Contribution of the η to a lattice calculation of $K_{\mathrm{L}} \longrightarrow \mu^+\mu^-$ decay

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Outline

- Introduction
- Study of η on lattice
- Methods to remove η
- Preliminary results
- Conclusion and outlook

Introduction

- Long distance 2γ exchange contribution to $K_{\mathrm{L}} \to \mu^+\mu^-$:

$$\mathcal{A}_{K_{\mathrm{L}}\to\mu^+\mu^-} = \int d^4x d^4r \; K_{\mu\nu}(r) \langle J_\mu(r) J_\nu(0) \mathcal{H}_w(x) | K_{\mathrm{L}} \rangle \quad (1)$$

$$\mathcal{H}_w = \mathcal{C}_1 \mathcal{Q}_1 + \mathcal{C}_2 \mathcal{Q}_2 \quad (2)$$

$$\mathcal{Q}_1 = (\bar{s}_a d_a)_{V-A} (\bar{u}_b u_b)_{V-A} \quad (3)$$

$$\mathcal{Q}_2 = (\bar{s}_a d_b)_{V-A} (\bar{u}_b u_a)_{V-A} \quad (4)$$

$K_{\mu\nu}(r)$ EM kernel for QED_∞, $\mathcal{C}_i$ Wilson coefficients.

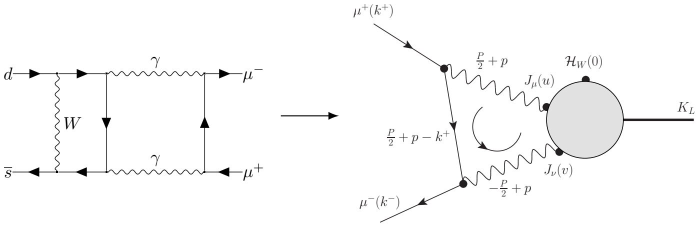

Figure 1: 2γ exchange contribution to $K_{\mathrm{L}} \to \mu^+\mu^-$

Introduction

- Unphysical intermediate state contribution:

$$\sum_n \int d^4r d^4x \; K_{\mu\nu}(r) \langle J_\mu(r) J_\nu(0) | n \rangle \langle n | \mathcal{H}_w(x) | K_\mathrm{L} \rangle \frac{e^{(M_K - M_n)\delta}}{M_K - M_n} \quad (5)$$

  δ is time separation between $\mathcal{H}_w$ and the earliest EM current.

- $n = \pi$: $M_K - M_\pi \approx 0.365 a^{-1} \longrightarrow$ exponentially growing term.
- $n = \eta$: $M_K - M_\eta \approx -0.06 a^{-1} \longrightarrow$ slowly decaying term.

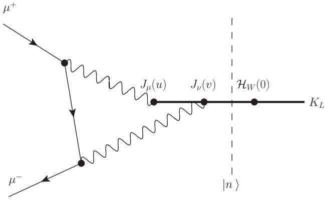

Figure 2: intermediate state contribution of π and η

- Preliminary GEVP study:

$$C(t) = \begin{pmatrix} \langle O_l(t)O_l^\dagger(0)\rangle & \langle O_s(t)O_l^\dagger(0)\rangle \\ \langle O_l(t)O_s^\dagger(0)\rangle & \langle O_s(t)O_s^\dagger(0)\rangle \end{pmatrix} \quad (6)$$

$$O_l = \frac{i}{\sqrt{2}}(\overline{u}\gamma_5 u + \overline{d}\gamma_5 d) \quad (7)$$

$$O_s = i\overline{s}\gamma_5 s \quad (8)$$

Generalized Eigenvalue Problem (GEVP) solves for the η and η' mass:

