Eigenspectra of Minimally Doubled Fermions

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Motivation and previous work

- By calculating the eigenspectra of the Dirac operator, we want to see how the background gauge field topology is expressed ,i.e., index theorem
- There is a theoretical approach to the index of the staggered fermions but it does not give an integer value from the beginning and requires a renormalization depending upon the ensemble of the gauge fields [Smit and Vink, NPB 286 (1987) 485; NPB 298 (1988) 557]
- In 2010, David Adams showed that, with addition of a flavoured mass term, index of the staggered fermions correctly represents the topological charge up to a factor coming from flavours using the spectral flow of a certain hermitian version of the Dirac operator. [Adams, PRL 104 (2010) 141602]

Motivation and previous work

- There have some studied done afterwards using flavored mass terms for staggered and minimally doubled fermions in 2-dim to study index theorem [Creutz, Kimura, and Misumi, JHEP 12 (2010) 041; Forcrand, Kurkela, Panero, JHEP 04 (2012) 142; Durr and Weber, PRD 105, 114511 (2022)]
- Study done for staggered fermion in 2-dim and 4-dim by Follana et al. [Azcoiti, Follana, Vaquero, and Carlo, PLB 744 (2015) 303–308]
- In this work, we have obtained the index for minimally doubled fermions, viz., Karsten-Wilczek and Borici-Creutz fermions in 4-dim with an SU(3) background gauge field by obtaining their eigenspectra

Minimally doubled fermion

- Examples of MDF are Borici-Creutz (BC) and Karsten-Wilczek (KW) fermions
- They have minimum no. of species - two excitations, respect chiral symmetry.
- BC-dirac operator in d-dimension:

$$D_{BC}(x,y) = \sum_{\mu} \gamma_{\mu} \nabla_{\mu}(x,y) + \frac{i}{2} \sum_{\mu} \gamma'_{\mu} \Box_{\mu}(x,y)$$

$$D_{BC}(p) = \underbrace{\sum_{\mu} i\gamma_{\mu} \sin p_{\mu} - 2i \sum_{\mu} \gamma'_{\mu} \left[\sin(p_{\mu}/2)\right]^2}_{\text{Two zeros : (i) } p_{\mu} = 0, \text{ (ii) } p_{\mu} = \frac{\pi}{2}}$$

$$\text{where } \gamma'_{\mu} = \frac{2}{\sqrt{d}}\Gamma - \gamma_{\mu}, \Gamma = \frac{1}{\sqrt{d}} \sum_{\mu} \gamma_{\mu}$$

Minimally doubled fermion

- KW-dirac operator in d-dimension:

$$D_{KW}(x,y) = \sum_{\mu} \gamma_\mu \nabla_\mu(x,y) - \frac{i}{2}\gamma_d \sum_{\mu=1}^{d-1} \Box_\mu(x,y)$$

$$D_{KW}(p) = \underbrace{\sum_{\mu} i\gamma_\mu \sin p_\mu + 2i\gamma_d \sum_{\mu=1}^{d-1} [\sin(p_\mu/2)]^2}_{\text{Two zeros : (i) } p_\mu = 0, \text{ (ii) } p_\mu = 0, \mu \neq d \,\&\, p_d = \pi}$$

SU(3) gauge field with a topological charge

- In 4-dim with a simple SU(3) gauge field having a non-zero topological charge on lattice is as follows: [Smit and Vink, NPB 286 (1987) 485; Gattringer and Hip, NPB 536 (1999) 363-380]

$$U_1(x) = exp(-i\omega_1 a x_2 \tau)$$

$$U_2(x) = \begin{cases} 1, & \text{for } x_2 = 0, a, ..., (N-2)a \\ exp(i\omega_1 L x_1 \tau), & \text{for } x_2 = (N-1)a \end{cases}$$

$$U_3(x) = exp(-i\omega_2 a x_4 \tau)$$

$$U_4(x) = \begin{cases} 1, & \text{for } x_4 = 0, a, ..., (N-2)a \\ exp(i\omega_2 L x_3 \tau), & \text{for } x_4 = (N-1)a \end{cases}$$

$$\text{where } \; L = Na, \;\; \omega_i a^2 = \frac{2\pi n_i}{L^2}, \;\; Q = 2n_1 n_2, \;\; \tau = \text{Gell-Mann matrix}, \;\; x = (x_1, x_2, x_3, x_4)$$

