$\Delta I = 1/2$ process of $K \to \pi\pi$ decay on multiple ensembles with periodic boundary conditions

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K → ππ & Direct CPV

CP odd CP even

|K$_L$> = |K$_2$> + ε|K$_1$>

ε′ direct CPV

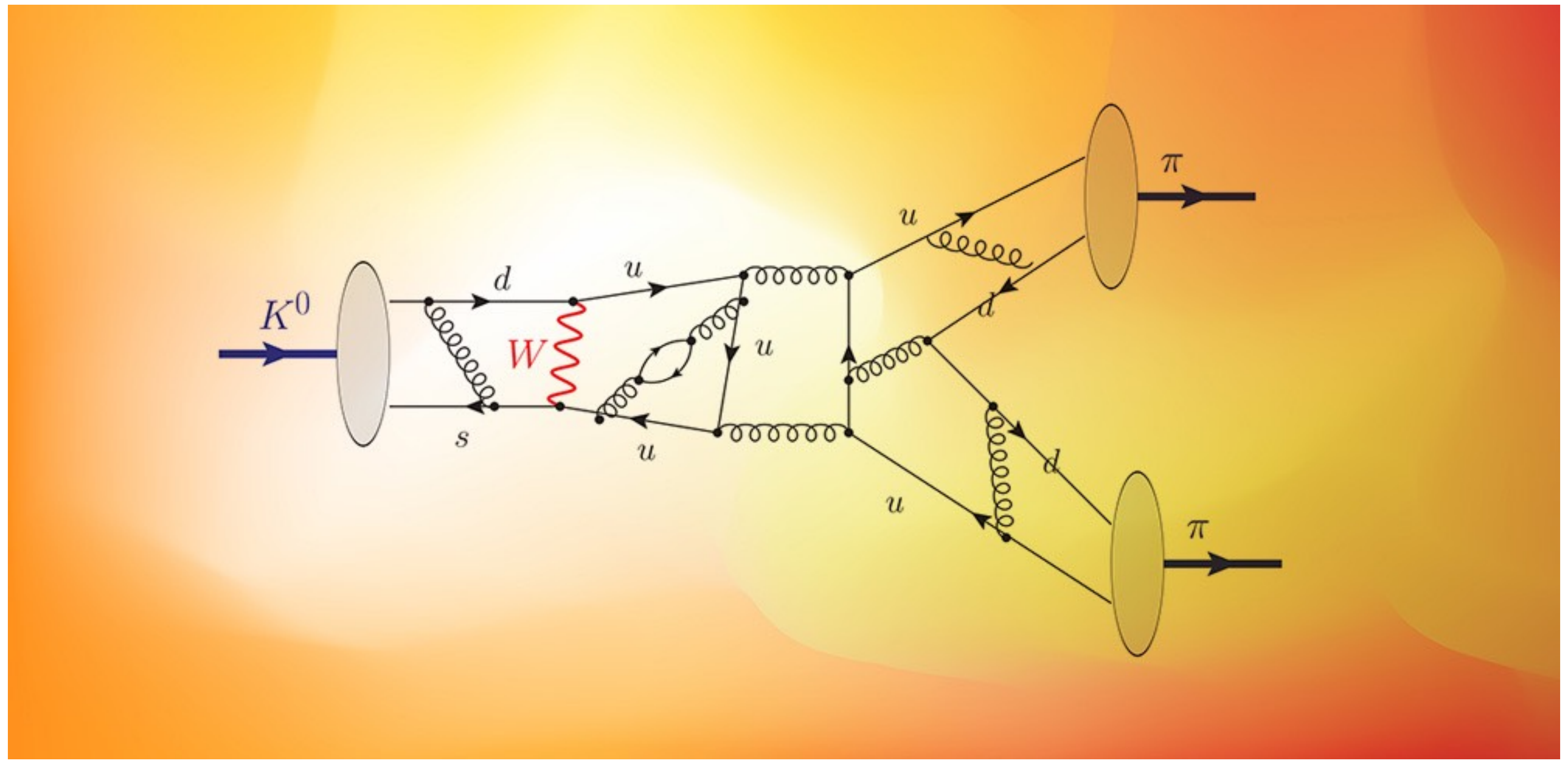
ε indirect CPV

|ππ>

CP even

$$\frac{\Gamma(K_L\to\pi^0\pi^0)}{\Gamma(K_S\to\pi^0\pi^0)}\Big/\frac{\Gamma(K_L\to\pi^+\pi^-)}{\Gamma(K_S\to\pi^+\pi^-)} = 1 - 6\times\mathrm{Re}(\epsilon'/\epsilon)$$

- ε′ vs ε
  - Re (ε′/ε)$_{exp}$ = 16.6(2.3) x 10^{-4} (KTeV, NA48)
  - Explained by SM?



$K \to \pi\pi$ Amplitude and ε'

ππ phase shifts at m_K

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\left[\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\right]\right\} \qquad (\omega = \mathrm{Re}\,A_2/\mathrm{Re}\,A_0)$$

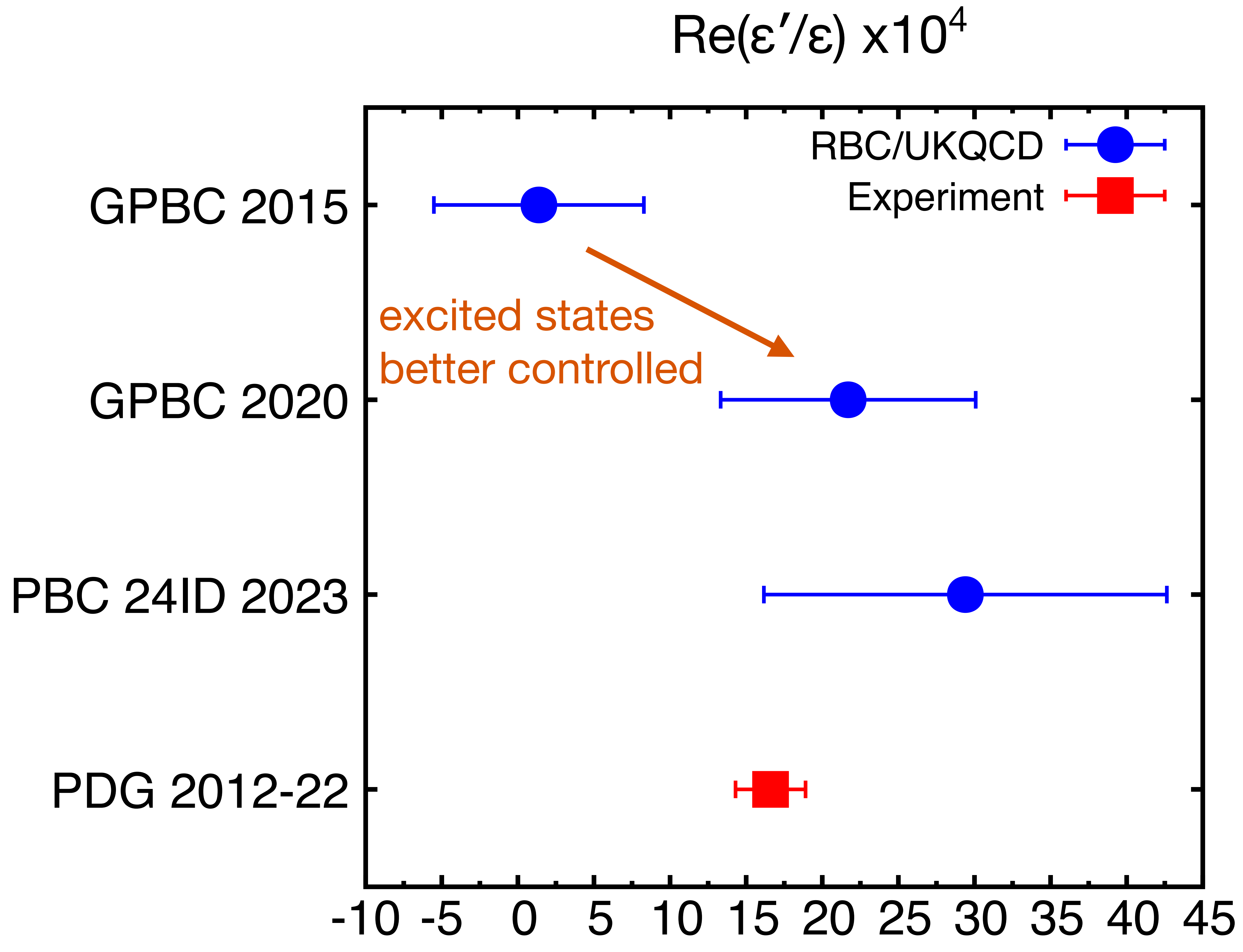
Lellouch-Lüscher finite volume correction

