LATTICE 2024 LIVERPOOL

Lattice 2024

Generalized boost transformations in finite volumes and application to Hamiltonian methods

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Outline

- Motivation for moving frame
- QC in the moving-frame finite volume
- Three-momentum transformation
- How to apply in HEFT
- The test in S-wave of $\pi\pi$ scattering
- Summary

Motivation for moving frame

T matrix
(Phase Shifts, inelasticity)

Lüscher's Method

M. Luscher, NPB 354, 531 (1991).

Lattice Spectrum

Partial Wave Analysis

Lattice QCD

Experimental Observable
(Differential Cross Sections)

QCD theory
Non-perturbative

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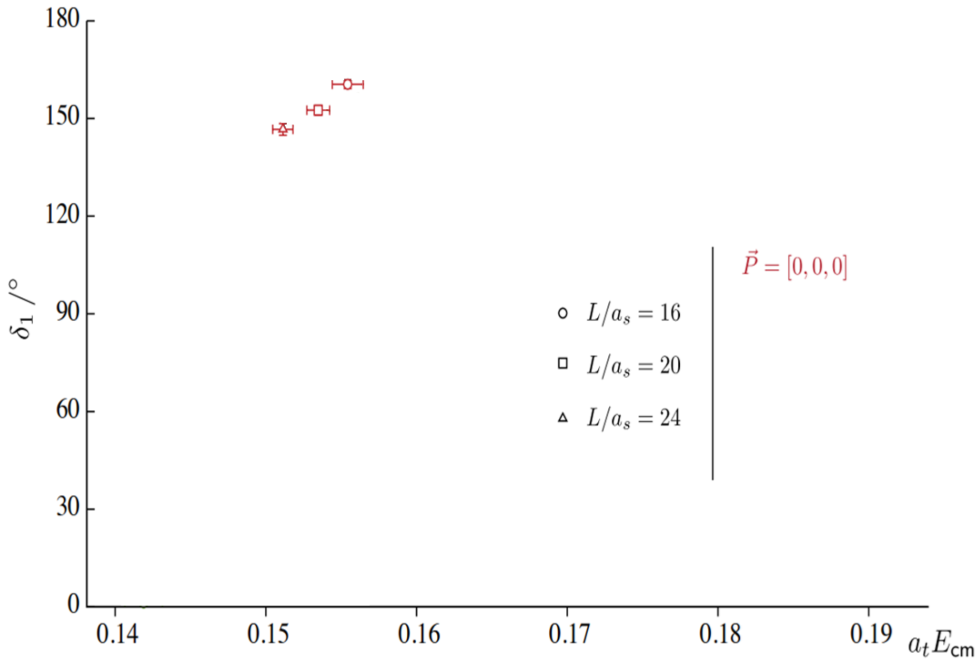

Hadron Spectrum Collaboration
PRD 87 (2013) 3, 034505

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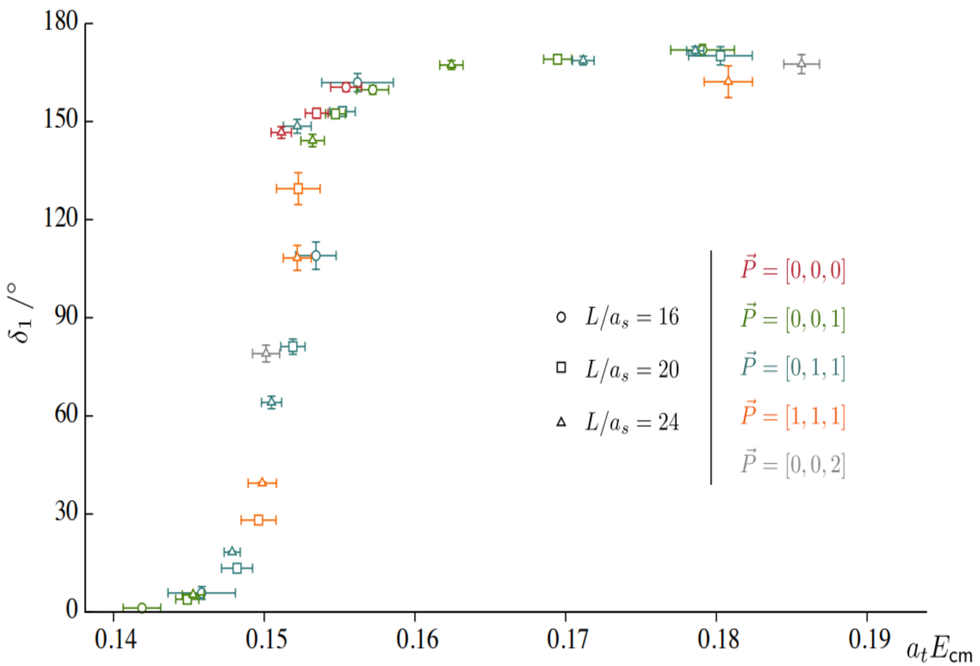

(1) More lattice spectra.