SU(3) gauge field with a topological charge

- The plaquette elements are constant:

$$U_{\mu\nu}(x) = U_\mu(n)U_\nu(x+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger(x)$$

with

$$U_{12} = exp(-i\omega_1\tau)$$
$$U_{34} = exp(-i\omega_2\tau)$$

and all other plaquettes

$$U_{\mu\nu} = 1$$

- Discritized field strength tensor $F_{\mu\nu}(x)$ upto O(a):

$$F^{plaq}_{\mu\nu}(x) = \frac{1}{2}(U_{\mu\nu}(x) - U^\dagger_{\mu\nu}(x))_{AH} + O(a^2)$$

- Charge calculation:

$$Q(x) = -\sum_x q(x), \text{ where } q(x) = \frac{1}{4\pi^2}\mathrm{Tr}(F_{12}F_{34} - F_{13}F_{24} + F_{23}F_{14})$$

SU(3) gauge field with a topological charge

- Roughening of the smooth gauge fields:

$$U_\mu(x)_{rough} = U_\mu(x)_{(\delta)} U_\mu(x)_{old}$$

where $U_\mu(x)_{old}$ are smooth links previously defined and $U_\mu(x)_{(\delta)}$ are SU(3) elements in the vicinity of 1

$$U_\mu(x)_{(\delta)} = exp(i \sum_j \theta_j \lambda_j)$$

$\theta_j(x)$ is a random number uniformly distributed in $(-\delta\pi, \delta\pi)$

- Next we have used $O(a^2)$-improved field strength tensor to calculate charge of the roughened gauge configs
- For small δ, topological charge still remains closer to the charge of smooth gauge config

Index theorem and Spectral flow

- The index is defined as the difference between the number of zero modes of the massless Dirac operator with positive and negative chirality, n_+ and n_-:

$$\text{index(D)} = n_+ - n_-$$

- Index theorem :

$$\text{index(D)} = (-1)^{d/2} Q, \quad \text{where } Q = \text{topological charge}$$

- We can calculate the zero-mode chiralities, but there is another way called spectral flow. We need a certain hermitian version of the Dirac operator :

$$H(m) = \gamma_5 (D + m)$$

$$\text{Zero-modes of D : eigenvalues of H(m)} = \lambda(m) = \pm m \rightarrow \pm\text{chirality}$$

- $\lambda(m) = \pm m$ will cross the origin with slopes ± 1 depending on $\pm$ chirality as mass varies [Adams, PRL 104 (2010) 141602; Creutz, Kimura, and Misumi, JHEP 12 (2010) 041]

Index theorem and Spectral flow

- For lattice fermions such as MDF, the index cancel between doublers, so we cannot see the eigenvalue flow with just simple mass term.
- We need a flavoured mass term (point-splitting method)
- Hermitian dirac operator for BC- and KW-fermion with different flavoured mass terms:

$$H_{BC}(m) = \gamma_5(D_{BC} + m[(2C_{sym} - 1) \otimes 1])$$

$$H_{KW}(m) = \gamma_5(D_{KW} + m[C_{sym} \otimes 1])$$

$$\text{where } C_{sym} = \frac{1}{d!}\sum_{perm} C_1 C_2 \ldots C_d$$

$$C_\mu(x,y)\psi(y) = \frac{1}{2}[U_\mu\psi(x+\hat{\mu}) + U^\dagger_{\hat{\mu}}(x-\hat{\mu})\psi(x-\hat{\mu})]$$

[Creutz, Kimura, and Misumi, JHEP 12 (2010) 041; Creutz, PoS, LATTICE2010 (2010) 078; Durr and Weber, PRD 105, 114511 (2022)]

Algorithm used to calculate eigenvalues

- We have used Kalkreuter-Simma algorithm to calculate eigenvalues [Kalkreuter and Simma, Computer Physics Communications 93 (1996) 33-47]
- It is a variant of Conjugate-Gradient method
- It can be used for hermitian matrix whose eigenvalues are bounded from below
- It uses CG-method coupled with intermediate diagonalizations to calculate eigenvalues and eigenvectors
- We have implemented this algorithm in publicly available MILC-code

