Formulation of $\mathrm{SU}(N)$ Lattice Gauge Theories with Schwinger Fermions

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Based on 2112.02090, 2312.17734 and 2408.xxxxx

In collaboration with

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Overview

- Goal: study QCD using lattice models with finite dimensional local Hilbert space
- Our approach: *qubit regularization*[1,2,3,4] of traditional lattice models
- This talk: reformulate the theory using Schwinger fermions; reproduce the physics (IR and UV) of 2d $\mathrm{SU}(N)$ QCD

[1] H. Singh and S. Chandrasekharan, 2019, *Phys. Rev. D* arXiv: **1905.13204** (hep-lat)
[2] T. Bhattacharya et al., 2021, *Phys. Rev. Lett.* arXiv: **2012.02153** (hep-lat)
[3] H. Liu and S. Chandrasekharan, 2022, *Symmetry* arXiv: **2112.02090** (hep-lat)
[4] S. Maiti et al., 2024, *PoS* arXiv: **2401.10157** (hep-lat)

Outline

Review of Schwinger boson representation of $\mathrm{SU}(2)$

- $a^{\alpha\dagger} : \alpha = 1,2$ are boson creation operators that transform as an $\mathrm{SU}(2)$ doublet:

$$[a^\alpha, a^\beta] = [a^{\alpha\dagger}, a^{\beta\dagger}] = 0, \quad [a^\alpha, a^{\beta\dagger}] = \delta^{\alpha\beta}.$$

- The generators of $\mathrm{SU}(2)$ are defined as $J^a := \frac{1}{2} a^{\alpha\dagger}\sigma^a_{\alpha\beta} a^\beta \implies [J^a, J^b] = \mathrm{i}\varepsilon^{abc} J^c$
- The number operator $\hat{n} := a^{1\dagger}a^1 + a^{2\dagger}a^2$ distinguishes different irreps of $\mathrm{SU}(2)$ (number of boxes in the Young diagram $\square\square\cdots\square$), and is related to the Casimir operator as

$$\sum_a (J^a)^2 = \frac{\hat{n}}{2}\left(1 + \frac{\hat{n}}{2}\right)$$

- $|0\rangle$ satisfies $a^\alpha|0\rangle = 0$ for $\alpha = 1,2 \implies$ the trivial representation of $\mathrm{SU}(2)$

$$|jm\rangle = |\frac{1}{2}(n_1 + n_2), \frac{1}{2}(n_1 - n_2)\rangle \propto \left(a^{1\dagger}\right)^{n_1}\left(a^{2\dagger}\right)^{n_2}|0\rangle.$$

- Applications in LGT, e.g., loop-string-hadron formulation: $\mathrm{SU}(2)$ [5] and $\mathrm{SU}(3)$ [6,7]

[5] I. Raychowdhury and J. R. Stryker, 2020, *Phys. Rev. D* arXiv: 1912.06133 (hep-lat)
[6] S. V. Kadam et al., 2023, *Phys. Rev. D* arXiv: 2212.04490 (hep-lat)
[7] S. V. Kadam et al., 2024, arXiv: 2407.19181 (hep-lat)

Introducing Schwinger fermion representation of $\mathrm{SU}(N)$

- $c^{\alpha\dagger}:\alpha=1,\cdots,N$ are fermion creation operators that transform in the fundamental representation of $\mathrm{SU}(N)$:

$$\{c^\alpha, c^\beta\} = \{c^{\alpha\dagger}, c^{\beta\dagger}\} = 0, \quad \{c^\alpha, c^{\beta\dagger}\} = \delta^{\alpha\beta}.$$