$$C(t)V_n(t,t_0) = \lambda_n(t,t_0)C(t_0)V_n(t,t_0) \quad (9)$$

$$M_n(t,t_0) = \ln\frac{\lambda_n(t,t_0)}{\lambda_n(t+a,t_0)} \quad (10)$$

$$t = t_0 + a \quad (11)$$

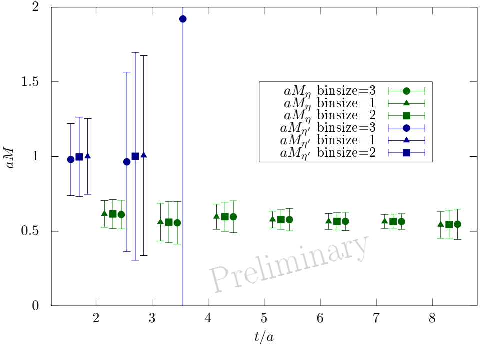

Figure 3: GEVP results on lattice with $1/a = 1.015$ GeV and physical quark masses

- Masses determined by GEVP*:

$$aM_\eta = 0.569(0.019) \qquad aM_{\eta'} = 0.996(0.060) \qquad (12)$$

*aM_η is obtained with fitting range $t \in [4, 8]$, $aM_{\eta'}$ is obtained with fitting range $t \in [2, 4]$

Study of η on lattice

- η-related 3 point functions
  - O_s : better statistics and overlap with η.
  - Only η and η' matter at large t.
  - Two-state fit:

$$A(t) = A_\eta e^{-M_\eta t} + A_{\eta'} e^{-M_{\eta'} t} \tag{13}$$

M_η and $M_{\eta'}$ are fixed in the ansatz from GEVP in Eqs.(12).

  - Preliminary results for 3 point functions*:

	lattice	$\eta \to \mu^+\mu^-$ exp. †
$\int d^4x \, \langle \eta \| \mathcal{Q}_1(x) \| K_\mathrm{L} \rangle$	-0.0118(18)	—
$\int d^4x \, \langle \eta \| \mathcal{Q}_2(x) \| K_\mathrm{L} \rangle$	0.00243(89)	—
$\int d^4x \, \langle \eta \| \bar{s}d + \bar{d}s(x) \| K_\mathrm{L} \rangle$	1.22(32)	—
$\mathrm{Re} \int d^4r \; K_{\mu\nu}(r) \langle J_\mu(r) J_\nu(0) \| \eta \rangle$	0.0104(37)	0.0146(52)
$\mathrm{Im} \int d^4r \; K_{\mu\nu}(r) \langle J_\mu(r) J_\nu(0) \| \eta \rangle$	0.0270(76)	$0.0254(5 \times 10^{-5})$

Table 1: 3 point function results from two-state fit

*Different methods are explored, the two-state fit described here allows the most precise determination. $K_{\mu\nu}(r)$ in $\int d^4r \; K_{\mu\nu}(r) \langle J_\mu(r) J_\nu(0) | \eta \rangle$ uses M_K instead of M_η, which will be corrected.

†Experimental results are converted to the same convention as lattice results.

Methods to remove η

- Method 1: direct subtraction
  - Subtract unphysical η contribution at time slice δ:

$$-\mathcal{A}_\eta \frac{e^{(M_K - M_\eta)\delta}}{M_K - M_\eta} \qquad (\delta > 0) \tag{14}$$

with overlap $\mathcal{A}_\eta$:

$$\mathcal{A}_\eta = \int d^4 r d^4 x \; K_{\mu\nu}(r) \langle J_\mu(r) J_\nu(0) | \eta \rangle \langle \eta | \mathcal{H}_w(x) | K_\mathrm{L} \rangle \tag{15}$$

  - Need to avoid $O(a)$ error when doing direct subtraction.
  - Direct subtraction can also remove unphysical π contribution.