Results

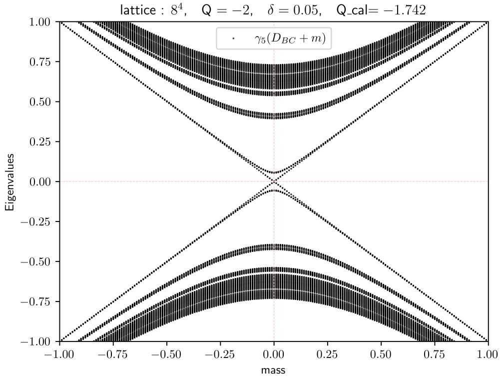

Figure: 1(a) BC-fermion with just mass

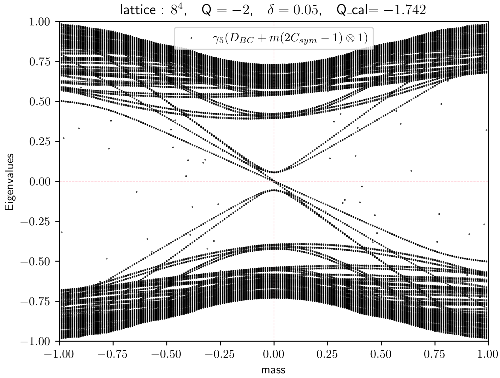

Figure: 1(b) BC-fermion with flavoured mass

- In 1(b) figure, there are two doubled crossings with negative slope, i.e., $n_+ = 0$, $n_- = 2 \times 2$, for $Q = -2$ following $n_+ - n_- = 2(-1)^{d/2} Q$

Results

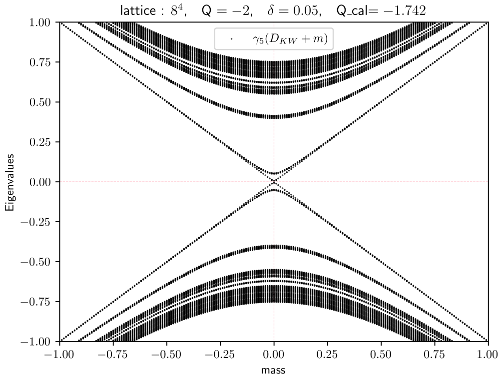

Figure: 2(a) KW-fermion with just mass

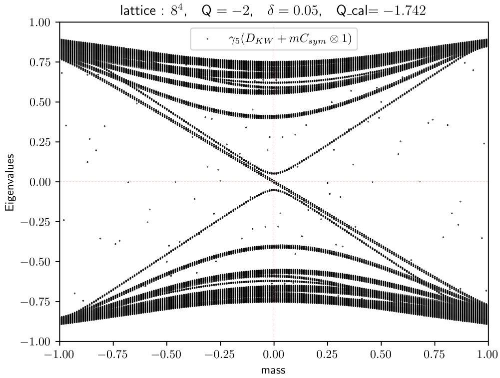

Figure: 2(b) KW-fermion with flavoured mass

- In 2(b) figure, there are two doubled crossings with negative slope, i.e., $n_+ = 0$, $n_- = 2 \times 2$, for $Q = -2$ following $n_+ - n_- = 2(-1)^{d/2}Q$

Conclusion

- We verified the index theorem in 4-dim with a background SU(3) gauge field using eigenspectra of minimally doubled fermions, viz., BC- and KW-fermions

References

- R. G. Edwards, U. M. Heller, R. Narayanan, "Spectral flow, condensate and topology in lattice QCD", Nucl. Phys. B 535 (1998) 403-422
- S. Dürr, J. H. Weber, "Minimally doubled fermions and topology in 2D", Proc. Sci., LATTICE2021 (2021) 556
- S. Dürr, J. H. Weber, "Topological properties of minimally doubled fermions in two spacetime dimensions", Phys. Rev. D 105, (2022), 114511
- V. Azcoiti, G. Di Carlo, E. Follana, and A. Vaquero, "Topological index theorem on the lattice through the spectral flow of staggered fermions", Phys. Lett. B 744, 303 (2015)

Thank You

Back up slides

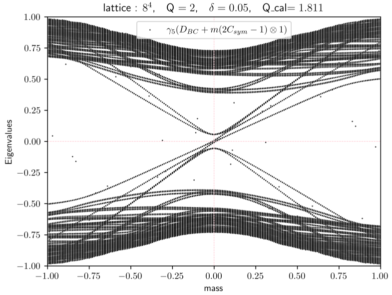

Figure: BC-fermion with flavoured mass

- With positive topological charge
- In this figure, there are two doubled crossings with positive slope, i.e., $n_+ = 2 \times 2$, $n_- = 0$, for $Q = 2$ following $n_+ - n_- = 2(-1)^{d/2}Q$

Back up slides

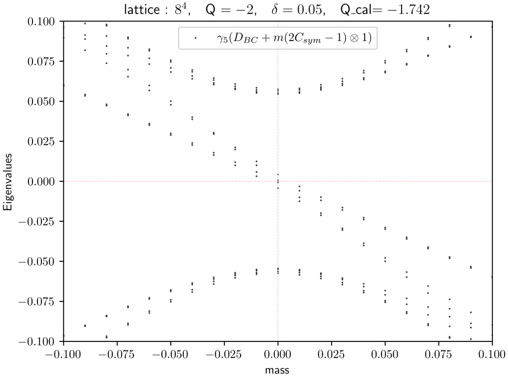

Figure: BC-fermion with flavoured mass

- Zoomed in to see doubled crossings