- The generators of $\mathrm{SU}(N)$ are defined as $Q^a := c^{\alpha\dagger}T^a_{\alpha\beta}c^\beta \implies [Q^a, Q^b] = \mathrm{i}f^{abc}Q^c$, where $[T^a, T^b] = \mathrm{i}f^{abc}T^c$
- The number operator $\hat{k} := \sum_{\alpha=1}^N c^{\alpha\dagger}c^\alpha$ distinguishes different irreps of $\mathrm{SU}(N)$ (number of boxes in the Young diagram $(\square\square\cdots\square)^T$), and is related to the Casimir operator as

$$\sum_a (Q^a)^2 = \frac{N+1}{2N}(N-\hat{k})\hat{k}$$

- $|0\rangle$ satisfies $c^\alpha|0\rangle = 0$ for $\alpha = 1, 2 \implies$ the trivial representation

$$|\alpha_1\cdots\alpha_k\rangle = c^{\alpha_k\dagger}\cdots c^{\alpha_1\dagger}|0\rangle.$$

Hamiltonian LGT and qubit-regularization of the Hilbert space

Kogut-Susskind Hamiltonian

electric field — magnetic field: — fermion hopping

$$H = \frac{g^2}{2}\sum_{\langle i,j\rangle}\left(L_{ij}^{a2} + R_{ij}^{a2}\right) - \frac{1}{4g^2}\sum_{\square}\left(W_\square + W_\square^\dagger\right) + t\sum_{\langle i,j\rangle}\left(c_i^{\alpha\dagger} U_{ij}^{\alpha\beta} c_j^\beta + \mathrm{h.c.}\right)$$

Qubit regularization: same Hamiltonian, but truncates the link Hilbert space:

	Traditional	Qubit regularization
Hilbert space	$L^2(G) = \bigoplus_{\lambda\in\widehat{\mathrm{SU}(N)}} V_\lambda \otimes V_\lambda^*$ (Peter-Weyl theorem)	$\mathcal{H}_Q := \bigoplus_{\lambda\in Q} V_\lambda \otimes V_\lambda^*$ (Symmetry is preserved)
Irreps	$\widehat{\mathrm{SU}(N)}$: Young diagrams with at most $N-1$ rows	$Q = \{\circ, \square, \begin{array}{c}\square\\\square\end{array}, \begin{array}{c}\square\\\square\\\square\end{array}, \cdots, \overline{\begin{array}{c}\square\\\square\end{array}}, \overline{\square}\}$

Reasons for Q-scheme (anti-symmetric)

$$Q = \left\{\circ,\ \square,\ \begin{array}{c}\square\\[-4pt]\square\end{array},\ \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array},\ \cdots,\ \overline{\begin{array}{c}\square\\[-4pt]\square\end{array}},\ \overline{\square}\right\}$$

- Contains all N-ality: string tensions at large distance are dictated by N-ality (screening)
- Smallest quadratic Casimir among each N-ality: minimize $\frac{g^2}{2}\left(L^{a2} + R^{a2}\right)$
- Anti-symmetric representations: formulation using Schwinger fermions

Schwinger fermion representation of the link variables

- Link algebra:

$$[L_i^a, L_j^b] = \mathrm{i} f^{abc} L_i^c \delta_{ij}, \quad [R_i^a, R_j^b] = \mathrm{i} f^{abc} R_i^c \delta_{ij}, \quad [L_i^a, R_j^b] = 0,$$
$$[L_i^a, U_{jk}^{\alpha\beta}] = -T_{\alpha\gamma}^a U_{jk}^{\gamma\beta} \delta_{ij}, \quad [R_i^a, U_{jk}^{\alpha\beta}] = U_{jk}^{\alpha\gamma} T_{\gamma\beta}^a \delta_{ik},$$