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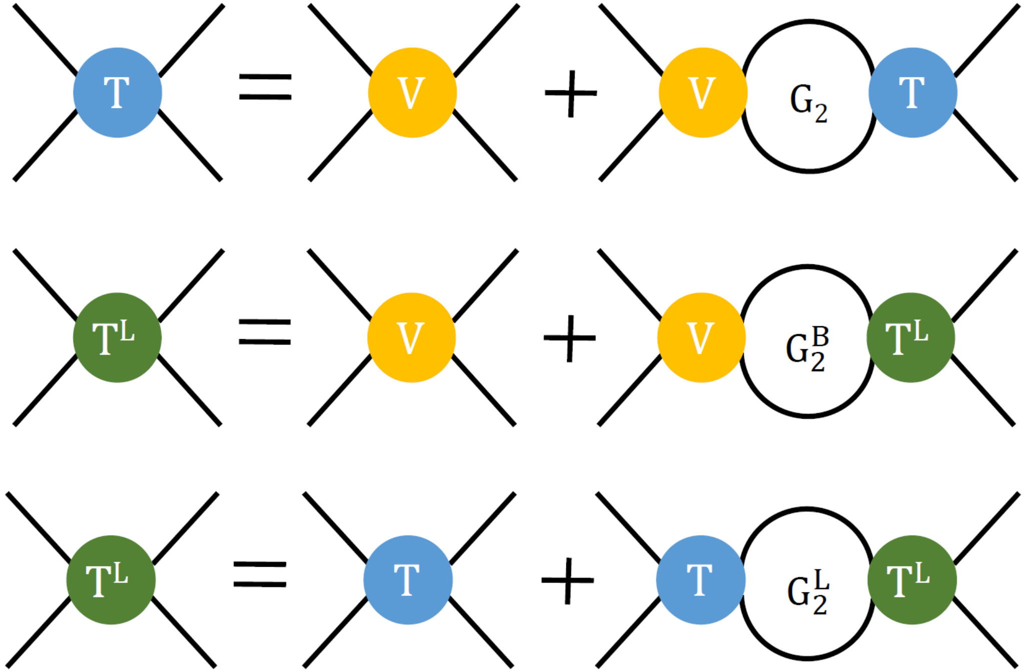
Motivation

In a multibody system, each subsystem should possess momentum. Therefore, the formalism in the finite volume of a moving system is crucial.

For example, 3-body system in the rest frame, any 2-body should have the momentum.

(1) More lattice spectra.
(2) Subsystem of multibody system

OC in the finite volume of rest frame



(A) (B) (C)

$$T = V + V G_2 T,$$

$$T^L = V + V G_2^B T^L\,,$$

$$T^L = T + T G_2^L T^L, \quad G_2^L \equiv G_2^B - G_2.$$

Lüscher, Commun.Math. Phys. 105, 153 (1986).
Kim, Sachrajda and Sharpe
NPB 727 218 (2005)
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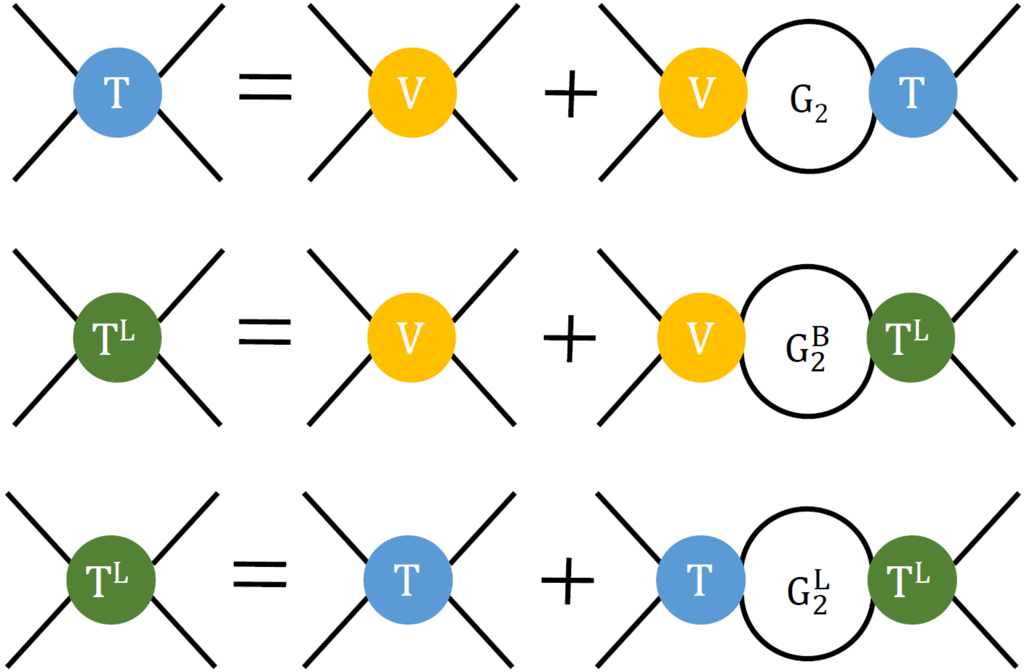

(A) $$T = V + VG_2T,$$

(B) $$T^L = V + VG_2^BT^L\,,$$

(C) $$T^L = T + TG_2^LT^L, \quad G_2^L \equiv G_2^B - G_2.$$

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$$T^L(\mathbf{p}_f^*,\mathbf{p}_i^*;E^*) = T(\mathbf{p}_f^*,\mathbf{p}_i^*;E^*) + i\left(\frac{1}{L^3}\sum_{\mathbf{k}^*} - \int\frac{d^3k^*}{(2\pi)^3}\right)\frac{T(\mathbf{p}_f^*,\mathbf{k}^*;E^*)}{4\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}\frac{T^L(\mathbf{k}^*,\mathbf{p}_i^*;E^*)}{E^* - \omega_1(\mathbf{k}^*) - \omega_2(\mathbf{k}^*) + i\epsilon}$$

After PW Quantization Condition

$$[T^L(E^*)] = \left([T(E^*)]^{-1} - [F(E^*)]\right)^{-1}$$

$$[F(E^*)]_{lm,l'm'} = \left(\frac{1}{L^3}\sum_{\mathbf{k}^*} - \int\frac{d^3k}{(2\pi)^3}\right)\frac{i}{4\omega_1(\mathbf{k}^*)\,\omega_2(\mathbf{k}^*)}\frac{Y_{lm}(\hat{k}^*)Y_{l'm'}^*(\hat{k}^*)\left(\frac{|\mathbf{k}^*|}{q}\right)^{l+l'}}{E^* - (\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k})^*) + i\varepsilon}$$

