The Road to Computing the Hadronic Tensor from Euclidean Correlators

Lattice 2024 Parallel Talk

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Summary

- Observables of Interest
- The Inverse Problem: Spectral Reconstruction with Hansen-Lupo-Tantalo
- Results: R-Ratio from Domain Wall
- Intricacies of Staggered Spectral Reconstruction
- Conclusions

Observables of Interest

Cross Section for Neutrino-Nucleon Interaction

- $d\sigma \propto L^{\mu\nu} \boxed{W_{\mu\nu}}$

$$W_{\mu\nu} \propto \int d^4x\, e^{iq\cdot x} \langle H, p | J_\mu(x) J_\nu(0) | H, p \rangle.$$

- Lattice observable: Euclidean hadronic tensor $W_{\mu\nu}^{\text{Euc.}}$

$$W_{\mu\nu}^{\text{Euc.}}(\tau) = \int d\omega\, e^{-\omega\tau} W_{\mu\nu}(\omega)$$

Work so far is with $|H\rangle = |\pi\rangle$ and electromagnetic currents $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d$.

R-Ratio

- $R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$
- Lattice observable: vector two-point function $V(\tau)$

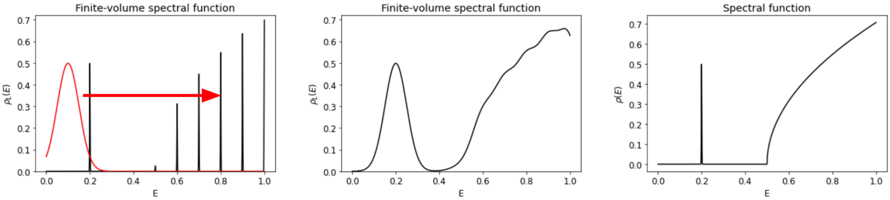
$$V(\tau) = \frac{1}{3} \sum_{i=1,2,3} \langle J_i(\tau) J_i(0) \rangle = \frac{1}{12\pi^2} \int_0^\infty d\omega e^{-\omega\tau} \omega^2 R(\omega^2)$$

The “Problem” with the Inverse Problem

- Generically our correlator is of the form

$$C(\tau) = \int_0^\infty d\omega \; e^{-\omega\tau} \rho_L(\omega). \tag{1}$$

- Correlators are computed on a finite lattice
- The Inverse Laplace transform amounts to a problem in analytic continuation
- Smeared quantities are expected to converge to their infinite-volume counterparts after an ordered limit: first to infinite volume and then zero smearing width



$$\rho(\omega) = \lim_{\sigma \to 0} \lim_{L \to \infty} \rho_L^\sigma(\omega)$$

The Hansen-Lupo-Tantalo Method [arXiv:1903.06476]

- Modification of the Backus-Gilbert Method
- Constructs an output smearing function that approximates an input target smearing function
  1. Provide an *a priori* smearing function $\Delta_\sigma^{\text{in}}(\omega)$ (e.g. a Gaussian)
  2. Using basis functions $b(\omega, \tau)$ and coefficients $\mathbf{g} = (g_0, ..., g_{\tau_{\text{max}}})$, assume the form of an output smearing function $\Delta_\sigma^{\text{out}}(\omega) = \sum_{\tau=1}^{\tau_{\text{max}}} g_\tau b(\omega, \tau)$
  3. Compute coefficients g by minimizing the functional $W[\mathbf{g}]$

$$W[\mathbf{g}] = \frac{A[\mathbf{g}]}{A[\mathbf{0}]} + \lambda B[\mathbf{g}] \tag{2}$$

$$A[\mathbf{g}] = \int_{\omega_0}^{\infty} d\omega e^{\alpha\omega} \left| \Delta_\sigma^{\text{in}}(\omega - \omega') - \Delta_\sigma^{\text{out}}(\omega') \right|^2 \tag{3}$$

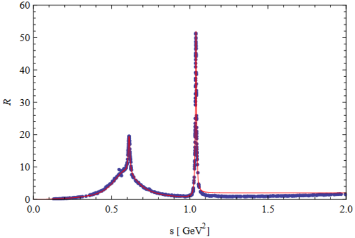
$$B[\mathbf{g}] = B_{\text{norm}} \sum_{\tau_1, \tau_2 = 1}^{\tau_{\text{max}}} g_{\tau_1} g_{\tau_2} \text{Cov}(\tau_1, \tau_2), \quad B_{\text{norm}} = \frac{1}{C(a)^2} \tag{4}$$