- Method 2: adding $c_s(\bar{s}d + \bar{d}s)$
  - Ward identity:

$$\begin{pmatrix} d \\ s \end{pmatrix} \longrightarrow (\mathcal{I}_{2\times 2} + \epsilon T) \begin{pmatrix} d \\ s \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

$$\langle (m_d - m_s)(\bar{s}d + \bar{d}s)\mathcal{O}_{\mu\nu} + i\partial_\lambda(\bar{s}\gamma_\lambda d - \bar{d}\gamma_\lambda s)\mathcal{O}_{\mu\nu} \rangle = 0 \tag{17}$$

$$\mathcal{O}_{\mu\nu} = J_\mu J_\nu \hat{K}_{\mathrm{L}} \tag{18}$$

  - Redefine $\mathcal{H}_w$ to eliminate η:

$$\mathcal{H}'_w = \sum_{i=1}^{2} \mathcal{C}_i \left\{ \mathcal{Q}_i + c_{si}(\bar{s}d + \bar{d}s) \right\} \tag{19}$$

$$c_{s1} = -\frac{\langle \eta | \mathcal{Q}_1 | K_{\mathrm{L}} \rangle}{\langle \eta | \bar{s}d + \bar{d}s | K_{\mathrm{L}} \rangle} = 9.65(3.23) \times 10^{-3} \tag{20}$$

$$c_{s2} = -\frac{\langle \eta | \mathcal{Q}_2 | K_{\mathrm{L}} \rangle}{\langle \eta | \bar{s}d + \bar{d}s | K_{\mathrm{L}} \rangle} = -1.99(1.10) \times 10^{-3} \tag{21}$$

Methods to remove η

- Method 3: adding $c_s(\bar{s}d + \bar{d}s)$ to isospin 0 part
  - Isospin decomposition with $N_f = 2+1$:

$$J_\mu^{I=1} = \bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d \propto |I=1\rangle \quad (22)$$

$$J_\mu^{I=0} = \bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d + 2\bar{s}\gamma_\mu s \propto |I=0\rangle \quad (23)$$

$$(J_\mu J_\nu)^{I=1} = \frac{1}{12}\left(J_\mu^{I=0} J_\nu^{I=1} + J_\mu^{I=1} J_\nu^{I=0}\right) \quad (24)$$

$$(J_\mu J_\nu)^{I=0\&I=2} = \frac{1}{4} J_\mu^{I=1} J_\nu^{I=1} + \frac{1}{36} J_\mu^{I=0} J_\nu^{I=0} \quad (25)$$

  - Only isospin 0 part has non-zero overlap with η.
  - Add $c_s(\bar{s}d + \bar{d}s)$ to isospin 0 part, and modify Ward identity:

$$\left\langle (m_d - m_s)(\bar{s}d + \bar{d}s)\mathcal{O}'_{\mu\nu} + i\partial_\lambda(\bar{s}\gamma_\lambda d - \bar{d}\gamma_\lambda s)\mathcal{O}'_{\mu\nu} \right\rangle = \mathrm{c.t.} \quad (26)$$

$$\mathcal{O}'_{\mu\nu} = (J_\mu J_\nu)^{I=0\&I=2}\,\hat{K}_\mathrm{L} \quad (27)$$

  - Contact terms in Eqs.(26) are 3 point functions that can be precisely determined on the lattice.

Methods to remove η

- Method 4: adding $d_s(\bar{s}d + \bar{d}s)$ to remove $1/a^2$ divergence
  - $1/a^2$ power divergence in weak Hamiltonian can be removed:

$$\tilde{\mathcal{H}}_w = \sum_{i=1}^{2} \mathcal{C}_i \left\{ \mathcal{Q}_i + d_{si}(\bar{s}d + \bar{d}s) \right\} \quad (28)$$

$$d_{s1} = -\frac{\langle \pi | \mathcal{Q}_1 | K_{\mathrm{L}} \rangle}{\langle \pi | \bar{s}d + \bar{d}s | K_{\mathrm{L}} \rangle} = 3.724(43) \times 10^{-4} \quad (29)$$

$$d_{s2} = -\frac{\langle \pi | \mathcal{Q}_2 | K_{\mathrm{L}} \rangle}{\langle \pi | \bar{s}d + \bar{d}s | K_{\mathrm{L}} \rangle} = 19.635(46) \times 10^{-4} \quad (30)$$

  - Need to directly subtract η from $\tilde{\mathcal{H}}_w$ by Eqs.(14) and (15).