Renormalization matrix

$$A_I = F\frac{G}{\sqrt{2}}V_{us}^*V_{ud}\sum_{i,j}\underbrace{[z_i(\mu)+\tau y_i(\mu)]}_{\text{Wilson coefs. pQCD}}\underbrace{Z_{ij}(\mu)}_{\text{LQCD (+pQCD)}}\underbrace{\langle(\pi\pi)_I|Q_j^{lat}|K\rangle}_{\text{LQCD}}$$

- Matrix elements $< (\pi\pi)_I \,|\, Q_i^{lat} \,|\, K >$ from 3pt correlation functions
- A_2 amplitude has been determined very precisely [PRD91,074502 (2015)]
- A_0 challenging — disconnected diagrams, power divergences – **main focus**

Earlier calculations at physical m_π & m_K



PRL 115,212001

PRD 102,054509

PRD 102,094517

- G-parity Boundary Conditions (GPBC)
  - $a^{-1} \approx 1.38$ GeV
  - Efforts started by early 2000s
  - Led by C. Kelly

- Periodic Boundary Conditions (PBC)
  - newer project
  - important for introducing EM/IB effects
  - $a^{-1} \approx 1.02$ GeV
  - Led by MT

- **$a^{-1} \approx 1.38$ GeV almost done, wrapping up**
- **starting calculation at $a^{-1} \approx 1.73$ GeV**

Systematic errors in 2020

- Systematic errors on Im A_0

Finite lattice spacing	12%
Wilson coefficients	12%
Lelloch-Lüscher FV correction	1.5%
Residual FV correction	7%
Parametric error	6%
Off-shellness	5%
Renormalization	4%
Missing G_1 operator	3%
TOTAL	21%

Improvement desired

Improvement study underway

- Additional systematic error on ε′ **Hope to compute near future (PBC appear necessary)**
  - ε′ could be significantly affected by EM/IB effects ($\Delta I = 1/2$ rule → ~20%)

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This talk:
Showing our first attempt at continuum extrapolation

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  - ε′ could be significantly affected by EM/IB effects (ΔI = 1/2 rule → ~20%)

Lattice setup

- RBC/UKQCD's 2+1-flavor MDWF ensembles at physical pion & kaon masses
  - 24^3 x 64, a^{-1} = 1.0 GeV, 439 configurations
  - 32^3 x 64, a^{-1} = 1.4 GeV, 470 configurations
- All-to-all quark propagators
  - 2,000 low modes for light quarks (no low mode for strange)
  - high-mode part: spin, color and time dilutions => 4x3x64 = 768 inversions
- Sample AMA in use (fewer configurations for exact)

Matrix elements

- For extraction of ME

$$M^{\mathrm{eff}}(t_2, t_1) = C^{(3)}(t_2, t_1)\left[\frac{e^{E^{\pi\pi}t_2}e^{E^K t_1}}{C^{\pi\pi}(t_2)C^K(t_1)}\right]^{1/2} \xrightarrow{\text{large } t_1 \ \& \ t_2} M$$

$C^{\pi\pi}$: 2-pt function of ππ operator

C^K : 2-pt function of kaon operator

$C^{(3)}$: K → ππ 3-pt function

Matrix elements

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Matrix elements

- For extraction of ME

$$M_n^{\mathrm{eff}}(t_2,t_1) = C_n^{(3)}(t_2,t_1)\left[\frac{e^{E_n^{\pi\pi}t_2}e^{E^K t_1}}{C_n^{\pi\pi}(t_2)C^K(t_1)}\right]^{1/2} \xrightarrow{\text{large } t_1 \ \& \ t_2} M_n \ (= \langle \pi\pi(\cancel{270\ \mathrm{MeV}}\ E_n) \,|\, H_W \,|\, K \rangle)$$

$C_n^{\pi\pi}$: 2-pt function of ππ operator that couples well with only n-th state

C^K : 2-pt function of kaon operator

$C_n^{(3)}$: K → ππ 3-pt function with n-th ππ operator used in $C_n^{\pi\pi}$

- Energy-conserving process found for excited ππ state – confronting PBC approach

Energy of 2 pions in rest frame with PBC

	momentum (non-interacting ππ's case)	Energy
n = 0	(0,0,0)	$2m_\pi$ (+ interaction)
n = 1	$2\pi/L$ x (1,0,0)	could be $\approx m_K$
n = 2	$2\pi/L$ x (1,1,0)	

Variational method [Lüscher, 1990]

- Solving GEVP (Generalized Eigenvalue Problem)

$$C(t)v_n(t,t_0) = \lambda_n(t,t_0)C(t_0)v_n(t,t_0)$$

$C(t)$: N x N correlator matrix $C_{ab}(t) = \langle O_a(t)O_b(0)^\dagger \rangle$

  - $O'_n = \sum_a v^*_{n,a} O_a$ couples mostly with n-th state
  - $\lambda_n(t,t_0) = e^{-E_n(t-t_0)}$

- ππ operators used in this work:
  - $\pi_{p=(0,0,0)}\pi_{p=(0,0,0)}$
  - $\pi_{p=(0,0,1)}\pi_{p=(0,0,-1)}$
  - $\pi_{p=(0,1,1)}\pi_{p=(0,-1,-1)}$
  - $\pi_{p=(1,1,1)}\pi_{p=(-1,-1,-1)}$
  - σ ~ $\bar{u}u + \bar{d}d$: turned out to be very important to extract ~500-MeV state **(e.g. PRD 102,054509)**
  - KK ~ $\overline{K}K + K^+K^-$: for checking further excited-state contamination

Re-based GEVP

- Re-based GEVP
  - Large size GEVP at short time separations
  - Switch to smaller-size GEVP at larger time any eigenvalue is becoming zero-consistent

- Example:

$$\begin{pmatrix} O_{(000)} \\ O_{(001)} \\ O_{(011)} \\ O_{(111)} \\ O_{\sigma} \end{pmatrix} \xrightarrow{\text{5x5 GEVP at t = 2}} \left\{ \begin{array}{l} O_0^{t=2} \\ O_1^{t=2} \\ O_2^{t=2} \\ O_3^{t=2} \\ O_4^{t=2} \end{array} \right.$$

- $O_0^{t=2}$ - couples well with the ground st.
- $O_1^{t=2}$ - couples well with the 1st excited st.
- $O_2^{t=2}$ - …
- $O_4^{t=2}$ - incorporating 5th op and hence 5th state can spoil the signal of lower states at larger time separations