QC in the finite volume of moving frame

$$\int \frac{d^3k^*}{(2\pi)^3} \to \int \frac{d^3k^r}{(2\pi)^3}\mathcal{J}^r \to \frac{1}{L^3}\sum_{\mathbf{k}^r}\mathcal{J}^r$$

$$\left(\frac{1}{L^3}\sum_{\mathbf{k}^*} - \int \frac{d^3k^*}{(2\pi)^3}\right) \to \left(\frac{1}{L^3}\sum_{\mathbf{k}^r} - \int \frac{d^3k^r}{(2\pi)^3}\right)\mathcal{J}^r$$

$$T^{r,L}(\mathbf{p}_f^*, \mathbf{p}_i^*; E^*) = T(\mathbf{p}_f^*, \mathbf{p}_i^*; E^*) + i\left(\frac{1}{L^3}\sum_{\mathbf{k}^r} - \int \frac{d^3k^r}{(2\pi)^3}\right)\mathcal{J}^r \frac{T(\mathbf{p}_f^*, \mathbf{k}^*; E^*)}{4\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)} \frac{T^{r,L}(\mathbf{k}^*, \mathbf{p}_i^*; E^*)}{E^* - \omega_1(\mathbf{k}^*) - \omega_2(\mathbf{k}^*) + i\epsilon}$$

Quantization Condition

$$[T^{r,L}(E^*;\mathbf{P})] = \left([T(E^*)]^{-1} - [F(E^*;\mathbf{P})]\right)^{-1}$$

q is the on-shell three-momentum of E^*

$$\det\left([\cot\delta(q)] + [M(q;\mathbf{P})]\right) = 0\,,$$

$$[M(q;\mathbf{P})]_{lm,l'm'} = \frac{16\pi^2}{q}\left(\frac{1}{L^3}\sum_{\mathbf{k}} - \mathcal{P}\int \frac{d^3k^r}{(2\pi)^3}\right)\mathcal{J}^r \frac{Y_{lm}(\hat{\mathbf{k}}^*)Y^*_{l'm'}(\hat{\mathbf{k}}^*)\left(\frac{|\mathbf{k}^*|}{q}\right)^{l+l'}}{q^2 - k^{*\,2}}$$

Three-momentum transformation

$$\int \frac{d^3k^*}{(2\pi)^3} \to \int \frac{d^3k^r}{(2\pi)^3}\mathcal{J}^r \to \frac{1}{L^3}\sum_{\mathbf{k}^r}\mathcal{J}^r$$

The relation $\vec{k}^r$ & $\vec{k}^*$???

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Moving Frame $(E, \vec{P})$

$\vec{k}_1 = \vec{k}^r$

$\vec{k}_2 = \vec{P} - \vec{k}^r$

Failed to fix the relation

Because of off-shell

Rest Frame $(E^*, \vec{0})$

$\vec{k}_1 = \vec{k}^*$

$\vec{k}_2 = -\vec{k}^*$

$$\mathbf{k}^r = (k^r_{\parallel}, \mathbf{k}^r_{\perp}) = (\gamma\,\beta\,b^* + \gamma\,k^*_{\parallel}, \mathbf{k}^*_{\perp}) \equiv \mathcal{A}\,\mathbf{k}^*_{\parallel} + \mathcal{B}\,\mathbf{P} + \mathbf{k}^*_{\perp}\,,$$

$$\beta = \frac{|\mathbf{P}|}{\sqrt{a^{*\,2} + \mathbf{P}^2}}\,, \qquad \mathcal{A} = \gamma = \frac{\sqrt{a^{*\,2} + \mathbf{P}^2}}{a^*}\,, \qquad \mathcal{B} = \frac{b^*}{a^*}\,.$$

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$$\left(\frac{1}{L^3}\sum_{\mathbf{k}^r} - \int\frac{d^3k^r}{(2\pi)^3}\right)$$ ➡ Singularity term + O(e^{-mL})

two particles are both on-shell,
$a^* = E^*$ and $b^* = \omega_1$

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$$a^* = E^*(q) \;\text{ or }\; \omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)\,,$$
$$b^* = \omega_1(q) \;\text{ or }\; \omega_1(\mathbf{k}^*)\,.$$

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Three typical transformation formalisms

Kim, Sachrajda and Sharpe NPB 727 218 (2005) $r = \mathbf{KSS}$

$$a^* = E^*(q)\,, \quad b^* = \omega_1(\mathbf{k}^*)\,, \quad \mathcal{J}^r = \frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k}^r)}$$

$$\mathbf{k}^r = \frac{E(q)}{E^*(q)}\mathbf{k}_\parallel^* + \frac{\omega_1(\mathbf{k}^*)}{E^*(q)}\mathbf{P} + \mathbf{k}_\perp^*\,,$$

$$\mathbf{k}^* = \frac{E(q)}{E^*(q)}\mathbf{k}_\parallel^r - \frac{\omega_1(\mathbf{k}^r)}{E^*(q)}\mathbf{P} + \mathbf{k}_\perp^r$$

The first particle is always on-shell, while second one is not.