The Hansen-Lupo-Tantalo Method

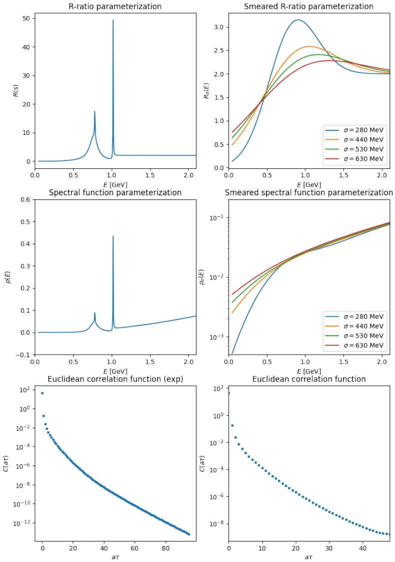
- Basis functions and τ_{max}:
  - Infinite temporal lattice $b(\omega, \tau) = e^{-\omega\tau}$, $\tau_{\text{max}} = T - 1$
  - Finite temporal lattice $b(\omega, \tau) = e^{-\omega\tau} + e^{-\omega(T-\tau)}$, $\tau_{\text{max}} = T/2$
- The smeared finite-volume spectral function is a linear combination of the input corrleator $C(\tau)$

$$\begin{aligned}
\rho_L^\sigma(\omega') &= \int_0^\infty d\omega\ \Delta_\sigma(\omega, \omega')\rho_L(\omega) \\
&= \int_0^\infty d\omega \left(\sum_{\tau=1}^{\tau=\tau_{\text{max}}} g_\tau(\omega') b(\omega, \tau) \right) \rho_L(\omega) \\
&= \sum_{\tau=1}^{\tau=\tau_{\text{max}}} g_\tau(\omega') \int_0^\infty d\omega\ b(\omega, \tau)\rho_L(\omega) \\
&= \sum_{\tau=1}^{\tau=\tau_{\text{max}}} g_\tau(\omega') * C(\tau)
\end{aligned}$$

R-Ratio Parameterization (Bernecker & Meyer)



- arXiv:1107.4388 [hep-lat]
- Comparison point for lattice data
  - $R(\omega^2) \to \rho(\omega) = \frac{12\pi^2}{\omega^2} R(\omega^2)$
  - Exact Gaussian smearing taken at different smearing widths
  - Euclidean correlator constructed from parameterization and run through HLT used as a comparison of lattice data



Ensemble Information

- Domain Wall (RBC/UKQCD):

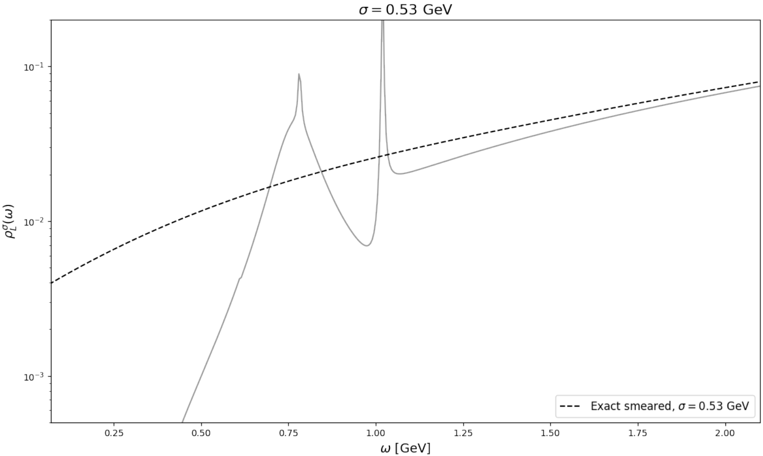
$\approx a$ [fm]	$N_s^3 \times N_t$	$\approx M_\pi$ [MeV]	N_{configs}	L_s
0.114	$48^3 \times 96$	139	112	24

- Staggered (MILC):

$\approx a$ [fm]	$N_s^3 \times N_t$	$\approx M_{\pi,\text{P}}$ [MeV]	N_{configs}	N_{low}	N_{high}
0.12	$48^3 \times 64$	135	98	4000	192