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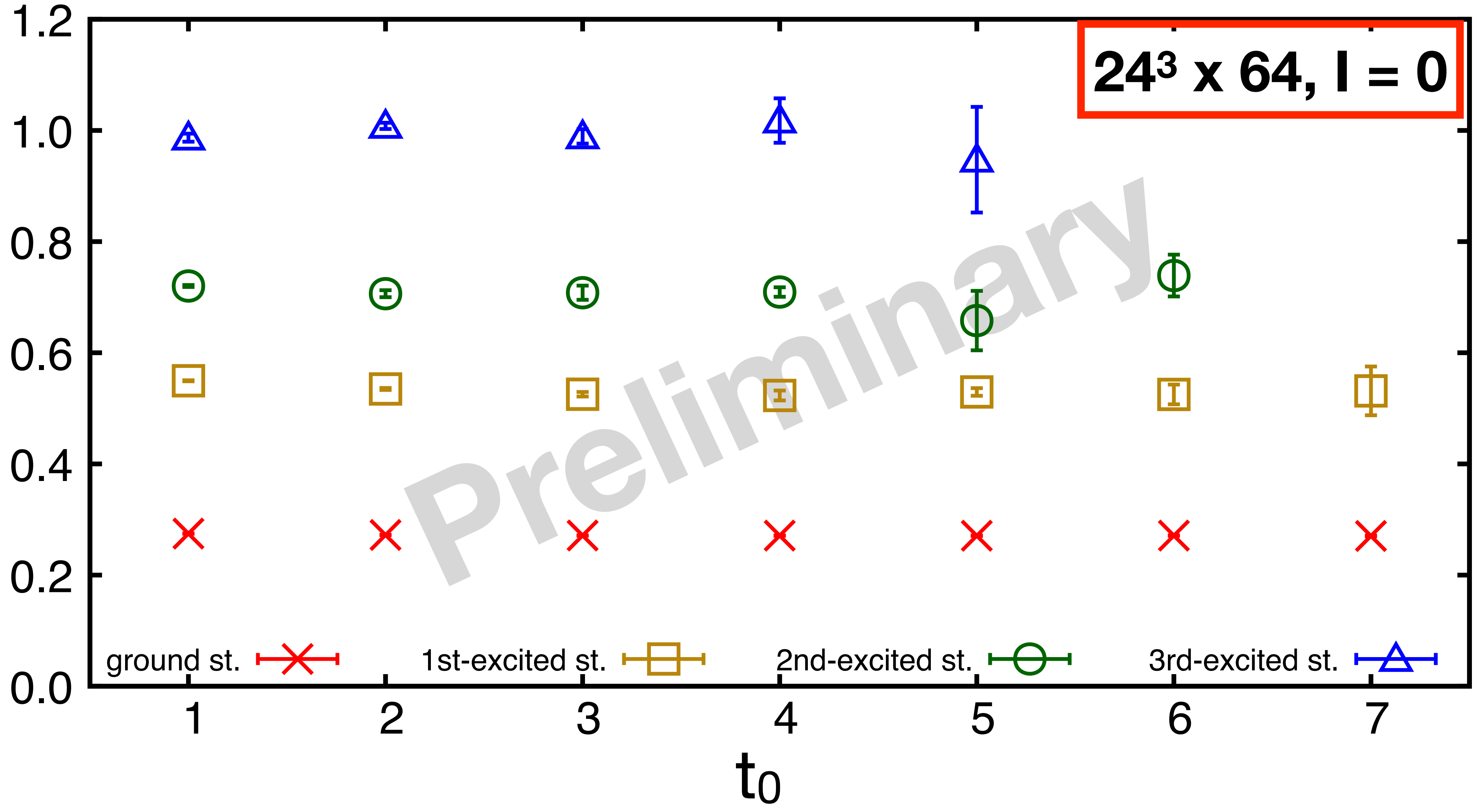
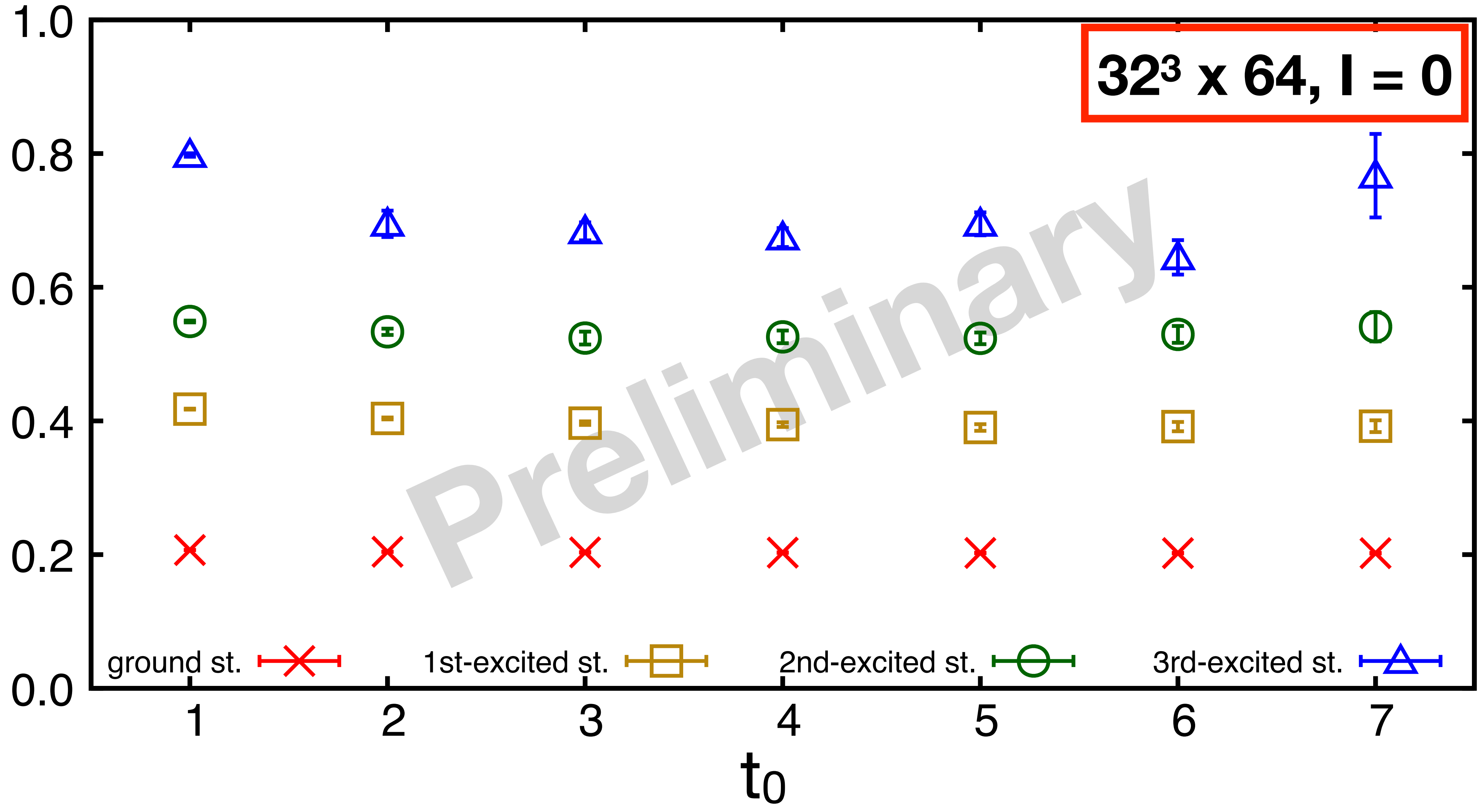
$$\begin{pmatrix} O_0^{t=2} \\ O_1^{t=2} \\ O_2^{t=2} \\ O_3^{t=2} \end{pmatrix} \xrightarrow{\textbf{4x4 GEVP at t = 3, ...}} \left\{ \begin{array}{l} \bullet\ \text{3 operators to form next basis} \\ \bullet\ \text{1 operator to be excluded from next basis} \end{array} \right.$$

