Split-even approach to the rare kaon decay $K \to \pi\ell^+\ell^-$

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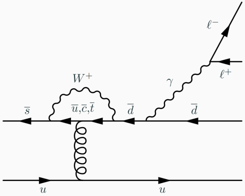
Rare Decays

- Rare kaon decay $K \to \pi\ell^+\ell^-$ is a $s \to d$ quark FCNC
- Heavily suppressed in the SM
  $\to$ good probe for BSM physics



- Focus on charged kaon decay $K^+ \to \pi^+\ell^+\ell^-$
  Neutral kaon decay also of interest

- Other interesting processes include: $\Sigma^+ \to p\ell^+\ell^-$ RH [Lattice 2023]
  $K \to \ell^+\ell^-$ E. Chao [11:55 Fri]
  C. Hu [12:15 Fri]
  $K \to \pi\nu\bar{\nu}$, etc.

- Split-even method is an alternate estimator for quark loop diagrams

- Long distance part $K^+ \to \pi^+\gamma^*$ given by amplitude

$$\mathcal{A}_\mu^+ = \int d^4x \left\langle \pi^+(\mathbf{p}) \right| T[H_W(x) J_\mu(0)] \left| K^+(\mathbf{k}) \right\rangle$$

- J_μ is the Electromagnetic current
- H_w is the $s \to d$ effective weak Hamiltonian

$$H_W = \frac{G_f}{\sqrt{2}} V_{us} V_{ud}^* \left[C_1(Q_1^u - Q_1^c) + C_2(Q_2^u - Q_2^c) + ... \right]$$

with 4-quark operators

$$Q_1^q = (\bar{d}\gamma^{L\mu} s)(\bar{q}\gamma_\mu^L q) \qquad Q_2^q = (\bar{d}\gamma^{L\mu} q)(\bar{q}\gamma_\mu^L s)$$

Minkowski Amplitude

- Amplitude definition

$$\mathcal{A}_\mu^+ = \int d^4x \left\langle \pi^+(\boldsymbol{p}) \right| T[H_W(x)J_\mu(0)] \left| K^+(\boldsymbol{k}) \right\rangle$$

- Form factor decomposition

$$\mathcal{A}_\mu^+ = -i\frac{G_F}{(4\pi)^2}V^+(z)\left[q^2(k+p)_\mu - (M_K^2 - M_\pi^2)q_\mu\right]$$

$$z = q^2/M_K^2$$

$$V^+(z) = a^+ + b^+z + V_{\pi\pi}^+(z)$$

- Existing experimental values and theory estimates

$a_{\text{ex}}^+ = -0.575(13)$	$b_{\text{ex}}^+ = -0.722(43)$	[hep-ex 2209.05076]
$a_{\text{th}}^+ = -1.59(8)$	$b_{\text{th}}^+ = -0.82(6)$	[hep-ph 1906.03046]

- a^+ is discrepant $\rightarrow$ Need a first principles theory determination

Euclidean Correlators

- In a finite Euclidean space-time have access to 4-point function

$$\Gamma^{(4)}_{\mu}(t_\pi, t_H, t_K) = \int d^3\mathbf{x}\, \langle \phi_\pi(t_\pi, \mathbf{p})\, H_W(t_H, \mathbf{x}) J_\mu(0)\, \phi^\dagger_K(t_K, \mathbf{k})\rangle$$
$$= Z_{K\pi}(t_\pi, t_K)\, \hat{\Gamma}^{(4)}_{\mu}(t_H)$$

  with meson interpolators ϕ_π and ϕ_K
- Integrate amputated 4-point function within windows $t_H \in [-T_a, 0]$ and $[0, T_b]$