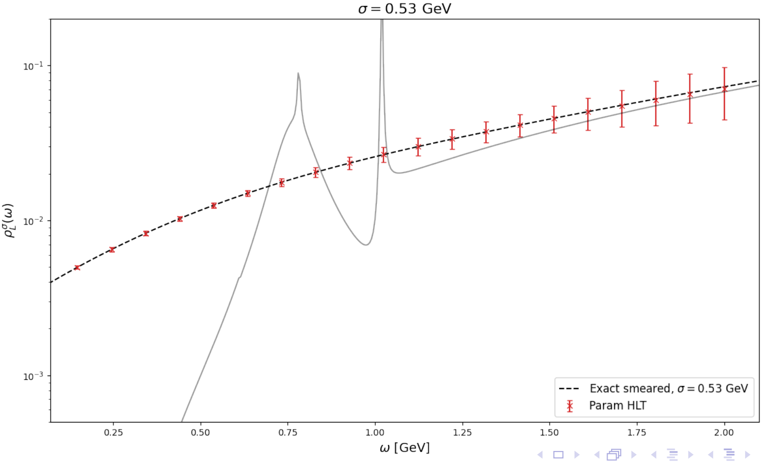
Domain Wall Results

- Black-dashed corresponds to a Gaussian smearing of the spectral density of the parameterization: $\rho_L^\sigma(\omega^*) = \int_{\omega_0}^{\infty} d\omega \frac{12\pi^2}{\omega^2} \Delta_\sigma^{\mathsf{G}}(\omega, \omega^*) R(\omega^2)$



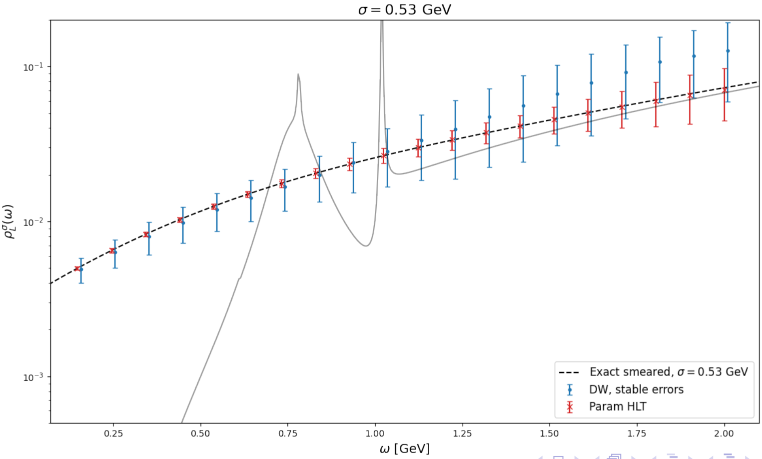
Domain Wall Results

- Red corresponds to creating a finite-temporal Euclidean correlator from the parameterization at the same times as the DW data with 0.1% uncorrelated errors



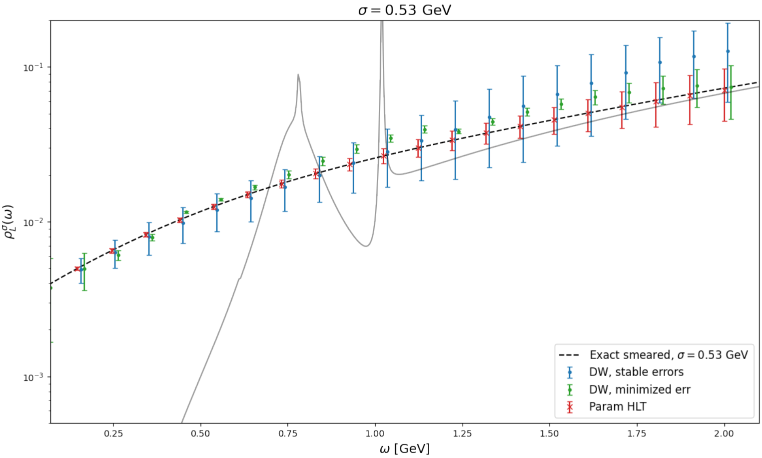
Domain Wall Results

- Blue corresponds to the spectral reconstruction from the DW lattice using the stability analysis of Alexandrou *et al.* [PhysRevD.107.074506] $(\Delta_{\text{tot}} = \sqrt{\Delta^2_{\text{stat}} + \Delta^2_{\text{syst}}})$