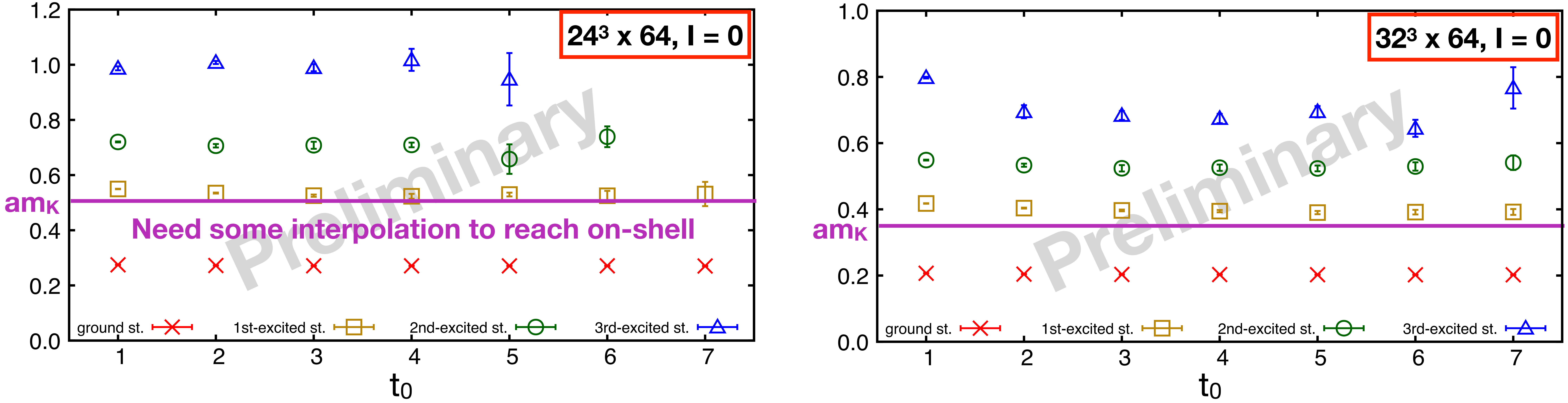
$aE^{\mathrm{eff}}_{\pi\pi}$ from ππ 2pt func & GEVP





- RGEVP works well
- 3rd excited state shown is from the 4-operator basis, the others from 3-operator basis
- Correction with non-interacting ππ system on lattice and continuum applied
  - Plateau appears at earlier time separations

$aE^{\text{eff}}_{\pi\pi}$ from ππ 2pt func & GEVP



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Effective matrix elements

$$M^{\mathrm{eff}}_{n,i}(t_2,t_1) = C^{(3)}_{n,i}(t_2,t_1)\left[\frac{e^{E^{\pi\pi}_n t_2}e^{E^K t_1}}{C^{\pi\pi}_n(t_2)C^K(t_1)}\right]^{1/2}$$