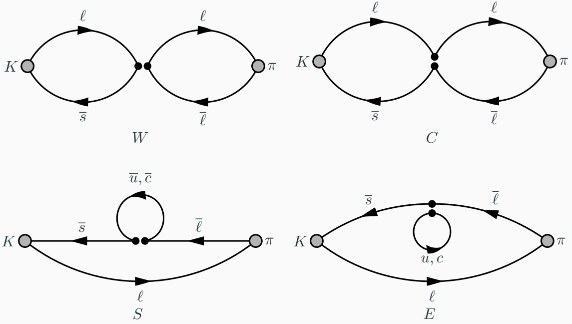
$$I^{\rho}_{\mu}(T_a) = -i \int_{-T_a}^{0} dt_H\, \hat{\Gamma}^{(4)}_{\mu}(t_H)$$
$$I^{\sigma}_{\mu}(T_b) = -i \int_{0}^{T_b} dt_H\, \hat{\Gamma}^{(4)}_{\mu}(t_H)$$

- Amplitude related to these integrated functions

$$\mathcal{A}_\mu \sim \lim_{T_a, T_b \to \infty} \left(I^{\rho}_{\mu}(T_a) + I^{\sigma}_{\mu}(T_b)\right)$$

  (see [hep-lat 1507.03094] for details hidden in $\sim$)

- To compute these correlators need Wick contraction topologies:



- Referred to as the Non-Loop (top) and Loop (bottom) type diagrams

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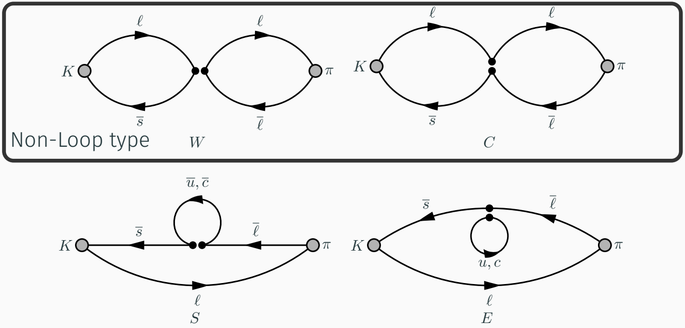

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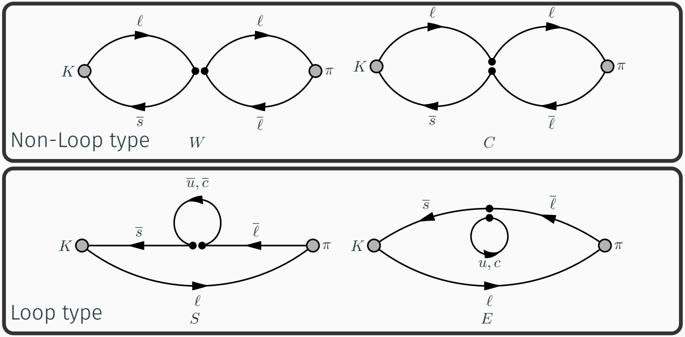

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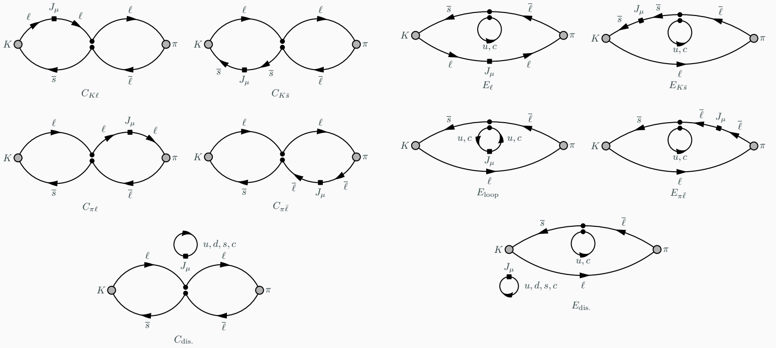
- 4-point function requires a current insertion on each leg



etc

- Currently neglect disconnected diagrams. Can also be computed using the split-even approach with techniques in talk by R. Hill [14:55 Fri]