- Each link is made of two Schwinger fermions $l^{\alpha\dagger}, r^{\alpha\dagger} : \alpha = 1, \cdots, N$

$$\{l^\alpha, l^{\beta\dagger}\} = \delta^{\alpha\beta}, \quad \{l^\alpha, l^\beta\} = \{l^{\alpha\dagger}, l^{\beta\dagger}\} = 0,$$
$$\{r^\alpha, r^{\beta\dagger}\} = \delta^{\alpha\beta}, \quad \{r^\alpha, r^\beta\} = \{r^{\alpha\dagger}, r^{\beta\dagger}\} = 0,$$
$$\{l^\alpha, r^{\beta\dagger}\} = \{l^{\alpha\dagger}, r^\beta\} = \{l^\alpha, r^\beta\} = \{l^{\alpha\dagger}, r^{\beta\dagger}\} = 0,$$

- Schwinger fermion representation of the link variables:

$$L^a = l^{\alpha\dagger} T_{\alpha\beta}^a l^\beta, \quad R^a = r^{\alpha\dagger} T_{\alpha\beta}^a r^\beta, \quad U^{\alpha\beta} = l^\alpha \frac{1}{\sqrt{\hat{k}_l \hat{k}_r}} r^{\beta\dagger}$$

- Peter-Weyl theorem, or conservation of electric flux $\implies k_l + k_r = N$.
- $|k_l = N; k_r = 0\rangle \equiv |k_l = 0; k_r = N\rangle$.

Schwinger fermion representation of the KS Hamiltonian

Kogut-Susskind Hamiltonian

electric field — magnetic field: — fermion hopping

$$H = \frac{g^2}{2}\sum_{\langle i,j\rangle}\frac{N+1}{N}(N-\hat{k})\hat{k} - \frac{1}{4g^2}\sum_{\Box}(W_\Box + W_\Box^\dagger) + t\sum_{\langle i,j\rangle}(c_i^{\alpha\dagger} l_i^\alpha)\frac{1}{\sqrt{\hat{k}_{l_i}\hat{k}_{r_j}}}(r_j^{\beta\dagger} c_j^\beta) + \text{h.c.}$$

where

$$W_\Box = l_1^{\alpha_1}\frac{1}{\sqrt{\hat{k}_{l_1}\hat{k}_{r_2}}} r_2^{\alpha_2\dagger} l_2^{\alpha_2}\frac{1}{\sqrt{\hat{k}_{l_2}\hat{k}_{r_3}}} r_3^{\alpha_3\dagger} l_3^{\alpha_3}\frac{1}{\sqrt{\hat{k}_{l_3}\hat{k}_{r_4}}} r_4^{\alpha_4\dagger} l_4^{\alpha_4}\frac{1}{\sqrt{\hat{k}_{l_4}\hat{k}_{r_1}}} r_1^{\alpha_1\dagger}$$

Advantages:

- manifestly gauge invariant on each site. Gauss law: conservation of quark number plus Schwinger fermion number $\mod N$.
- gauge invariant operators on each site are bosonic

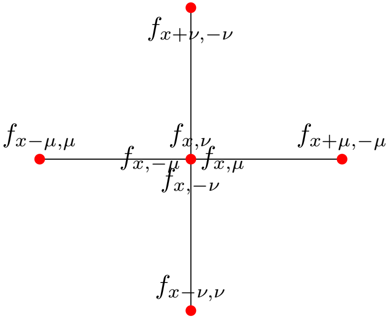
Gauge invariant operators

$$C^\dagger_{x,\mu} := c^{\alpha\dagger}_x f^{\alpha}_{x,\mu}, \quad F_{x,\mu\nu} := f^{\alpha\dagger}_{x,\mu} f^{\alpha}_{x,\nu}.$$

$F^\dagger_{x,\mu\nu} = F_{x,\nu\mu}$ and $F_{x,\mu\mu} = F^\dagger_{x,\mu\mu} = \hat{k}_{x,\mu}$ by definition.
These operators satisfies

$$\begin{aligned}
[C_\mu, C^\dagger_\nu] &= F_{\mu\nu} - \hat{n}\delta_{\mu\nu}. \\
[F_{\mu\nu}, F_{\rho\sigma}] &= F_{\mu\sigma}\delta_{\nu\rho} - F_{\rho\nu}\delta_{\mu\sigma} \\
[F_{\mu\nu}, C_\rho] &= C_\mu \delta_{\nu\rho}, \\
[F_{\mu\nu}, C^\dagger_\rho] &= -C^\dagger_\nu \delta_{\mu\rho},
\end{aligned}$$

which is the algebra of $\mathrm{U}(2d + n_f)$. ($\mathfrak{l} = F \oplus \hat{n}$ and $\mathfrak{p} = C$ form a Cartan decomposition.)