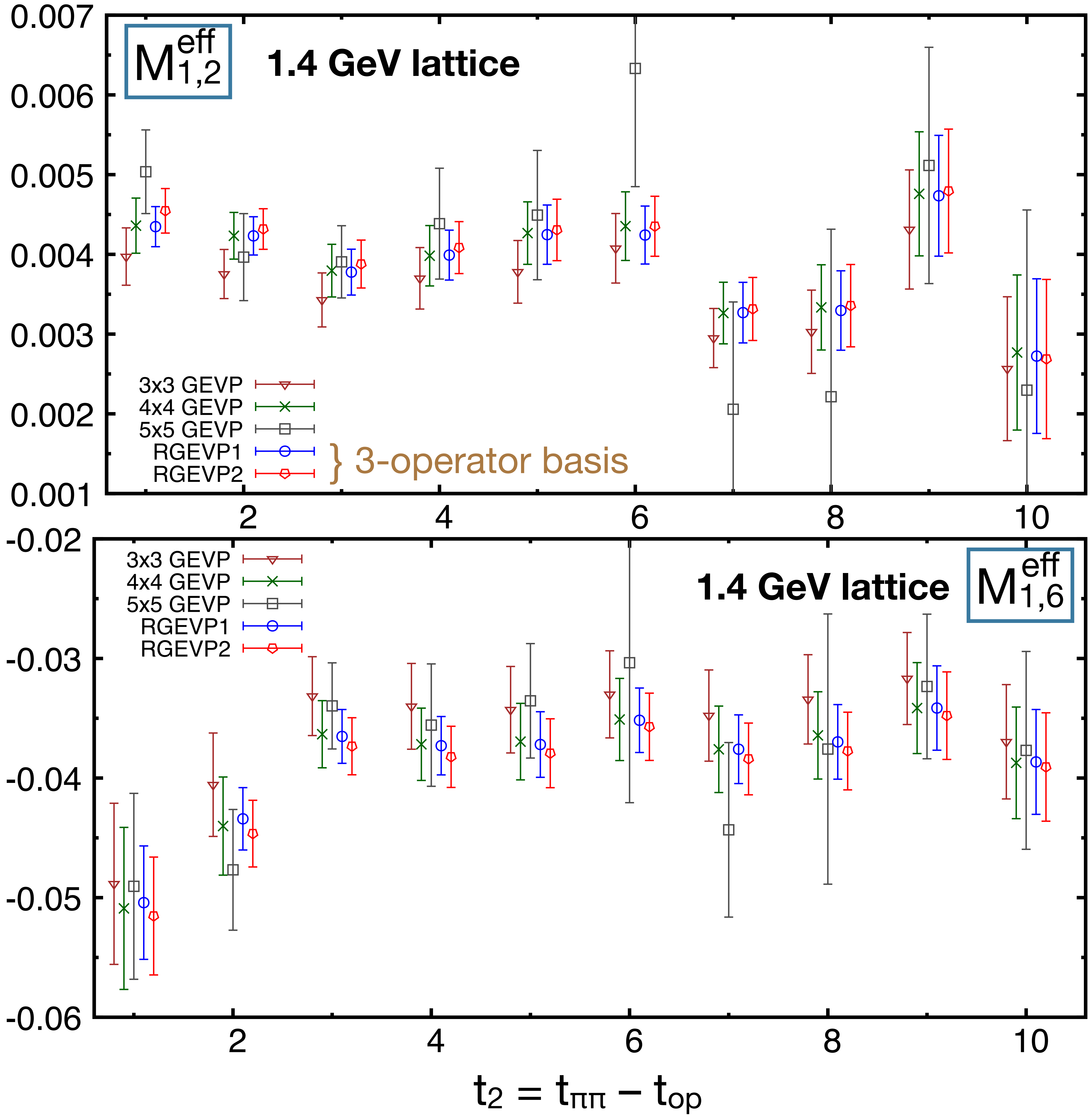
n: state index
i: operator index

$$\xrightarrow{\text{large } t_1 \,\&\, t_2} M_{n,i}$$

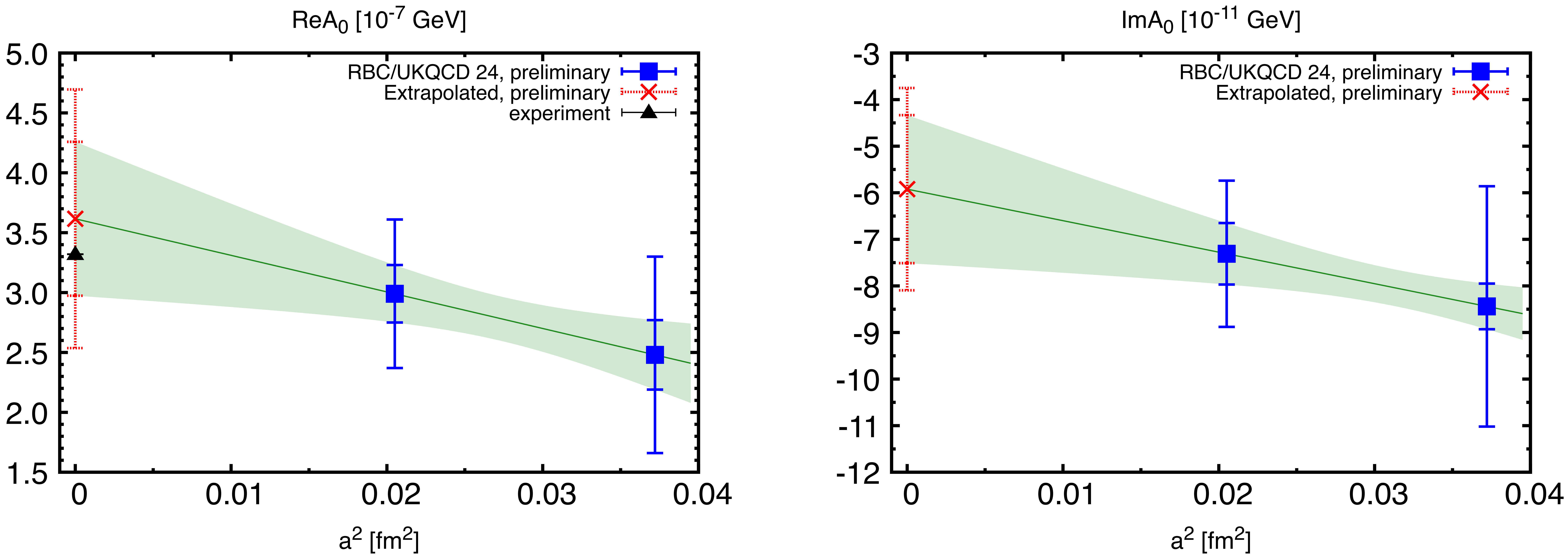
- Weighted average over $t_1 = t_{op} - t_K$ taken
- State n extracted by (R)GEVP eigenvector

$$C(t)v_n(t,t_0) = \lambda_n(t,t_0)C(t_0)v_n(t,t_0)$$

  - $t - t_0$ fixed to 2
  - No significant dependence on t_0 seen around current choice: $t_0 = 5$
- RGEVP (5→4→3) plateauing from t_2 = 3 or 4
  - smaller error than 4x4
  - potential excited-state contamination in 3x3
  - GEVP statistically near singular for 5x5