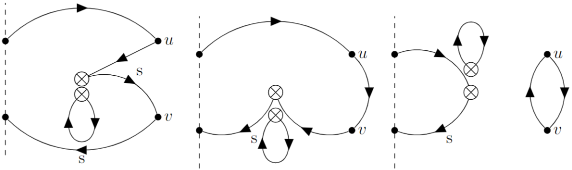

Figure 4: Diagrams with closed quark loop

Preliminary results

- Real part

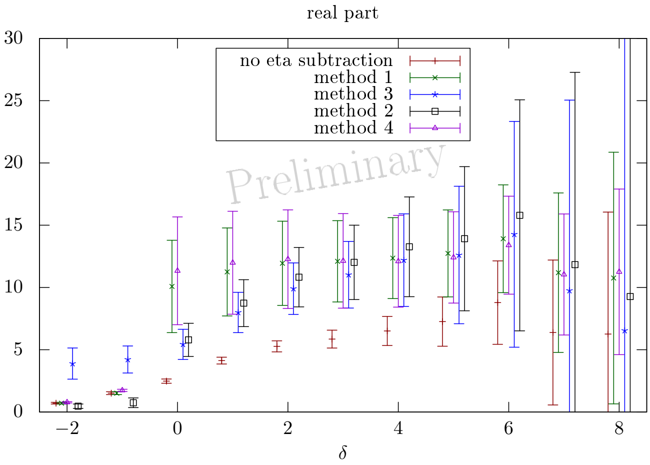

Figure 5: Real part of 2γ contribution to $K_{\mathrm{L}} \to \mu^+\mu^-$ amplitude

- Imaginary part

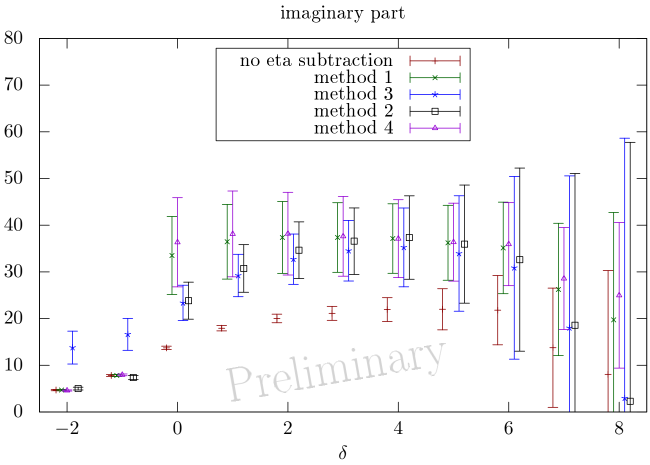

Figure 6: Imaginary part of 2γ contribution to $K_L \to \mu^+\mu^-$ amplitude

Preliminary results

- Lattice amplitudes can be related to experiments:

$$\left|\mathcal{A}_{K_{\mathrm{L}}\to\mu^+\mu^-}\right| = \frac{1}{e^4 V_{ud} V_{us} \frac{G_F}{\sqrt{2}}} \sqrt{\frac{8\pi M_K \Gamma_{K_{\mathrm{L}}\to\mu^+\mu^-}}{\beta}} \quad (31)$$

$$\beta = \sqrt{1 - \frac{4m_\mu^2}{M_K^2}} \quad (32)$$

- Unphysical η contribution:

$$\mathcal{A}^{\eta}_{K_{\mathrm{L}}\to\mu^+\mu^-}(\delta) = \frac{e^{(M_K - M_\eta)\delta}}{M_K - M_\eta} \int d^4r d^4x \, K_{\mu\nu}(r) \left\langle J_\mu(r) J_\nu(0) | \eta \right\rangle \left\langle \eta | \mathcal{H}_w(x) | K_{\mathrm{L}} \right\rangle \quad (33)$$