Where is N?

Since the Hamiltonian can be written solely in terms of gauge invariant operators C_μ and $F_{\mu\nu}$ that form $\mathrm{U}(2d + N_f)$ algebra, where is the information of the original gauge group $\mathrm{SU}(N)$?

Where is N?

Since the Hamiltonian can be written solely in terms of gauge invariant operators C_μ and $F_{\mu\nu}$ that form $\mathrm{U}(2d+N_f)$ algebra, where is the information of the original gauge group $\mathrm{SU}(N)$?

- Answer: It is hidden in the representation!

$$(C_\mu^\dagger)^{N+1} = 0, \ (F_{\mu\nu}^\dagger)^{N+1} = 0 \quad \text{when } \mu \neq \nu.$$

A trivial example:

$$C_\mu = |\mu\rangle\langle 0|, \quad C_\mu^\dagger = |0\rangle\langle\mu|, \quad F_{\mu\nu} = |\mu\rangle\langle\nu|,$$

Form a representation of the $\mathrm{U}(2d+1)$ algebra in the case of $N=1$.

Continuum theory and lattice theory

Full continuum theory

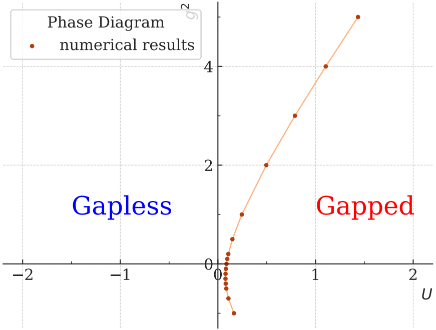
$$\mathcal{L} = \frac{1}{2\tilde{g}^2} \operatorname{tr} F^2 + \bar{\psi}^\alpha \mathrm{i} \not{D} \psi^\alpha + \sum_a \lambda^a J_L^a J_R^a$$

Full lattice theory

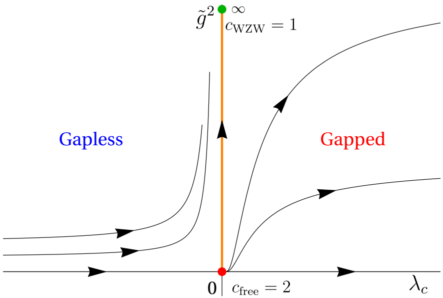
generalized Hubbard coupling

$$H = \frac{g^2}{2} \sum_{\langle i,j \rangle} \left(L_{ij}^{a2} + R_{ij}^{a2} \right) + t \sum_{\langle i,j \rangle} \left(c_i^{\alpha\dagger} U_{ij}^{\alpha\beta} c_j^{\beta} + \text{h.c.} \right) + U \sum_i n_i (N - n_i)$$

Phase Diagram (2312.17734)

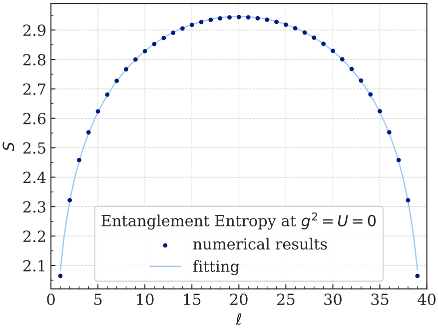


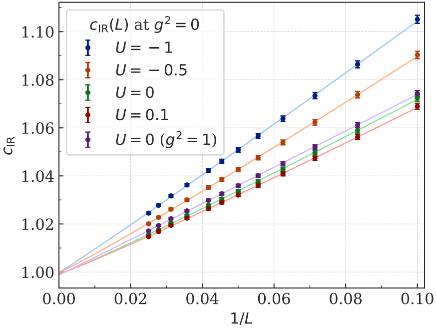
Phase diagram



Flow diagram

IR central charge (2312.17734)