$$\bar{M}_{00}^{\mathbf{KSS}}(q,\,\mathbf{P}) = \frac{4\pi}{q}\left[\frac{1}{L^3}\sum_{\mathbf{k}=\frac{2\pi}{L}\mathbf{n},\,\mathbf{n}\in\mathbb{Z}^3}\frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k})}\frac{e^{\alpha(q^2-\mathbf{k}^{*\,2})}}{q^2-\mathbf{k}^{*\,2}} - \mathcal{P}\int\frac{d^3k^*}{(2\pi)^3}\frac{e^{\alpha(q^2-\mathbf{k}^{*\,2})}}{q^2-\mathbf{k}^{*\,2}}\right]$$
$$-\frac{1}{\pi q L}\sum_{\mathbf{n}\in\mathbb{Z}^3,\mathbf{n}\neq 0}\int_0^\alpha dt\, e^{tq^2}\int dk^*\, e^{-tk^{*\,2}}$$
$$\times\cos\left[L\frac{\omega_1(\mathbf{k}^*)}{E^*(q)}\mathbf{n}\cdot\mathbf{P}\right]\frac{2k^*\sin\left[L\,D_{\mathbf{KSS}}\,k^*\right]}{D_{\mathbf{KSS}}}\,,$$

$$D_{\mathbf{KSS}} = \sqrt{\mathbf{n}^2 + \left(\frac{\mathbf{n}\cdot\mathbf{P}}{E^*(q)}\right)^2}\,.$$

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Rummukainen and Gottlieb NPB 450 397 (1997) $r = \mathbf{RG}$

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$$a^* = E^*(q)\,, \quad b^* = \frac{E^*(q)}{2} + \frac{m_1^2 - m_2^2}{2E^*(q)} = \omega_1(q)\,, \; \mathcal{J}^r = \frac{E^*(q)}{E(q)}$$

$$\mathbf{k}^r = \frac{E(q)}{E^*(q)}\mathbf{k}^*_\parallel + \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{E^{*\,2}(q)}\right)\mathbf{P} + \mathbf{k}^*_\perp\,,$$

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$$\times\int dk^*\,e^{-tk^{*\,2}}\frac{2k^*\sin\left[L\,D_{\mathbf{RG}}\,k^*\right]}{D_{\mathbf{RG}}}\,,$$
$$D_{\mathbf{RG}} = D_{\mathbf{KSS}} = \sqrt{\mathbf{n}^2 + \left(\frac{\mathbf{n}\cdot\mathbf{P}}{E^*(q)}\right)^2}\,.$$

The arrangement of energies follows two particles are both off-shell.

$$\bar{M}_{lm}^{\mathbf{RG}}(q,\,\mathbf{P}) = -\frac{1}{\pi qL}\frac{E^*(q)}{E(q)}\sqrt{4\pi}\mathcal{Z}_{lm}^{\mathbf{\Delta}}(1;\left(\frac{Lq}{2\pi}\right)^2)$$

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$$\bar{M}^{\mathbf{KSS}}_{00}(q,\,\mathbf{P}) = \frac{4\pi}{q}\left[\frac{1}{L^3}\sum_{\mathbf{k}=\frac{2\pi}{L}\mathbf{n},\,\mathbf{n}\in\mathbb{Z}^3}\frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k})}\frac{e^{\alpha(q^2-\mathbf{k}^{*2})}}{q^2-\mathbf{k}^{*2}} - \mathcal{P}\int\frac{d^3k^*}{(2\pi)^3}\frac{e^{\alpha(q^2-\mathbf{k}^{*2})}}{q^2-\mathbf{k}^{*2}}\right]$$
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Analytical Proof: They are the same!

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$$\bar{M}^{\mathbf{RG}}_{00}(q,\,\mathbf{P}) = \frac{4\pi}{q}\left[\frac{1}{L^3}\sum_{\mathbf{k}=\frac{2\pi}{L}\mathbf{n},\,\mathbf{n}\in\mathbb{Z}^3}\frac{E^*(q)}{E(q)}\frac{e^{\alpha(q^2-\mathbf{k}^{*2})}}{q^2-\mathbf{k}^{*2}} - \mathcal{P}\int\frac{d^3k^*}{(2\pi)^3}\frac{e^{\alpha(q^2-\mathbf{k}^{*2})}}{q^2-\mathbf{k}^{*2}}\right]$$
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$$-\frac{1}{\pi qL}\sum_{\mathbf{n}\in\mathbb{Z}^3,\mathbf{n}\neq 0}\cos\left[\frac{L\,\mathbf{n}\cdot\mathbf{P}}{2}\left(1+\frac{m_1^2-m_2^2}{E^{*\,2}(q)}\right)\right]\int_0^\alpha dt\, e^{tq^2}$$
$$\times\int dk^*\, e^{-tk^{*2}}\frac{2k^*\sin\left[L\,D_{\mathbf{RG}}\,k^*\right]}{D_{\mathbf{RG}}}\,,$$
$$D_{\mathbf{RG}} = D_{\mathbf{KSS}} = \sqrt{\mathbf{n}^2 + \left(\frac{\mathbf{n}\cdot\mathbf{P}}{E^*(q)}\right)^2}\,.$$

The arrangement of energies follows two particles are both off-shell.

$$\bar{M}^{\mathbf{RG}}_{lm}(q,\,\mathbf{P}) = -\frac{1}{\pi qL}\frac{E^*(q)}{E(q)}\sqrt{4\pi}\mathcal{Z}^{\mathbf{\Delta}}_{lm}(1;\left(\frac{Lq}{2\pi}\right)^2)$$

?????

$$Det[H(E) - EI] = 0$$

Three typical transformation formalisms

$r = \mathbf{LWLY}$

$$a^* = \omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*) \text{ and } b^* = \omega_1(\mathbf{k}^*)$$

$$\mathbf{k}^r = \frac{\sqrt{(\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*))^2 + \mathbf{P}^2}}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}\mathbf{k}^*_{\parallel} + \frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}\mathbf{P} + \mathbf{k}^*_{\perp}$$

$$\mathbf{k}^* = \frac{\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)}{\sqrt{(\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r))^2 - \mathbf{P}^2}}\mathbf{k}^r_{\parallel} - \frac{\omega_1(\mathbf{k}^r)}{\sqrt{(\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r))^2 - \mathbf{P}^2}}\mathbf{P} + \mathbf{k}^r_{\perp},$$

$$\mathcal{J}^r = \left|\frac{\partial \mathbf{k}^*}{\partial \mathbf{k}^r}\right| = \frac{\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)} \frac{\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)}{\omega_1(\mathbf{k}^r)\omega_2(\mathbf{P} - \mathbf{k}^r)}$$

New one, it has some benefits!

a^*, b^* are independent on E^*!
The boosted potential is still energy independent.
The eigenvectors form a complete orthonormal basis of the Hilbert space of the Hamiltonian.

