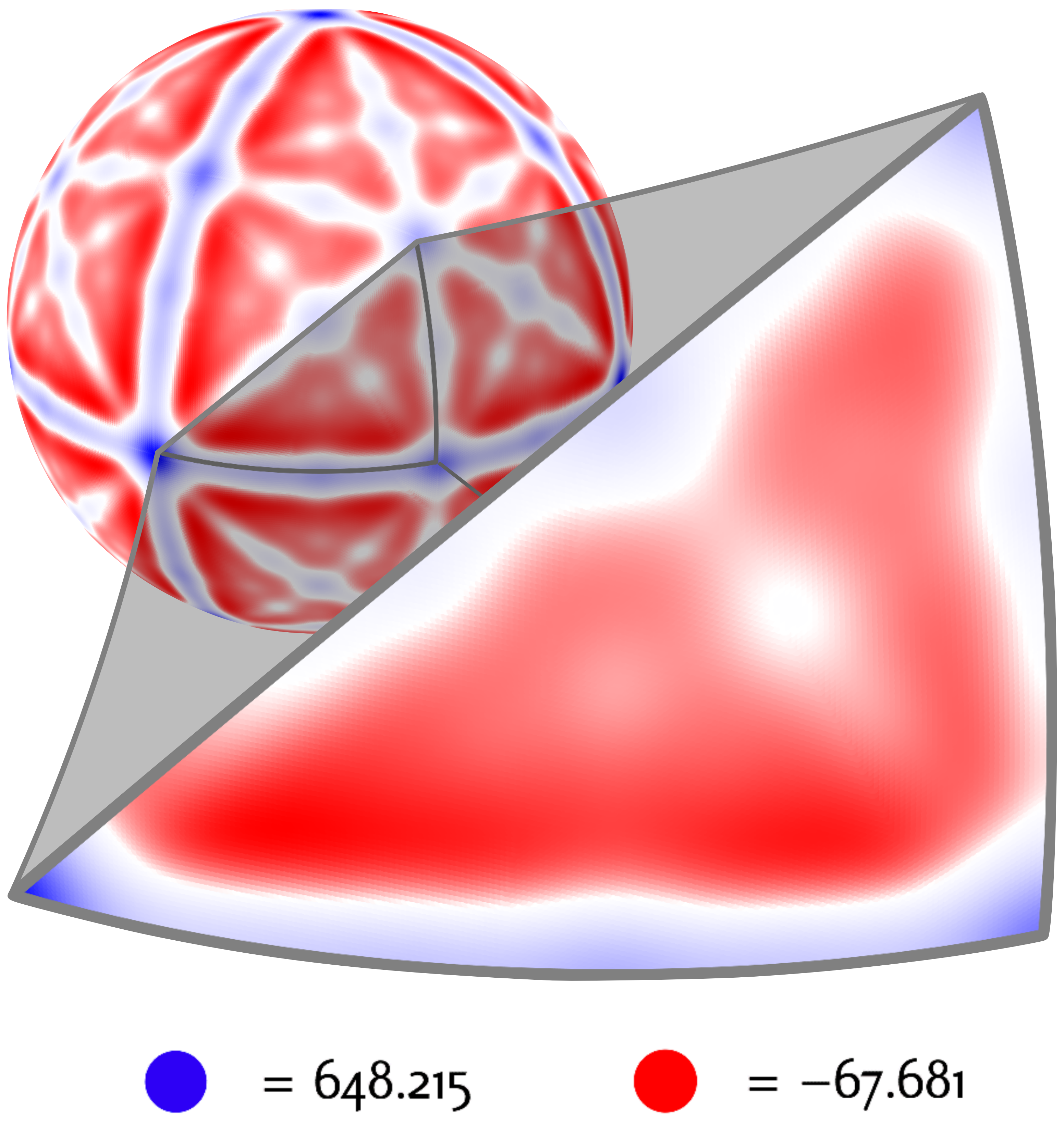
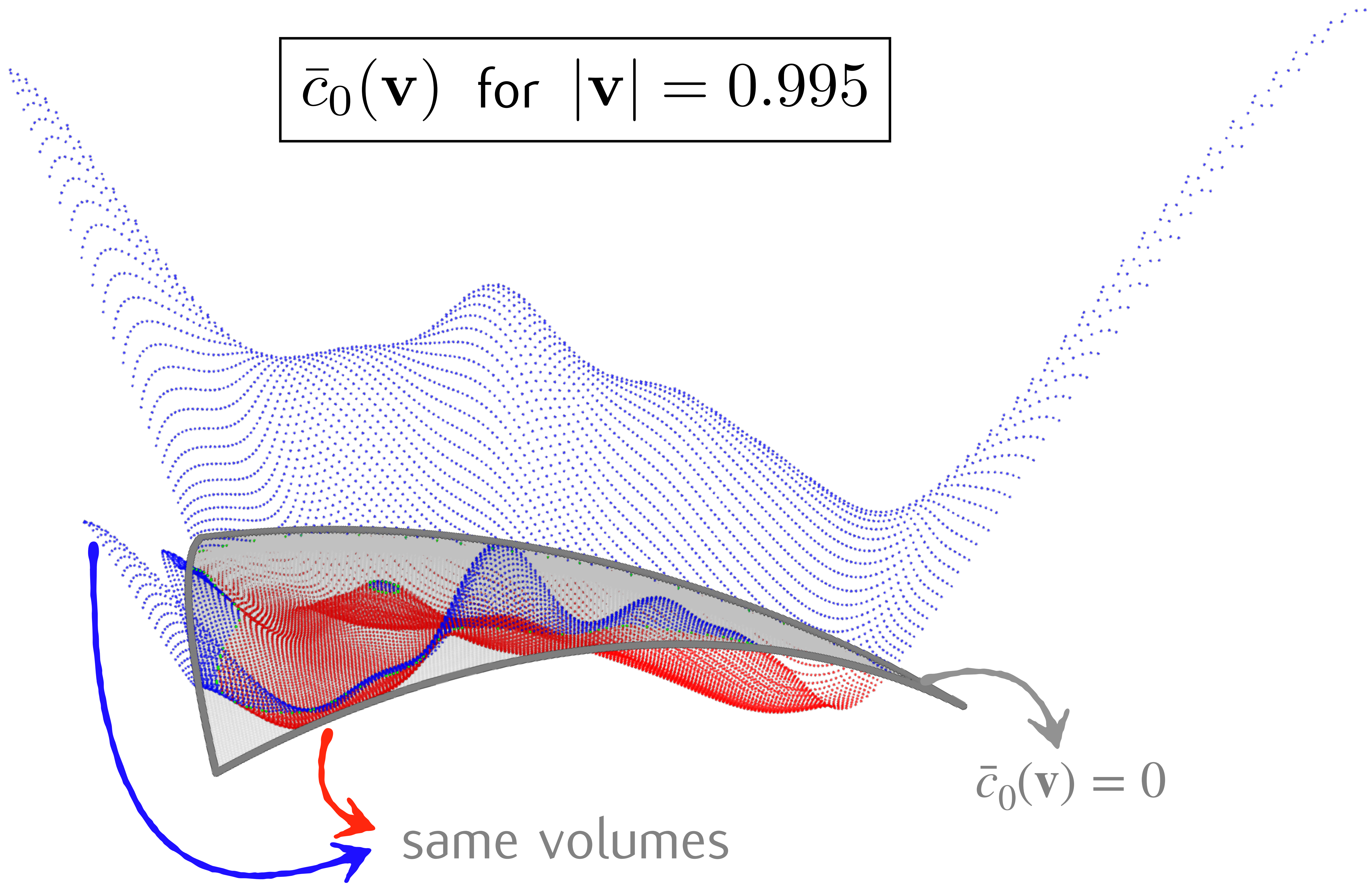