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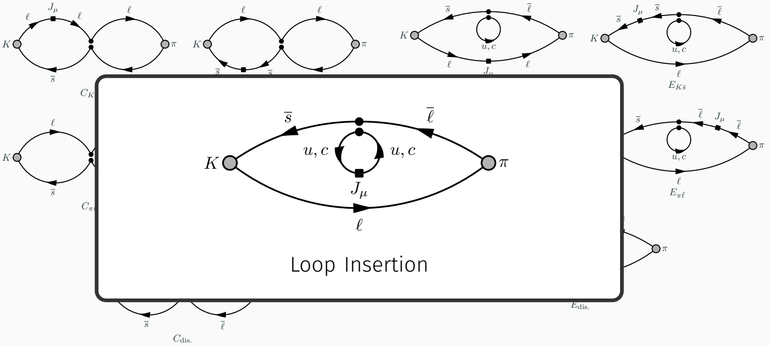

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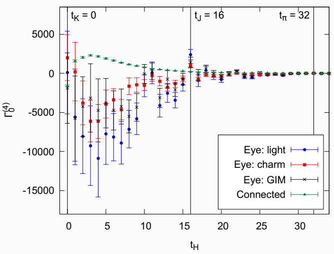
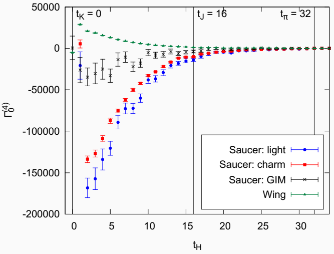
- Lattice result at physical point with DWF [RH hep-lat 2202.08795]
- Contributions to the 4pt function



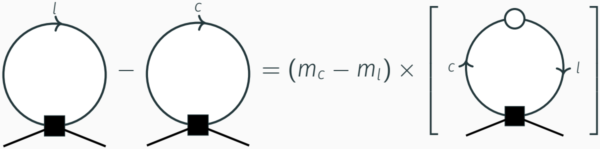


- Limited by the noise on the loop diagrams
- Result $V_{\text{lat}}^{+}(0.013(2)) = -0.87(4.44)$
- To be compared with experimental and theory results $V_{\text{exp}}^{+}(0) = a_{\text{exp}}^{+} = -0.578(16)$ and $a_{\text{th}}^{+} = -1.59(8)$
- Need new technique to tackle the loop noise

- Wilson fermion propagator difference:

$$D^{-1} - D'^{-1} = (m' - m)D^{-1}D'^{-1}$$

- Also holds true for Domain wall fermions [hep-lat 2301.03995]
- Apply to GIM subtraction of loop propagators in rare kaon decay

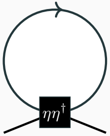


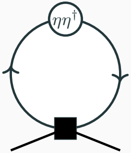
$$- \quad = (m_c - m_l) \times$$

where ■ = H_w , ○ = scalar

Split Even Loop Estimation

- Standard loop estimator puts stochastic noise sources (η) at the operator





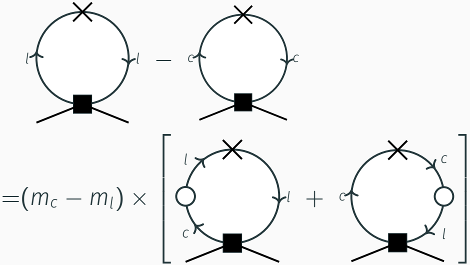
Split-even estimator puts noises at the scalar insertion [hep-lat 1903.10447]

- Can split the difference into multiple smaller steps with intermediate mass quarks: frequency splitting

$$l - c = l - c_1 + c_1 - c_2 + \ldots + c_N - c$$

- Frequency splitting allows more effort to be put into noisier parts of the spectrum
- Split-even estimator and frequency-splitting also important for unqueching QCD+QED R. Hill [14:55 Fri]

- Split-even loop estimator can be directly applied to L diagrams
- The LI diagram does not directly contain a difference of propagators
- Instead this can be written



