Two photon contribution to the $K_{\mathrm{L}} \to \mu^+\mu^-$ decay amplitude on a $1/a \approx 1$ GeV lattice

En-Hung Chao[1], Ceran Hu

Columbia University

August 2, 2024, Lattice 2024 @ Liverpool, UK

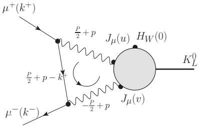
On behalf of the RBC/UKQCD collaboration
Based on on-going work with Norman Christ

[1]Speaker

Introduction

- **Motivation**: complementing the short-distance (SD) part of the $K_{\mathrm{L}} \to \mu^+\mu^-$ decay amplitude with the long-distance (LD) part for a precision test of the Standard Model $[\mathrm{Br}(K_{\mathrm{L}} \to \mu^+\mu^-) = 6.84(11) \times 10^{-9}]$ [BNL E871 Collab., PRL '00].
- **Goal**: determine the experimentally-inaccessible real part of the 2γ contribution to the $K_{\mathrm{L}} \to \mu^+\mu^-$ decay amplitude.
- **Formalism**: QED_∞ + effective weak Hamiltonian + lattice QCD

  [extension of 2208.03834]
- **Master formula**: [2406.07447]



$$\boxed{\mathcal{A}_{K_L\mu\mu} = \mathcal{A}^{\mathrm{I}} + \mathcal{A}^{\mathrm{II}}}$$

with the unphysical exponentially-growing states removed

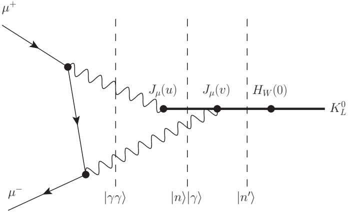
$$\mathcal{A}^{\mathrm{I}} = \int_{-T_v^-}^{T_v^+} dv_0 \int_V d^3\mathbf{v} \int_{v_0}^{T_u+v_0} du_0 \int_V d^3\mathbf{u}\, e^{M_K(u_0+v_0)/2} L_{\mu\nu}(u-v)\langle T\{J_\mu(u)J_\nu(v)\mathcal{H}_{\mathrm{W}}(0)K_L(t_i)\}\rangle' ,$$

and their physical contributions added back

$$\mathcal{A}^{\mathrm{II}} = -\sum_n \int_V d^3\mathbf{v} \int_0^{T_u} dw_0 \int_V d^3\mathbf{u}\, [\frac{e^{M_K w_0/2}}{M_K - E_n} L_{\mu\nu}(\mathbf{u}-\mathbf{v}, w_0) \times\langle T\{J_\mu(\mathbf{u}, w_0)J_\nu(\mathbf{v},0)\}|n\rangle\langle n| T\{\mathcal{H}_{\mathrm{W}}(0)K_L(t_i)\}\rangle .$$

Formalism

Time-ordering and Wick rotation



- Set an IR cutoff T and consider the possible intermediate states in the particular time-ordering $0 \leq v_0 \leq u_0$.
- The contribution from this time-ordering reads

$$\int_0^T du_0 \int_0^{u_0} dv_0 \int_{-\infty}^{\infty} dp_0 \; e^{i\left(\frac{M_K}{2}+p_0\right)u_0} e^{i\left(\frac{M_K}{2}-p_0\right)v_0}$$
$$\times \tilde{\mathcal{L}}^{\mu\nu}(p) e^{-iE_n u_0} e^{-i(E_{n'}-E_n)v_0} \langle 0 \,| J_\mu(0) | n \rangle \langle n | J_\nu(0) | n' \rangle \langle n' | \mathcal{H}_W(0) |\, K_L \rangle \,.$$

- Under Wick rotation $u_0 \leftarrow -iu_0$, it converges at $T \to \infty$ iff

$$\boxed{E_n' > M_K \quad (\mathrm{S1}) \qquad \text{and} \qquad E_n + \sqrt{\vec{p}^{\,2} + m_\gamma^2} \geq M_K \quad (\mathrm{S2})}$$

  Otherwise, unphysical exponential terms appear.

- Repeating the above analysis for all possible time-orderings and intermediate states, the two sources for the exponential terms are

  1. π^0 with zero spatial momentum, coming from K_{L} turned into π^0 by the weak Hamiltonian.
  2. $\pi\pi(\gamma)$ states with low kinetic energy, propagating between the electromagnetic currents.

Formalism

Limitations 1/3 [arXiv:2406.07447]

- In the case of non-interacting pions, (S2) is satisfied with $L \leq 10$ fm in the continuum
  $\Rightarrow$ errors from above this volume threshold?

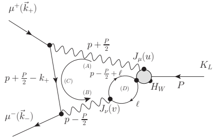
- Systematic errors due to finite-volume effects (FVEs);
  particular worry: the incomplete $\pi\pi$ spectrum

- With QED$_\infty$, in general the FVEs are expected to be exponentially suppressed; however, in the current case, LQCD+QED$_\infty$ does not conserve momentum $\Rightarrow$ (S2) is violated by $O(L^{-n})$ contributions.

  Claim: these effects are numerically small $\Leftarrow$ important check from the calculation by varying the IR cutoff R_{max} between the two EM currents.