Both particles are on-shell.

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For three-body, it avoids the negative energy or the velocity being larger than the light speed, Keep $E > |\vec{P}|$.

Blanton and Sharpe PRD 102, 054520 (2020)

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Blanton and Sharpe PRD 102, 054520 (2020)

Weak point, we caution the breaking of relativistic invariance of the three-particle divergence-free K matrix identified.

Blanton and Sharpe, PRD 103, 054503 (2021)

Introduction of HEFT

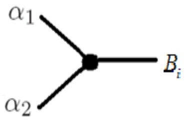
J. M. M. Hall etc. PRD 87(2013), 094510
J.-j. Wu etc. PRC90 (2014), 055206
Y. Li etc. PRD 101(2020), 114501
PRD 103(2021), 094518

$$H = H_0 + H_I$$

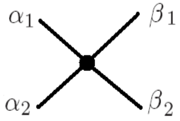
$|B_i\rangle$ bare state, bare mass m_i, $|\alpha(k_\alpha)\rangle$ non-interaction channels

$$H_0 = \sum_{i=1,n} |B_i\rangle m_i \langle B_i| + \sum_{\alpha} |\alpha(k_\alpha)\rangle \left[\sqrt{m_{\alpha 1}^2 + k_\alpha^2} + \sqrt{m_{\alpha 2}^2 + k_\alpha^2} \right] \langle \alpha(k_\alpha)|$$

$$H_I = \hat{g} + \hat{v} \qquad \hat{g} = \sum_{\alpha} \sum_{i=1,n} \left[|\alpha(k_\alpha)\rangle g_{i,\alpha}^{+} \langle B_i| + |B_i\rangle g_{i,\alpha} \langle \alpha(k_\alpha)| \right]$$



$$\hat{v} = \sum_{\alpha,\beta} |\alpha(k_\alpha)\rangle v_{\alpha,\beta} \langle \beta(k_\beta)|$$



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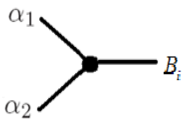

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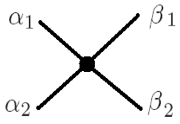

Continuum → Discrete

$$\int d\vec{k} \longrightarrow \sum_{i} \left(2\pi/L\right)^3 \qquad |\alpha(\vec{k}_\alpha)\rangle \longrightarrow \left(2\pi/L\right)^{-3/2} |\vec{k}_i, -\vec{k}_i\rangle_\alpha$$

$$\langle \beta(\vec{k}_\beta) | \alpha(\vec{k}_\alpha)\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta) \longrightarrow {}_\beta\langle \vec{k}_j, -\vec{k}_j | \vec{k}_i, -\vec{k}_i\rangle_\alpha = \delta_{\alpha\beta}\delta_{ij}$$

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$$H_I = \sum_{j} \left(2\pi/L\right)^{3/2} \sum_{\alpha} \sum_{i=1,n} \left[|\vec{k}_j, -\vec{k}_j\rangle_\alpha g_{i,\alpha}^{+} \langle B_i| + |B_i\rangle g_{i,\alpha} \,{}_\alpha\langle \vec{k}_j, -\vec{k}_j| \right] + \sum_{i,j} \left(2\pi/L\right)^3 \sum_{\alpha,\beta} |\vec{k}_i, -\vec{k}_i\rangle_\alpha v_{\alpha,\beta} \,{}_\beta\langle \vec{k}_j, -\vec{k}_j|$$

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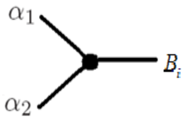

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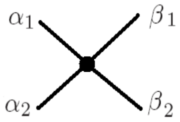

$$[H_0]_{N_c+1} = \begin{pmatrix} m_0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \epsilon_1(k_0) & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \epsilon_2(k_0) & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \epsilon_{n_c}(k_0) & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \epsilon_1(k_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Continuum → Discrete

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$$[H_I]_{N_c+1} = \begin{pmatrix} 0 & g_1^V(k_0) & g_2^V(k_0) & \cdots & g_{n_c}^V(k_0) & g_1^V(k_1) & \cdots \\ g_1^V(k_0) & v_{1,1}^V(k_0,k_0) & v_{1,2}^V(k_0,k_0) & \cdots & v_{1,n_c}^V(k_0,k_0) & v_{1,1}^V(k_0,k_1) & \cdots \\ g_2^V(k_0) & v_{2,1}^V(k_0,k_0) & v_{2,2}^V(k_0,k_0) & \cdots & v_{2,n_c}^V(k_0,k_0) & v_{2,1}^V(k_0,k_1) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ g_{n_c}^V(k_0) & v_{n_c,1}^V(k_0,k_0) & v_{n_c,2}^V(k_0,k_0) & \cdots & v_{n_c,n_c}^V(k_0,k_0) & v_{n_c,1}^V(k_0,k_1) & \cdots \\ g_1^V(k_1) & v_{1,1}^V(k_1,k_0) & v_{1,2}^V(k_1,k_0) & \cdots & v_{1,n_c}^V(k_1,k_0) & v_{1,1}^V(k_1,k_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$H_0 = \sum_{i=1,n} |B_i\rangle m_i \langle B_i| + \sum_{\alpha,i} |\vec{k}_i, -\vec{k}_i\rangle_\alpha \left[\sqrt{m_{\alpha_B}^2 + k_\alpha^2} + \sqrt{m_{\alpha_M}^2 + k_\alpha^2}\right]_\alpha \langle \vec{k}_i, -\vec{k}_i|$$