Preliminary results

sources/amplitudes ($\times 10^{-4}$ MeV^3)	$\mathrm{Re}\mathcal{A}_{K_\mathrm{L}\to\mu^+\mu^-}$	$\mathrm{Im}\mathcal{A}_{K_\mathrm{L}\to\mu^+\mu^-}$
experiment	1.53(0.14)	7.12(0.03)
short distance contribution	2.47(0.18)	—
no η subtraction	2.99(0.54)	10.09(1.17)
unphysical η contribution	2.69(1.55)	7.00(3.77)
results of method 1	5.68(1.49)	17.09(3.43)
results of method 2	6.10(1.84)	17.18(4.11)
results of method 3	5.61(1.71)	16.21(3.88)
results of method 4	5.57(1.69)	17.06(3.83)

Table 2: Comparison between the decay amplitudes from lattice and experiments

Preliminary results

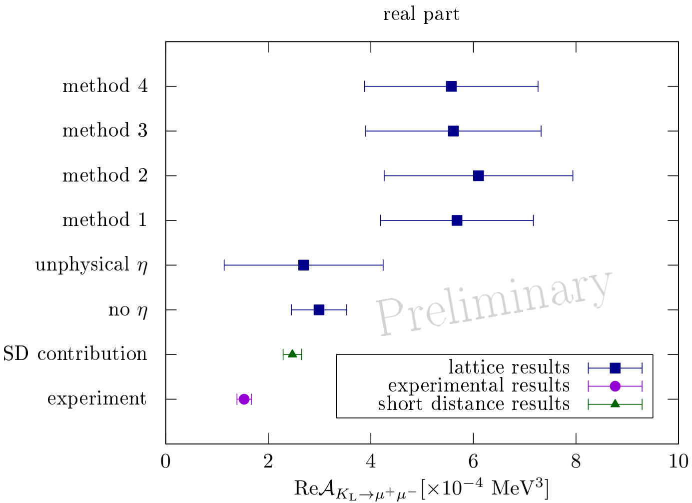

Figure 7: Real part results in Table 2

Preliminary results

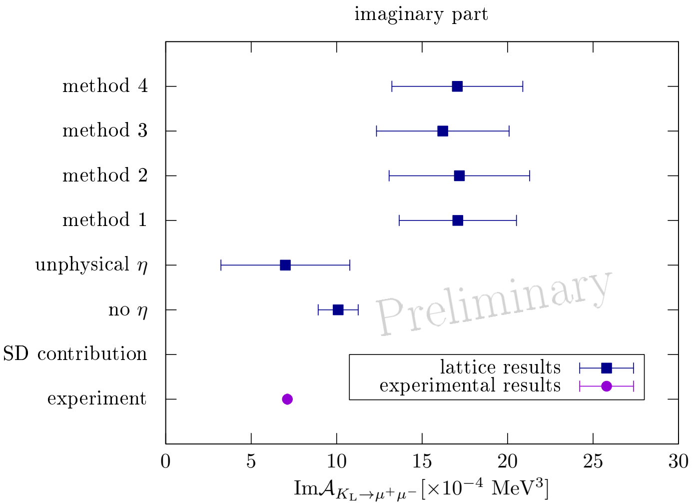

Figure 8: Imaginary part results in Table 2

Conclusion and outlook

- We perform a preliminary study of η on 24ID, including η mass and relevant 3 point functions.
- We propose 4 different methods to remove unphysical η contribution. Results from different methods show good consistency.
- We observe that large statistical errors are introduced from the poor determination of η-related quantities.
- We expect that on finer lattices, η-related quantities can be more precisely determined.
- We find a 2σ discrepancy between lattice computation and experimental results. A better understanding on systematic uncertainties and scaling behavior is needed.

Thanks for your attention!