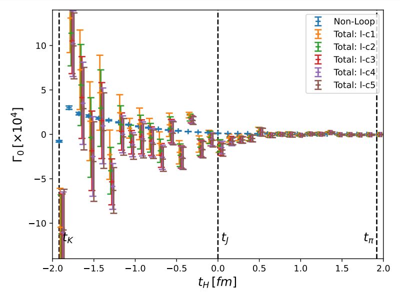
where × = J_μ

Ensemble details

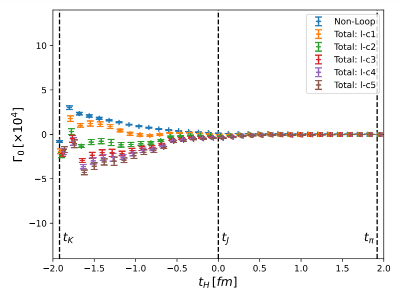
- RBC-UKQCD DWF ensemble $48^3 \times 96$, $m_\pi = 140\,\mathrm{MeV}$, $a^{-1} = 1730\,\mathrm{MeV}$
- Preliminary study using 10 configurations and 6 time translations
- Using inexact solves for the loop props. Still needs AMA correction
- 5 charm masses used:
  $am_1 = am_s = 0.0358$, $am_2 = 0.15$, $am_3 = 0.25$ $am_4 = 0.30$, $am_5 = 0.35$
- needed for frequency-splitting and extrapolation to physical charm mass $am_{\mathrm{phys}} \simeq 0.51$

4-point function

std-diff

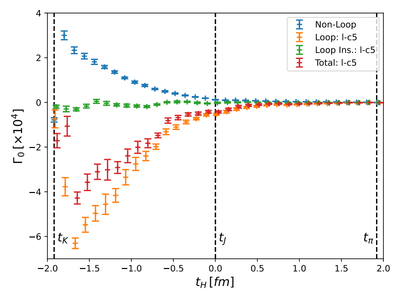


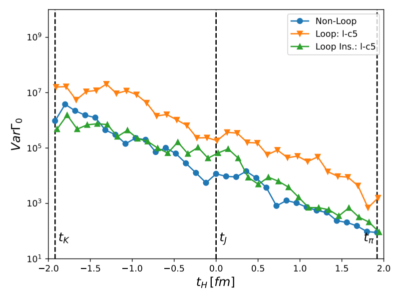
Split-even



- Massive error reduction ($5 - 25\times$) simply from the split-even estimator
- What is contributing the remaining variance?

4-point function breakdown



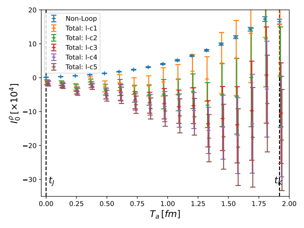


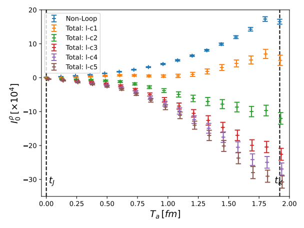
- Loop-Ins. diagrams only a small contribution
- They are also the most expensive
  $\rightarrow$ focus further efforts to the Loop diagrams