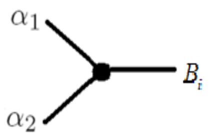
$$H_I = \sum_j \left(\frac{2\pi}{L}\right)^{3/2} \sum_\alpha \sum_{i=1,n} \left[|\vec{k}_j, -\vec{k}_j\rangle_\alpha g_{i,\alpha}^{+} \langle B_i| + |B_i\rangle g_{i,\alpha} {}_\alpha\langle \vec{k}_j, -\vec{k}_j| \right] + \sum_{i,j} \left(\frac{2\pi}{L}\right)^3 \sum_{\alpha,\beta} |\vec{k}_i, -\vec{k}_i\rangle_\alpha v_{\alpha,\beta} {}_\beta\langle \vec{k}_j, -\vec{k}_j|$$

$$(H_0 + H_I)\,|\Psi\rangle = E\,|\Psi\rangle$$

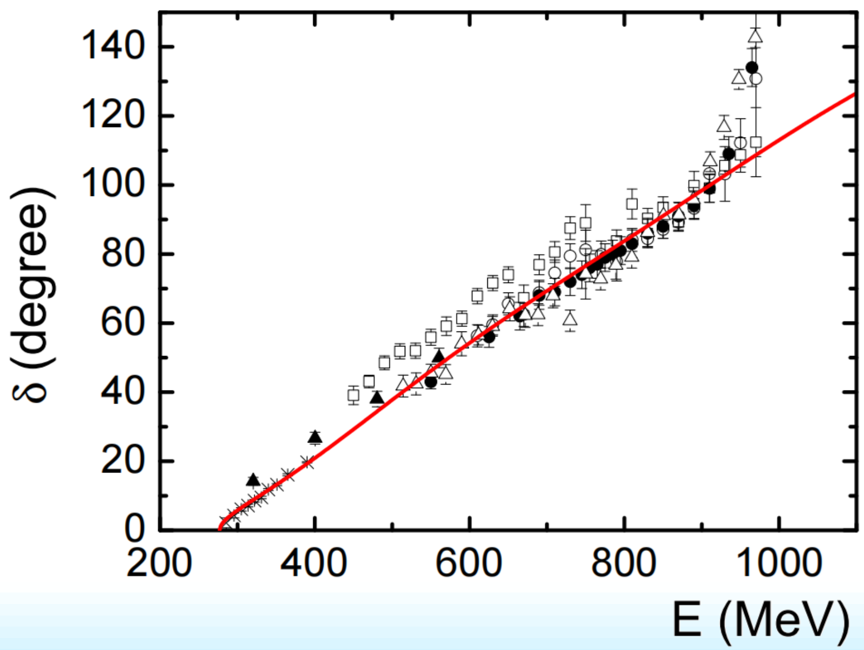
Eigen-Value ⟷ Lattice Spectrum

The test in S-wave of ππ scattering

$$g(|\vec{k}|) = \frac{g_{\pi\pi}}{\sqrt{m_\pi}} \frac{1}{(1+(c_{\pi\pi}\times k)^2)}$$



	l=0
m_B(MeV)	948.96
g_l	0.64698
c_l(fm)	0.43979



In the Finite Volume

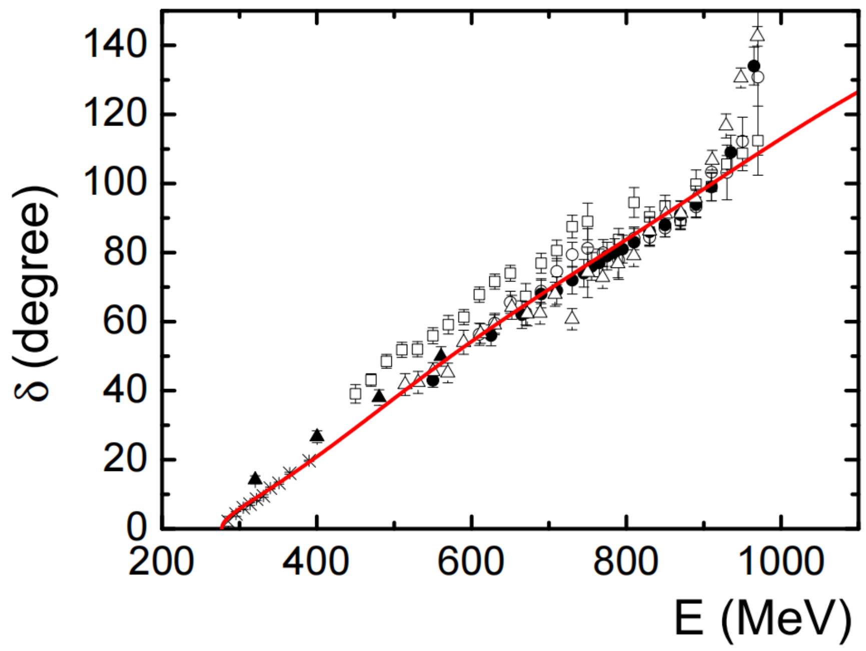
$$[H] = \begin{pmatrix} m_\sigma & g^{fin}(\vec{k}_1,\vec{P}) & g^{fin}(\vec{k}_2,\vec{P}) & \cdots \\ g^{fin}(\vec{k}_1,\vec{P}) & E^*(\vec{k}_1,\vec{P}) & 0 & \cdots \\ g^{fin}(\vec{k}_2,\vec{P}) & 0 & E^*(\vec{k}_2,\vec{P}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