Central charge extrapolation

IR central charge via entanglement entropy:

$$S = \frac{c_{\rm IR}}{3} \log\left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L}\right) + \text{const.}$$

between two subsystems with size ℓ and $L - \ell$.

$c_{\rm IR}(\infty)$ ranges from $0.9988(7)$ to $0.9998(9)$.

Conclusions

- Formulated $\mathrm{SU}(N)$ lattice gauge theories using Schwinger fermions.
- Remarkably, the resulting theory can be expressed purely in terms of gauge-invariant operators, which form a $\mathrm{U}(2d + N_f)$ algebra.
- This formulation applies to any $\mathrm{SU}(N)$ gauge group in any spacetime dimension.
- Reproduced the IR phases of 2d QCD using finite-dimensional local Hilbert space
- References:
  - 2112.02090: general idea of qubit regularization for $\mathrm{SU}(N)$ gauge theories.
  - 2312.17734: phase analysis in the continuum and numerical check on the lattice.
  - 2408.xxxxx: formulation of the theory using Schwinger fermions.

Thanks for attention!

The continuum theory of 2d QCD

$\mathrm{SU}(N)$ Yang-Mills theory coupled to single-flavor massless Dirac fermions

$$\mathcal{L}_0 = \frac{1}{2\tilde{g}^2}\,\mathrm{tr}\,F^2 + \bar{\psi}^\alpha \mathrm{i}\not{D}\psi^\alpha$$

Symmetries:

- $\tilde{g}^2 = 0$: free fermion, $\mathrm{O}(2N)_L \times \mathrm{O}(2N)_R$ chiral symmetry. ($\psi^\alpha = \frac{1}{\sqrt{2}}(\xi^{2\alpha-1} - \mathrm{i}\xi^{2\alpha})$)
- $\tilde{g}^2 > 0$: Gauge symmetry $\mathrm{SU}(N)$, $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ ($N = 2$); $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ ($N \geq 3$).

Bosonization:

- $\tilde{g}^2 = 0$: $\mathrm{U}(N)_1$ or $\mathrm{SO}(2N)_1$ WZW model. Central charge: $c = N$.
- $\tilde{g}^2 > 0$: $\mathrm{SO}(2N)_1/\,\mathrm{SU}(N)_1$ or $\mathrm{U}(N)_1/\,\mathrm{SU}(N)_1 \cong \mathrm{U}(1)_N$ coset WZW model.
  Central charge: $c = c(\mathrm{SO}(2N)_1) - c(\mathrm{SU}(N)_1) = N - (N-1) = 1$.
  - $N = 2$: $\mathrm{SO}(4) \cong \mathrm{SU}(2)_s \times \mathrm{SU}(2)_c$, coset is $\mathrm{SU}(2)_1$ WZW model in the charge sector.

Symmetries from the lattice

- Continuum symmetries forbid any relevant or marginal couplings.
- When regularizing the theory on the lattice using staggered fermions, $\mathrm{U}(1)_L \times \mathrm{U}(1)_R \to \mathrm{U}(1) \times \mathbb{Z}_2^\chi$ ($\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \to \mathrm{SU}(2) \times \mathbb{Z}_2^\chi$ for $N = 2$).
  $\mathbb{Z}_2^\chi$ forbids mass terms, but allows coupling between currents: $J^a_{L,R} := \frac{1}{2}\xi^T_{L,R} T^a \xi_{L,R}$