Integrated 4-point function

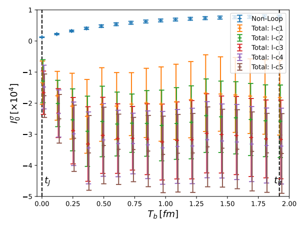
Std-diff | Split-even

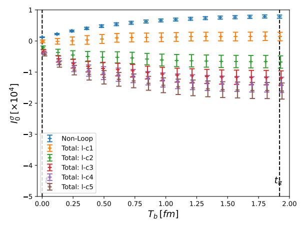
$\int_{-T_a}^{0} dt_H \, \Gamma_\mu$



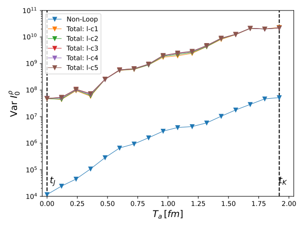
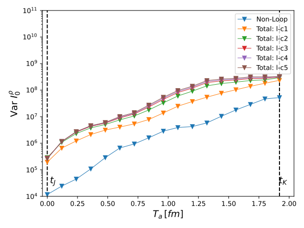
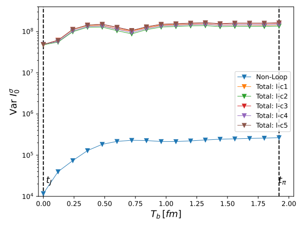
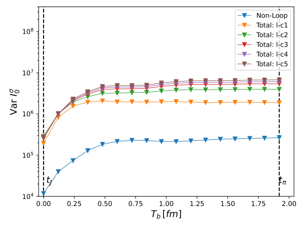


$\int_{0}^{T_b} dt_H \, \Gamma_\mu$





- These are the key quantities for future analysis of amplitude

	Std-diff	Split-even
$\int_{-T_a}^{0} dt_H \, \Gamma_\mu$		
$\int_{0}^{T_b} dt_H \, \Gamma_\mu$		

- Much of the variance also coming from the lighter mass differences
  → use more noises in this region (+ low mode averaging?)

Conclusions/Outlook

Conclusions

- Split-even estimator provides huge gains in Loop and Loop-Insertion diagrams
- LI diagram is only a small contribution and also the most expensive. We can reduce effort spent here
- Largest remaining contribution to the noise from the light part of the spectrum

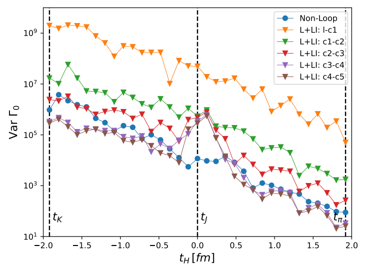
Outlook

- Investigation of LMA in the light mass region
- Include disconnected e.m. current loop
- Start full run using split-even estimator and frequency-splitting to significantly improve the $K^+ \to \pi^+\ell^+\ell^-$ at the physical point

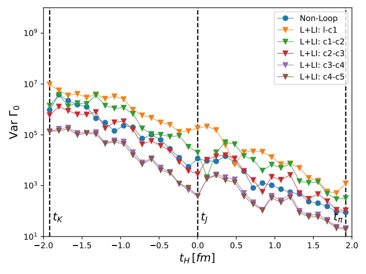
Backup

- Frequency split variance of Loop + Loop Ins. contributions

Std-diff

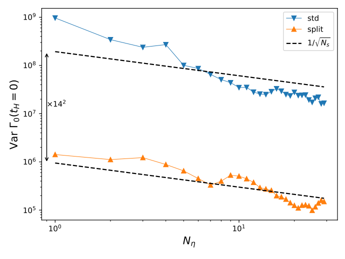


Split-even

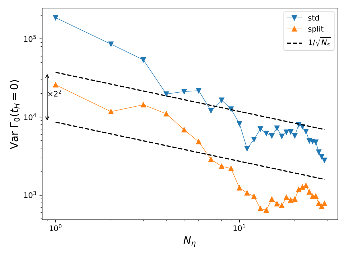


- Std-diff and split-even variance scaling for Loop diagrams

$l - c_1$

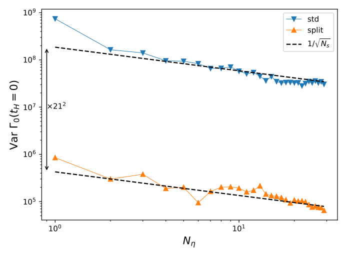


$c_4 - c_5$



- Std-diff and split-even variance scaling for Loop-Ins. diagram

$l - c_1$



$c_4 - c_5$