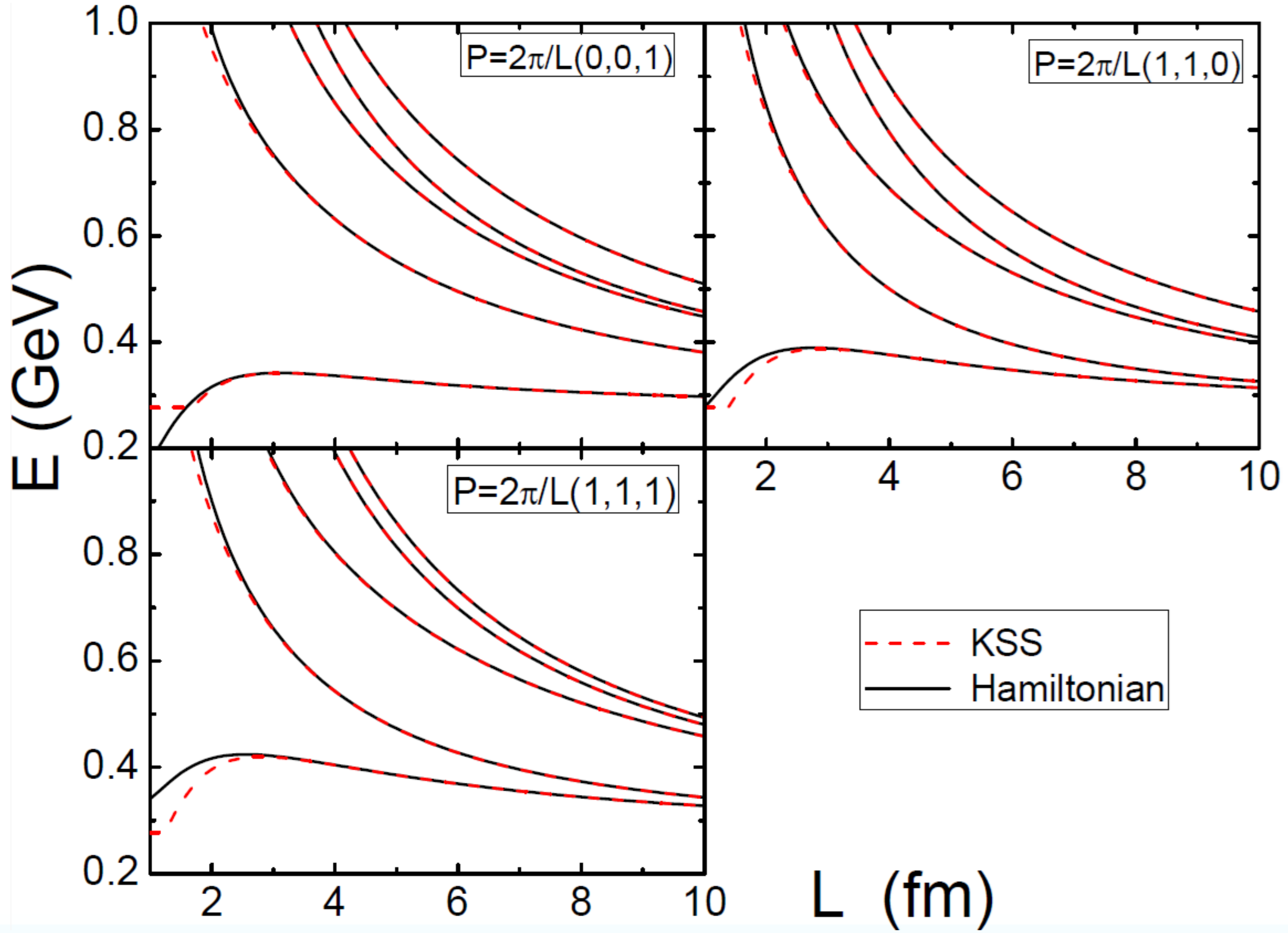
$$g^{fin}(\vec{k}_i,\vec{P}) = \left(\frac{2\pi}{L}\right)^{\frac{3}{2}} \frac{1}{\sqrt{4\pi}} g(|\vec{k}_i^*|) \left(\frac{\omega_{k_i^*}}{2}\frac{\omega_{k_i}+\omega_{Pk_i}}{\omega_{k_i}\omega_{Pk_i}}\right)^{\frac{1}{2}},$$

$$E^*(\vec{k}_i,\vec{P}) = \sqrt{(\omega_{k_i}+\omega_{Pk_i})^2 - \vec{P}},$$

$$Det[H-EI]=0$$

The test in S-wave of ππ scattering





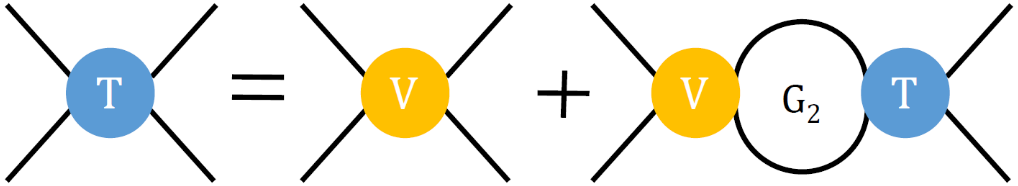
Summary

- **We explore the general formalism of momentum transformation in a finite volume.**

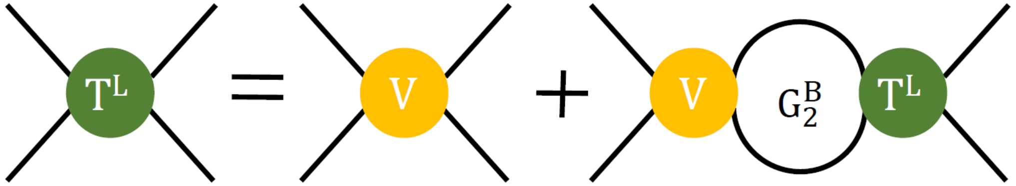
- **We discuss three different transformation methods, two of which have been investigated in previous studies. The third method is a novel approach that offers advantages for the Hamiltonian method and the study of three-body systems. All three methods are consistent within errors of O(e^{-mL}).**