Formalism

Limitations 2/3 [arXiv:2406.07447]

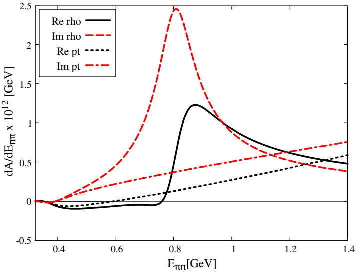


- Quantitative estimates of the CP-conserving $\pi\pi$ effects up to an energy of $E_{\pi\pi}$ from a spectral representation:

$$\mathcal{A}^{\pi\pi\gamma}_{K_L\mu\mu}(E_{\pi\pi}) = 4\pi C M_K^2 \int d^4p \, \frac{\mathbf{p}^2}{D(p)} \Pi(E_{\pi\pi}, p; P) \, ,$$

$$\Pi(E_{\pi\pi}, p; P) \equiv \int_{4M_\pi^2}^{E_{\pi\pi}^2 - \mathbf{p}^2} \frac{ds}{2\pi} \, s \, \eta(s) [F_\pi^V(s)]^* V^{\mathrm{pt}}_{K_L\pi\pi\gamma} \left[\frac{2}{\left(p - \frac{1}{2}P\right)^2 + s - i\varepsilon} \right] ,$$

- Valid for a point-like $K_\mathrm{L} \to \pi^+\pi^-\gamma$ vertex (fixed by experiment) and for a generic pion E&M form factor F_π^V.
- Two models for F_π^V are considered:
  - Point-like (scalar QED)
  - Gounaris-Sakuri
- **Caveat**: divergence as $E_{\pi\pi} \to \infty$, but irrelevant for our purpose.



Formalism

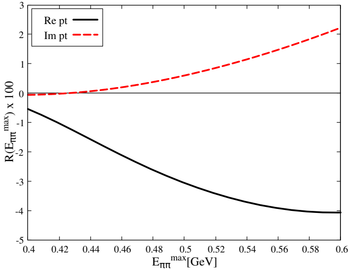
Limitations 3/3 [arXiv:2406.07447]

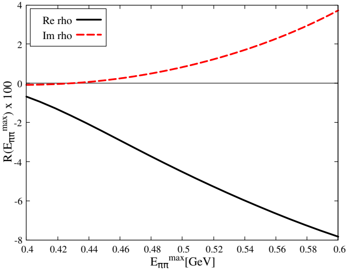
- With $E_{\pi\pi}^{\max} = 0.6$ GeV, we obtain the following ratios of the $\pi\pi$ contribution to the amplitude to the experimental results:

Model	$\mathrm{Re}\mathcal{A}/\mathrm{Re}\mathcal{A}_{\mathrm{exp}}$	$\mathrm{Im}\mathcal{A}/\mathrm{Im}\mathcal{A}_{\mathrm{exp}}$
pt	-0.041	0.022
GS	-0.078	0.037

$\Rightarrow$ both models give $< 10\%$ estimates.

- The ρ meson is generally well captured on the lattice $\Rightarrow$ the enhancement in the GS model in the energy region of interest may be included in the lattice result.



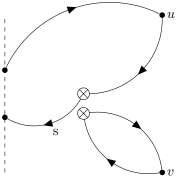


Numerical implementation

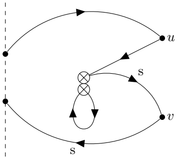
Contractions

- Wick-contractions for $\langle J_\mu(u) J_\nu(v) \mathcal{H}_{\rm W}(x) K_{\rm L}(t_K) \rangle$.

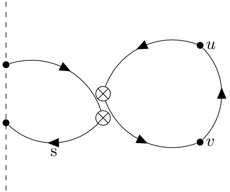
  Dashed line: $K_{\rm L}(t_K)$, crosses: $\mathcal{H}_{\rm W}(x)$, solid dots: $J_\mu(u)$ and $J_\nu(v)$



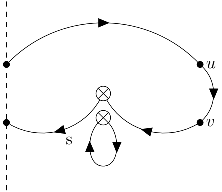
Type 1 diagram 1a



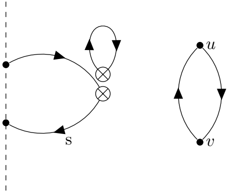
Type 2 diagram 1a



Type 3 diagram 1a



Type 4 diagram 1a



Type 5 diagram 1a

(*)We acknowledge Luchang Jin for generating the propagators used in this work.

Preliminary results

Parameter	Value
$L^3 \times T \times L_s$	$24^3 \times 64 \times 24$
m_π [MeV]	142
M_K [Mev]	515
a^{-1} [GeV]	1.023

- Lattice setup: Möbius Domain Wall fermion ensemble 24ID from the RBC/UKQCD collaboration.
- Current statistics:
  - O(100) configurations for the 4pt functions with O(500 − 2000) point sources with various variance reduction techniques (all-to-all, LMA+AMA).
  - O(700) configurations for the determination of the η mass $\Rightarrow$ principle source of error.
- Various strategies for treating the slowing decay η intermediate state (cf. talk by Ceran Hu).
- Summary table

$$\langle \mu^+\mu^- | \mathcal{H}_{\mathrm{W}}(0) | K_{\mathrm{L}} \rangle_{\mathrm{LD}} = \frac{G_{\mathrm{F}} e^4}{\sqrt{2}} |V_{us}||V_{ud}| \mathcal{A}$$

	$\lvert\mathrm{Re}\mathcal{A}\rvert \times 10^{-4}$ [MeV^3]	$\lvert\mathrm{Im}\mathcal{A}\rvert \times 10^{-4}$ [MeV^3]
24ID	5.68(1.49)	17.09(3.43)
SD	2.47(18)	—
exp.	1.53(14)	7.10(3)

- **Outlook** moving to the $48^3 \times 96$ $1/a \approx 1.73$ GeV physical pion ensemble, followed by a finer lattice for a continuum limit.