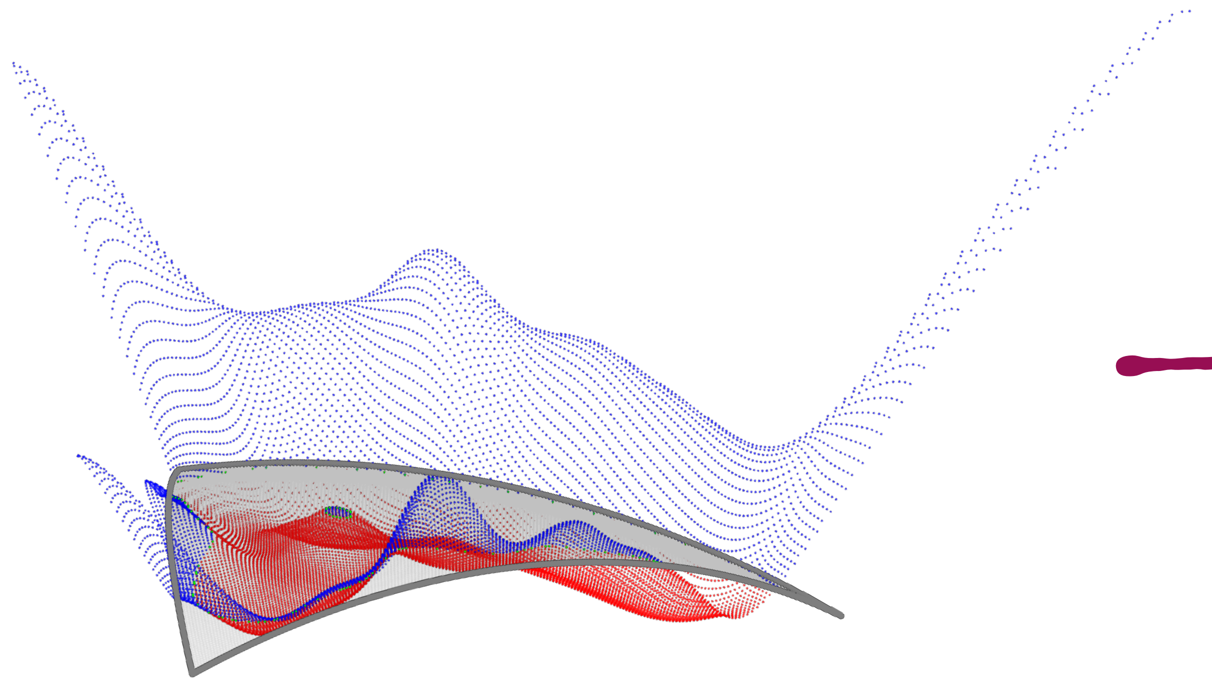
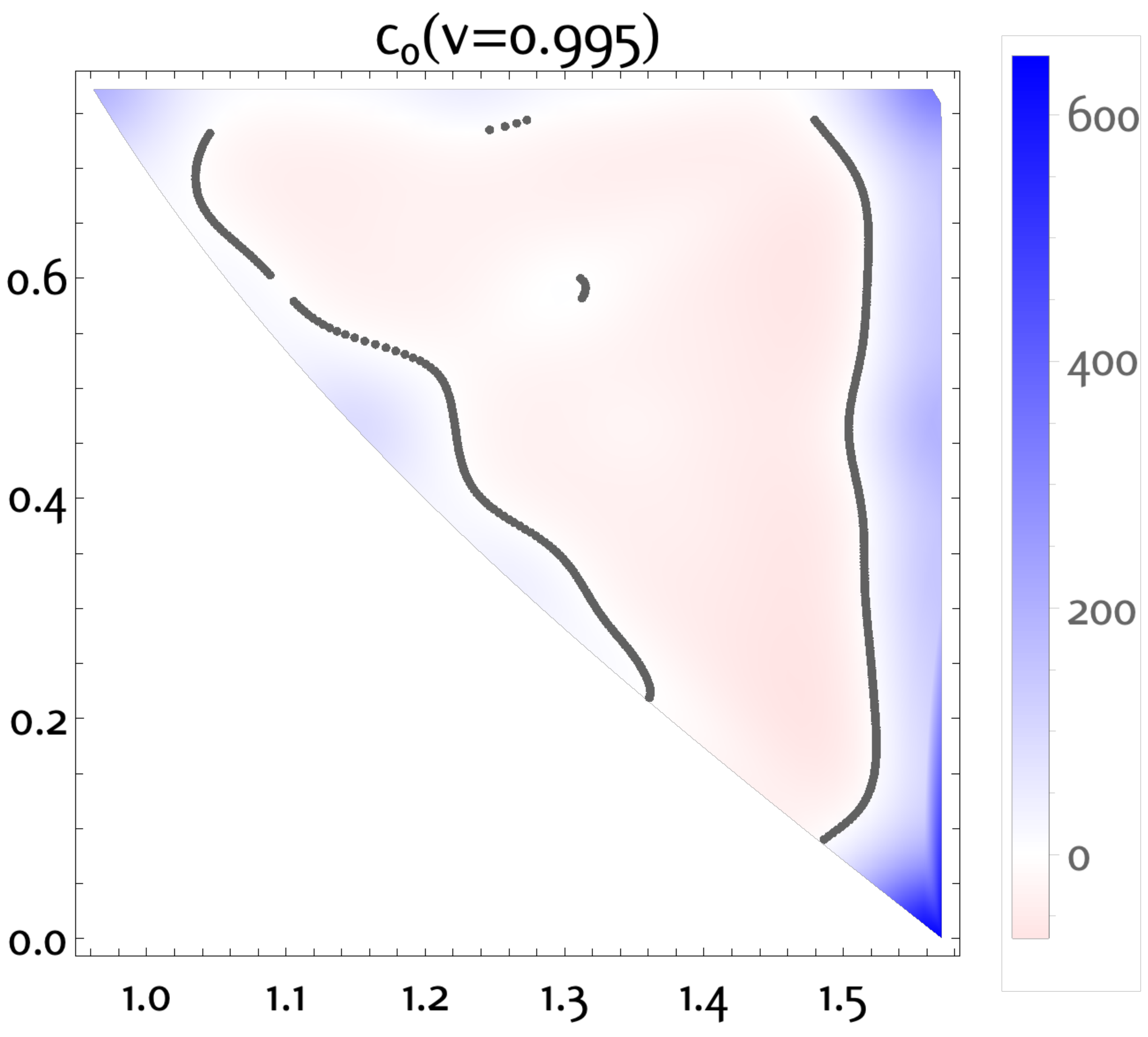




Velocity-dependent coefficients in $\mathrm{QED_r}$



"magic angles"



Stochastic direction average