- **Finally, we provide a comparison of the finite volume spectrum between the Hamiltonian and KSS methods based on the same phase shift of $\pi\pi$ scattering for S-wave interactions.**

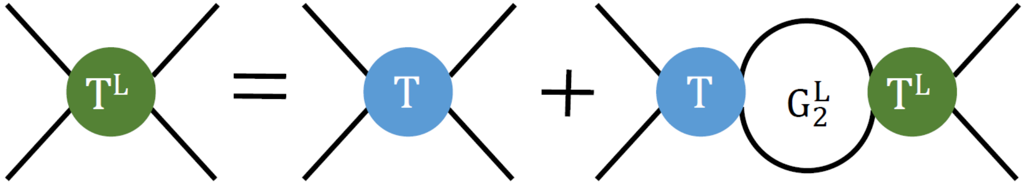
Thanks for attention!



(A)



(B)



(C)

$$T = V + VG_2T,$$

$$T^L = V + VG_2^B T^L ,$$

$$T^L = T + TG_2^L T^L,$$
$$G_2^L \equiv G_2^B - G_2.$$

The detailed derivation

$$T^L = V + V\left(G_2 + G_2^B - G_2\right)T^L = V + V\left(G_2 + G_2^L\right)T^L,$$

$$T - T^L = -(1 - VG_2)^{-1}VG_2^L T^L.$$

QC in the finite volume of rest frame

$$T = V + VG_2T,$$

$$T(p_f^*, p_i^*; P^*) = V(p_f^*, p_i^*; P^*) + \int \frac{d^4k^*}{(2\pi)^4} V(p_f^*, k^*; P^*) G_2(k^*; P^*) T(k^*, p_i^*; P^*)$$

$$T^L = V + VG_2^B T^L,$$

$$G_2^B(k^*, P^*) := \sum_{\mathbf{k}^* = \frac{2\pi\mathbf{n}}{L},\, \mathbf{n}\in\mathcal{Z}^3} \left(\frac{2\pi}{L}\right)^3 \delta^3(\mathbf{k}^* - \mathbf{k}) G_2(k, P^*)$$

$$T^L(p_f^*, p_i^*; P^*) = V(p_f^*, p_i^*; P^*) + \int \frac{d^4k^*}{(2\pi)^4} V(p_f^*, k^*; P^*) G_2^B(k^*, P^*) T^L(k^*, p_i^*; P^*)$$

$$T^L(p_f^*, p_i^*; P^*) = V(p_f^*, p_i^*; P^*) + \int \frac{dk_0^*}{2\pi} \frac{1}{L^3} \sum_{\mathbf{k}^* = \frac{2\pi\mathbf{n}}{L},\, \mathbf{n}\in\mathcal{Z}^3} V(p_f^*, k^*; P^*) G_2(k^*, P^*) T^L(k^*, p_i^*; P^*)$$

$$T^L = T + TG_2^L T^L, \qquad G_2^L \equiv G_2^B - G_2$$

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QC in the finite volume of rest frame

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$$G_2(k; P) = \frac{1}{(k^2 - m_1^2 + i\epsilon)}\frac{1}{((P-k)^2 - m_2^2 + i\epsilon)}$$

$$(1)\ \ k_0^* = -\omega_1(\mathbf{k}^*) + i\epsilon,$$

$$(2)\ \ k_0^* = P_0^* - \omega_2(\mathbf{k}^*) + i\epsilon$$

$$G_2(k^*; P^*) \to \frac{1}{-2\omega_1(\mathbf{P}^* - \mathbf{k}^*)}\frac{1}{P_0^* + \omega_1(\mathbf{k}^*) - \omega_2(\mathbf{k}^*)}\frac{(2\pi)i\,\delta(k_0^* + \omega_1(\mathbf{k}^*))}{P_0^* + \omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}$$
$$+ \frac{1}{2\omega_2(\mathbf{k}^*)}\frac{1}{P_0^* + \omega_1(\mathbf{k}^*) - \omega_2(\mathbf{k}^*)}\frac{(2\pi)i\,\delta(k_0^* - (P_0^* - \omega_2(\mathbf{k}^*)))}{P_0^* - \omega_1(\mathbf{k}^*) - \omega_2(\mathbf{k}^*) + i\epsilon}$$

$$T^L(\mathbf{p}_f^*, \mathbf{p}_i^*; E^*) = T(\mathbf{p}_f^*, \mathbf{p}_i^*; E^*) + i\left(\frac{1}{L^3}\sum_{\mathbf{k}^*} - \int \frac{d^3k^*}{(2\pi)^3}\right)\frac{T(\mathbf{p}_f^*, \mathbf{k}^*; E^*)}{4\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}\frac{T^L(\mathbf{k}^*, \mathbf{p}_i^*; E^*)}{E^* - \omega_1(\mathbf{k}^*) - \omega_2(\mathbf{k}^*) + i\epsilon}$$

Quantization Condition

$$[T^L(q; \mathbf{P})] = \left([T(q)]^{-1} - [F(q)]\right)^{-1}$$

$$[F(E^*)]_{lm,l'm'} = \left(\frac{1}{L^3}\sum_{\mathbf{k}^*} - \int \frac{d^3k}{(2\pi)^3}\right)\frac{i}{4\omega_1(\mathbf{k}^*)\,\omega_2(\mathbf{k}^*)}\frac{Y_{lm}(\hat{k}^*)Y^*_{l'm'}(\hat{k}^*)\left(\frac{|\mathbf{k}^*|}{q}\right)^{l+l'}}{E^* - (\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k})^*) + i\varepsilon}$$