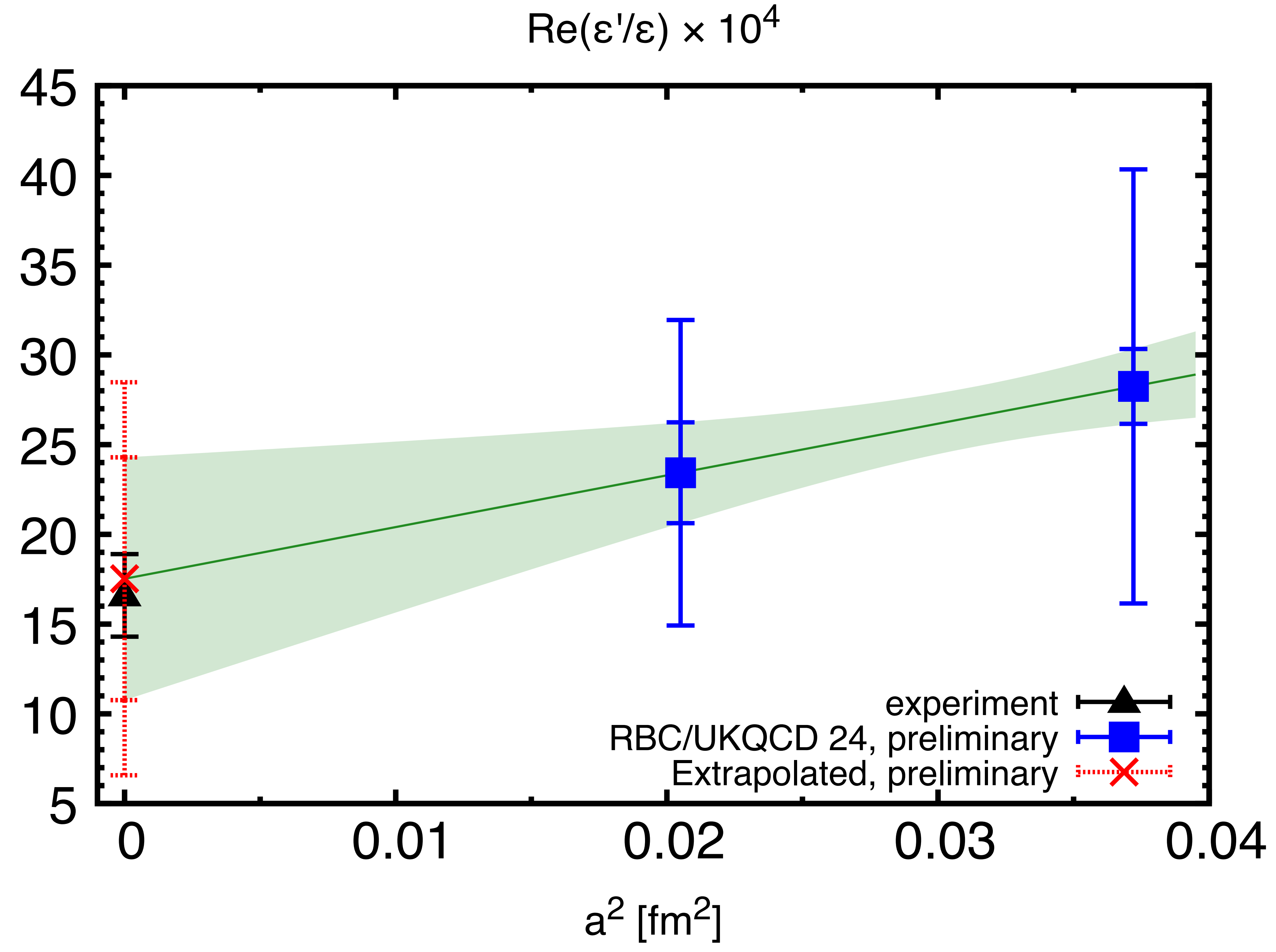
Result for A_0

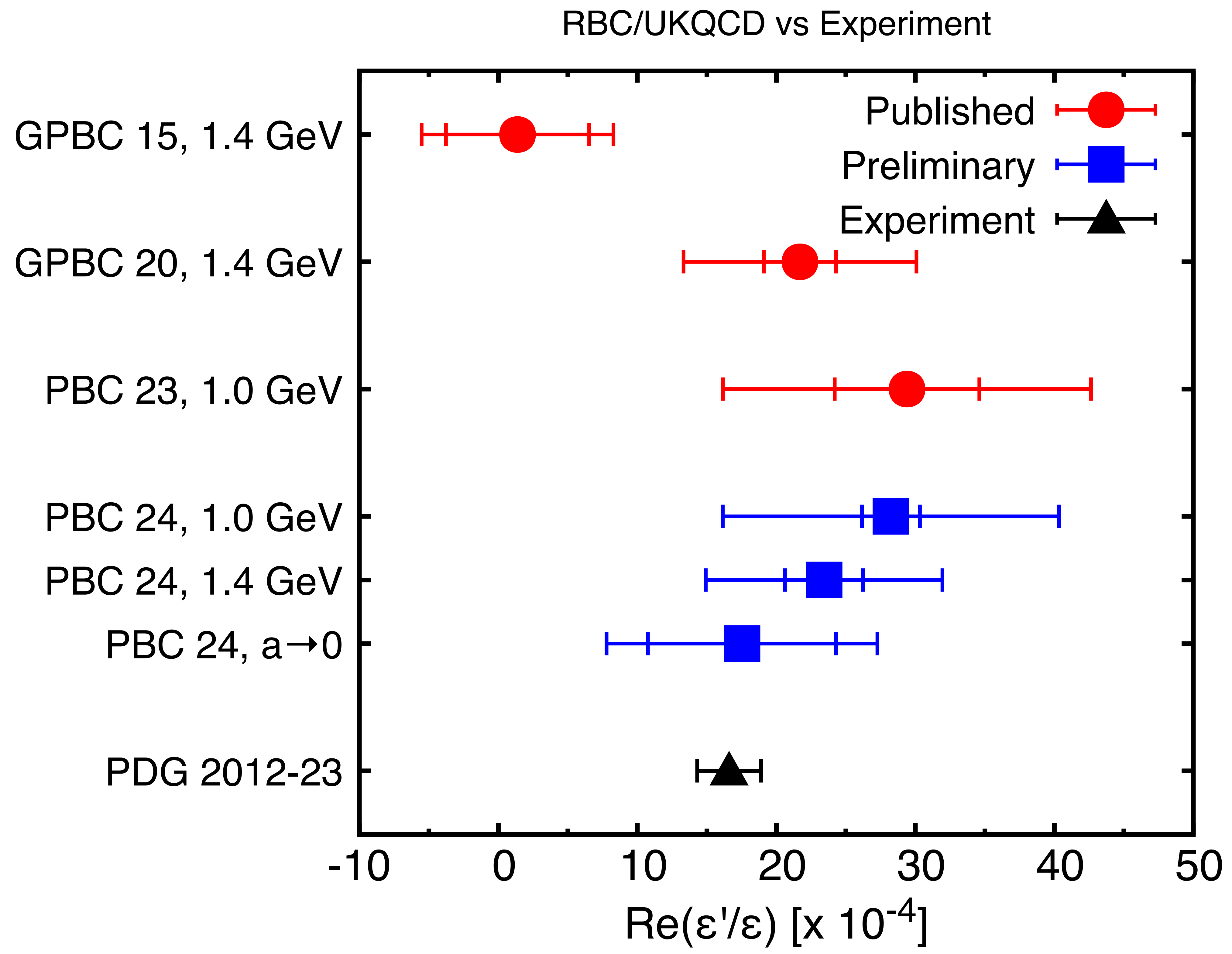


- $O(a^2)$ scaling violation potentially significant
  - Extrapolation with $c_0 + c_2\, a^2 + c_4\, a^4$ with a constraint $|c_2\, a^2| = 2\,|c_4\, a^4|$ at $a^{-1} = 1.0$ GeV corresponding to the coarser lattice did not change the result beyond statistical error

Result for ε′

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\left[\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\right]\right\}$$



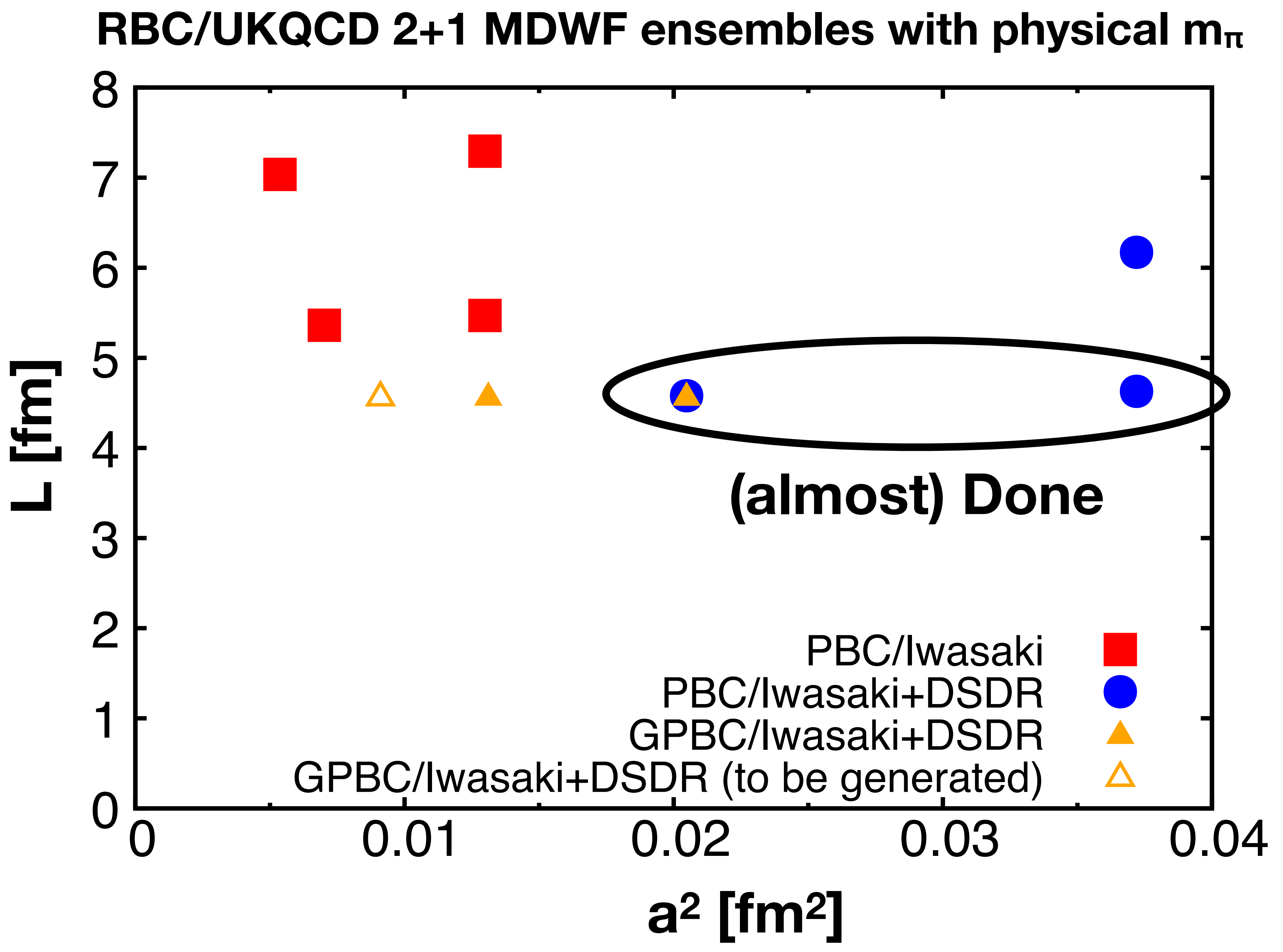


- 1.4 GeV lattice calculation with PBC and GPBC consistent
- Continuum extrapolation attempted

Summary & Outlook

Current main sources of systematic errors on ε′

- Finite lattice spacing
  - We did first attempt continuum extrapolation with multiple ensembles
  - 1.0 & 1.4 GeV ensembles
    → O(a^2) scaling violation potentially significant
  - Finer lattice calculations on-going & planned →→→



- Wilson coefficients
  - An improvement study underway, possibly to be incorporated in the upcoming paper
- EM/IB effects
  - Theoretical framework being developed *[Christ et al, PRD106, 014508 (2021)]*