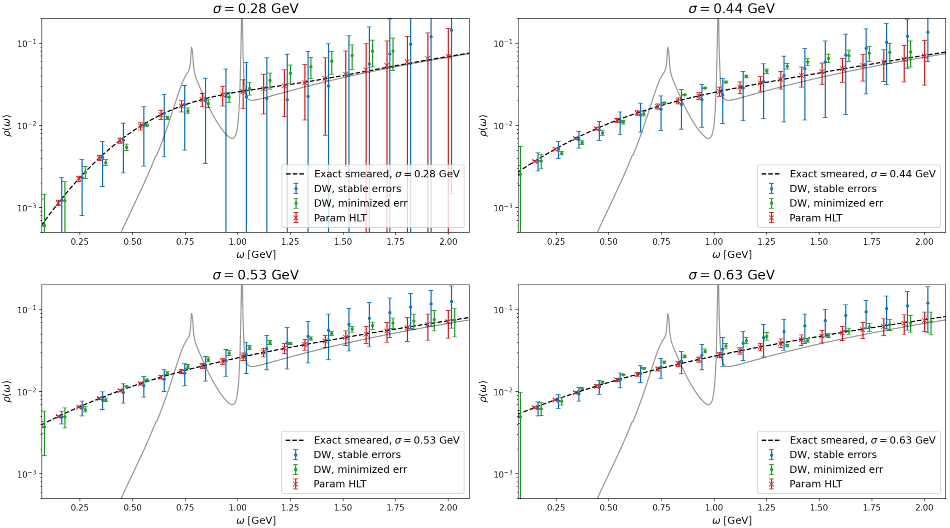


Domain Wall Results

- Green corresponds to the spectral reconstruction from the DW lattice, now with λ tuned upwards from stability to reduce the total error



Domain Wall Results



Staggered Spectrum

- Staggered operators couple to both positive and negative parity states such that the correlator form is

$$C(\tau) = \sum_{n=0} (-1)^{n\tau} \frac{\left|\langle\Omega|\mathcal{O}|n\rangle\right|^2}{2E_n} \left(e^{-E_n\tau} + e^{-E_n(N_\tau - \tau)}\right)$$

- This gives different spectral decompositions on the even and odd timeslices, and the definite parity spectral densities are linear combinations of these

$$\rho_+ = \frac{1}{2}(\rho_{\text{odd}} + \rho_{\text{even}})$$

$$\rho_- = \frac{1}{2}(\rho_{\text{odd}} - \rho_{\text{even}})$$

- Tastes also complicate interpretation of spectrum without continuum limit

Avenues for Improving Staggered Spectral Reconstruction with HLT

- Oscillating state subtraction (This talk)

$$C(\tau) \to C^{\text{non-osc.}}(\tau) = C(\tau) - C^{\text{osc.}}(\tau)$$

- Correlator interpolation (future work)

$$C(\tau) \to C^{\text{interp.}}(\tau) = \frac{1}{2}\Big(C^{\text{even interp.}}(\tau) + C^{\text{odd interp.}}(\tau)\Big)$$

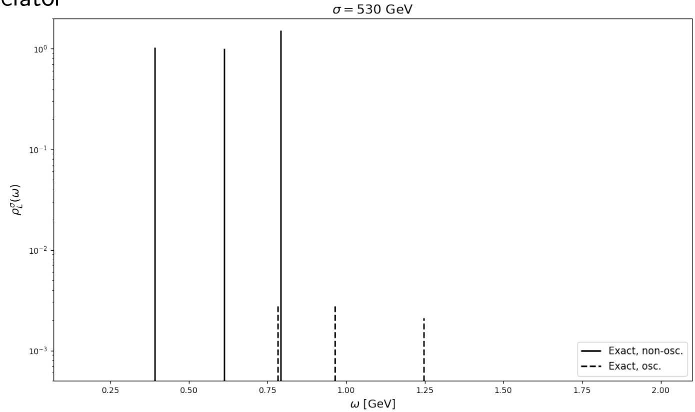
- Modified $b(\omega, \tau)$ (future work)

$$b(\omega, \tau) = e^{-\omega\tau} + e^{-\omega(T-\tau)}$$

$$\to \left(e^{-\omega\tau} + e^{-\omega(T-\tau)}\right) + (-1)^{\tau}\left(e^{-\omega\tau} + e^{-\omega(T-\tau)}\right)$$

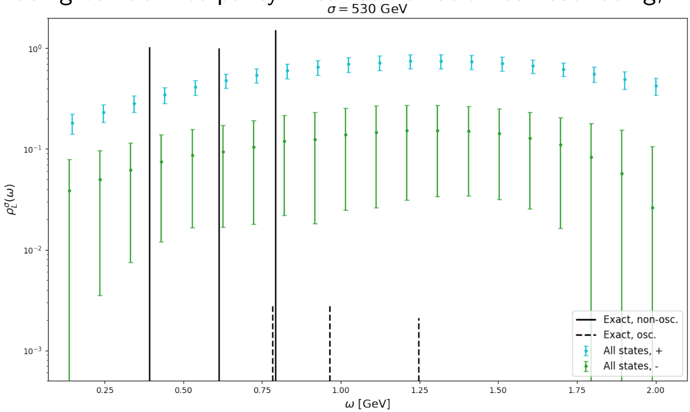
Oscillating State Subtraction: Staggered Toy Model