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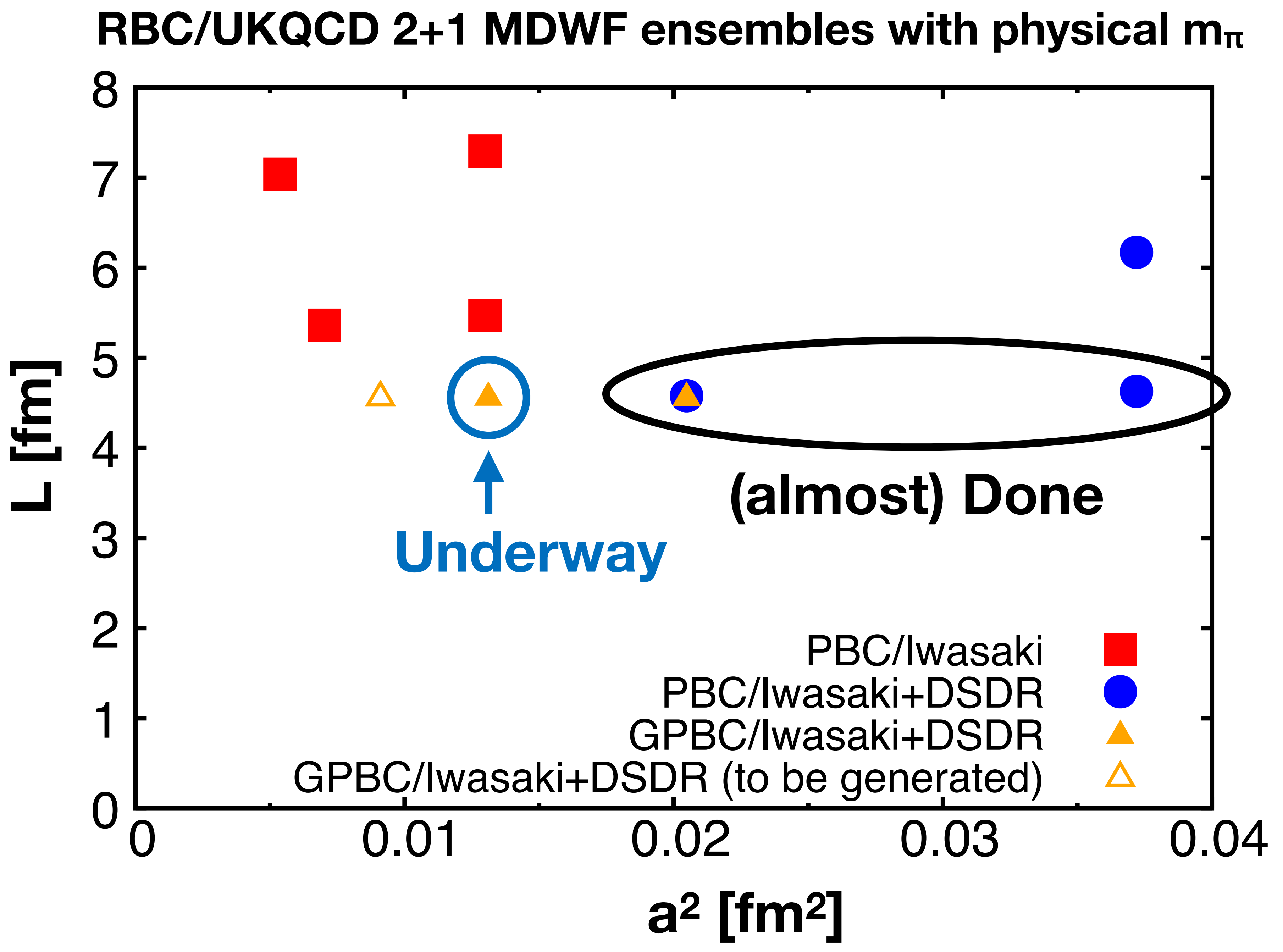

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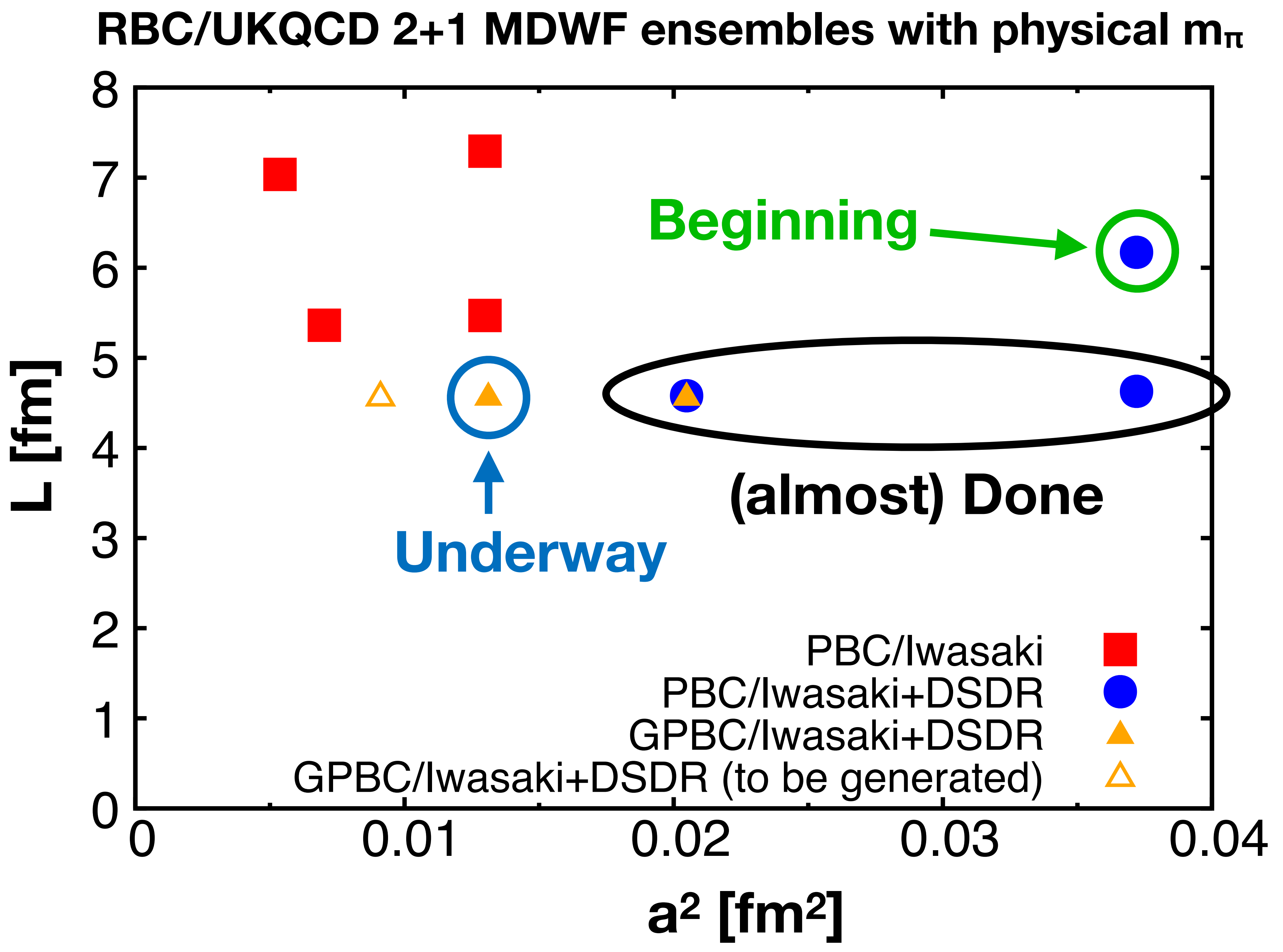