- Using Lepage's `corrfitter` python package, pull out the three lowest oscillating and non-oscillating states from our staggered vector two-point correlator

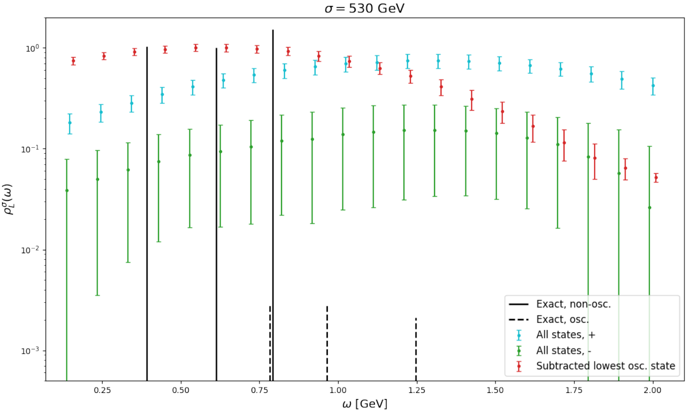


Oscillating State Subtraction: Staggered Toy Model

- Even using our definite-parity linear combination our oscillating,



Oscillating State Subtraction: Staggered Toy Model



What to Take Away

- Spectral reconstructions offer promising results for obtaining Minkowski observables from Euclidean correlators.
- More work is needed to fully realize the application of HLT to staggered fermions.
- Future Work
  - Further exploration of staggered spectral reconstruction analysis
  - Continue work towards spectral reconstruction of the hadronic tensor
  - Refine analysis pipeline to include continuum, infinite volume, and zero-smearing limits

Special thanks

- Alessandro de Santis
- USQCD compute resources at Fermilab

Extra Slides

Lattice Definitions

$$
\begin{aligned}
C^{\text{4-pt}}_{\mu\nu}(q,\tau,t_f-t_i) &= \sum_{x_f,x_i}\sum_{x_1,x_2} e^{-iq\cdot(x_2-x_1)}\langle\Omega|\mathcal{O}_\pi(x_f)J_\mu(x_2)J_\nu(x_1)\mathcal{O}^\dagger_\pi(x_i)|\Omega\rangle \\
&= \frac{|\langle\Omega|\mathcal{O}_\pi|\pi\rangle|^2}{(2M_\pi)^2}e^{-M_\pi(t_f-t_i)}\langle\pi|J_\mu(\tau,-q)J_\nu(0,q)|\pi\rangle + \ldots \\
C^{\text{2-pt}}(t_f-t_i) &= \sum_{x_f,x_i}\langle\Omega|\mathcal{O}_\pi(x_f)\mathcal{O}^\dagger_\pi(x_i)|\Omega\rangle = \frac{|\langle\Omega|\mathcal{O}_\pi|\pi\rangle|^2}{2M_\pi}e^{-M_\pi(t_f-t_i)} + \ldots
\end{aligned}
$$

R-Ratio Parameterization (Bernecker & Meyer)

$$R(s) = \theta(\sqrt{s} - 2m_{\pi^\pm})\theta(4.4m_{\pi^\pm})$$
$$\times \frac{1}{4}\left[1 - \frac{4m_{\pi^\pm}^2}{s}\right]^{3/2}(0.6473 + f_0(\sqrt{s}))$$
$$+ \theta(\sqrt{s} - 4.4m_{\pi^\pm})\theta(M_3 - \sqrt{s})\left(\sum_{i=1}^{2} f_i(\sqrt{s})\right)$$
$$+ f_3(\sqrt{s}) + 3\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right)\theta(\sqrt{s} - M_3)$$

$$f_i(\sqrt{s}) = \frac{C_i\Gamma_i^2}{4(\sqrt{s} - M_i)^2 + \Gamma_i^2}$$

	C_i	M_i/GeV	Γ_i/GeV
0	655.5	0.7819	0.0358
1	8.5	0.7650	0.130
2	11.5	0.7820	0.00829
3	50.0	1.0195	0.00426