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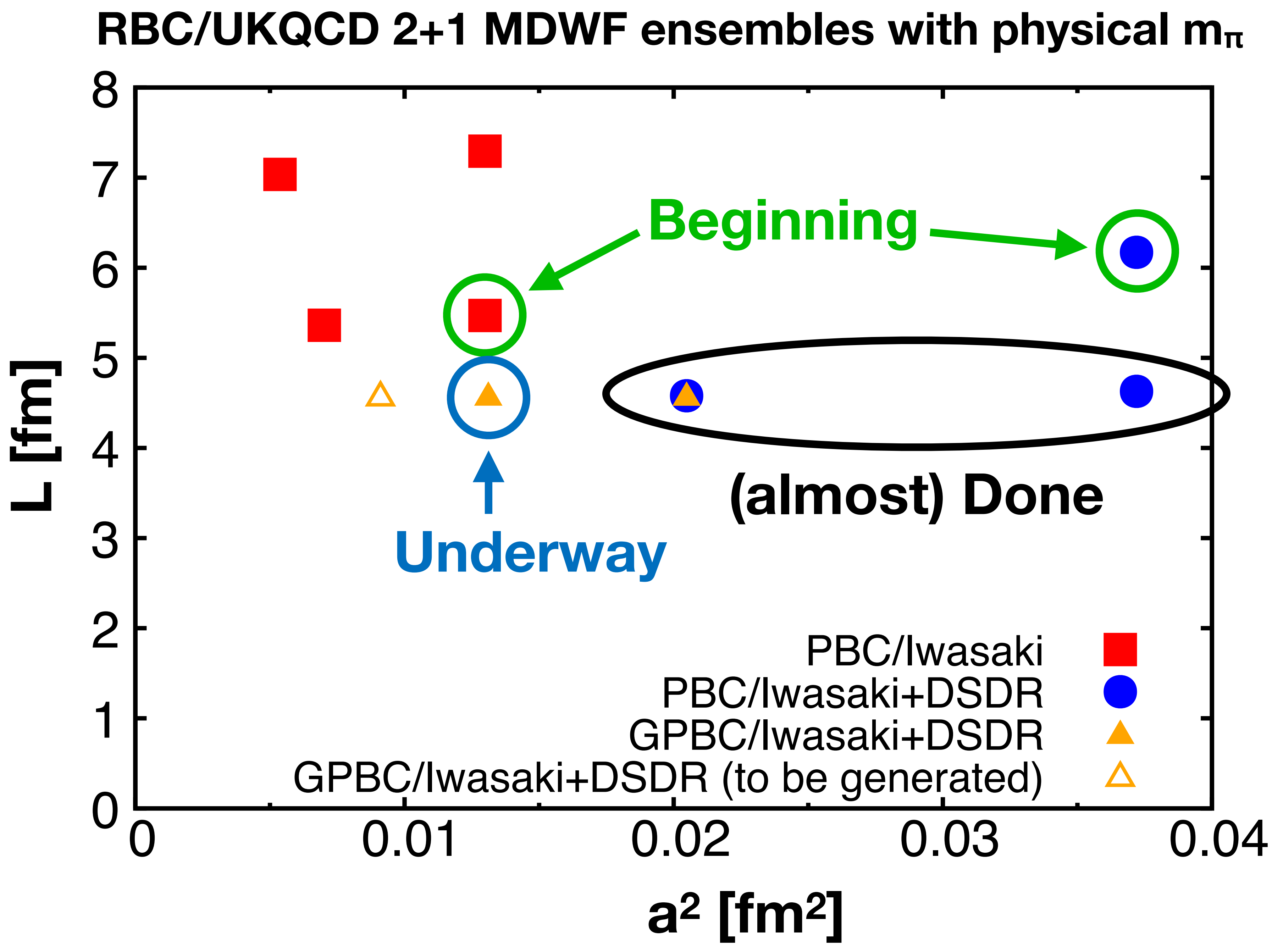

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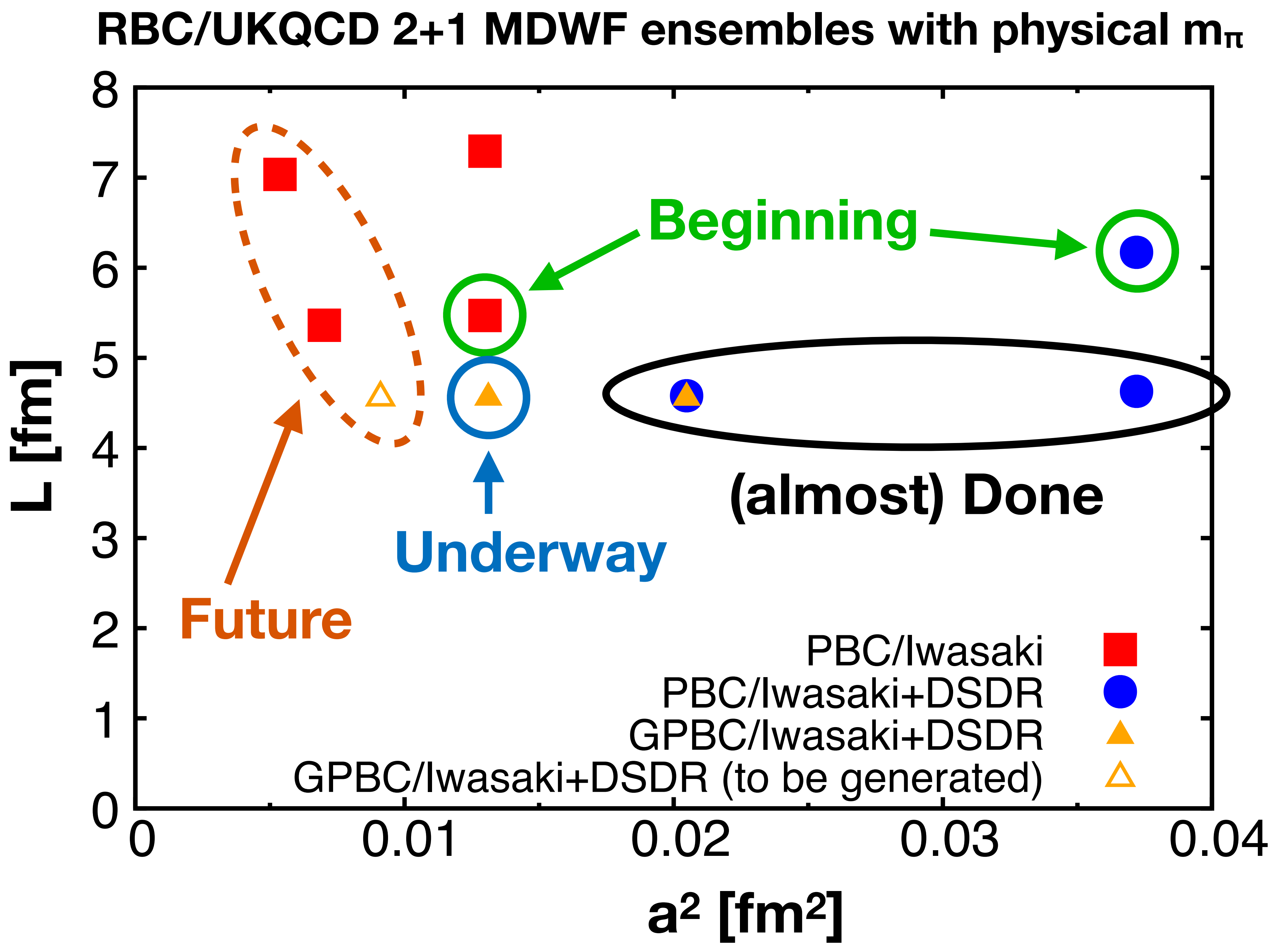

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