Full continuum theory

$$\mathcal{L} = \frac{1}{2\tilde{g}^2}\,\mathrm{tr}\,F^2 + \bar{\psi}^\alpha \mathrm{i}\not{D}\psi^\alpha + \sum_a \lambda^a J^a_L J^a_R$$

- $N \geq 3$: two independent couplings: λ_0 (Thirring coupling) and $\lambda_{\tilde{c}}$
- $N = 2$: one independent coupling: $\lambda_0 = \lambda_{\tilde{c}} = \lambda_c$

RG flow

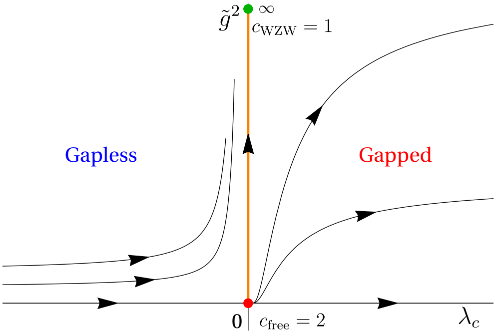
- $N \geq 3$:

$$\frac{\mathrm{d}\lambda_0}{\mathrm{d}\ln\mu} = -\frac{N-1}{2\pi}\lambda_{\tilde{c}}^2,$$
$$\frac{\mathrm{d}\lambda_{\tilde{c}}}{\mathrm{d}\ln\mu} = -\frac{1}{N\pi}\lambda_0\lambda_{\tilde{c}},$$

- $N = 2$:

$$\frac{\mathrm{d}\lambda_c}{\mathrm{d}\ln\mu} = -\frac{1}{2\pi}\lambda_c^2.$$

Mixed anomaly between $\mathrm{U}(1)$ and $\mathbb{Z}_2^\chi$ $\Longrightarrow$ the gapped phase should spontaneously break $\mathbb{Z}_2^\chi$, the translation-by-one-site symmetry on the lattice.



RG flow for $N = 2$

Phases in the strong coupling limit

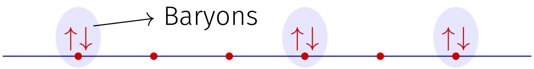
generalized Hubbard coupling

$$H = \frac{g^2}{2}\sum_{\langle i,j\rangle}\left(L^{a2}_{ij} + R^{a2}_{ij}\right) + t\sum_{\langle i,j\rangle}(c^{\alpha\dagger}_i U^{\alpha\beta}_{ij} c^{\beta}_j + \text{h.c.}) + U\sum_i n_i(N - n_i)$$

$g^2/t \gg 1$:

$$\frac{1}{2}\left(L^{a2}_{ij} + R^{a2}_{ij}\right)|k\rangle = \frac{N+1}{2N}k(N-k)|k\rangle,$$

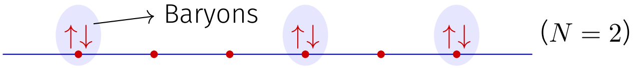
gauge links prefer $k = 0$ (trivial rep)



$(N = 2)$

Similar analysis for $-U/t \gg 1$.

Strong coupling expansion - spin-chain phase



When $g^2/t \gg 1$ or $-U/t \gg 1$, treat hopping terms as a perturbation:

$$XXZ \text{ spin chain:} \quad H_{\text{eff}} = \sum_{\langle i,j \rangle} J_{\perp}(X_i X_j + Y_i Y_j) + J_z(Z_i Z_j - 1)$$

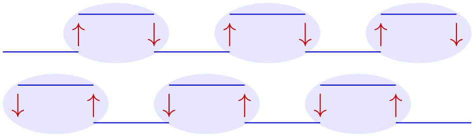
where

$$J_{\perp} = (-1)^{N-1} \frac{N}{2(N-1)!} \frac{t^N}{(\frac{N+1}{2N} g^2 + 2U)^{N-1}}, \quad J_z = \frac{N}{2(N-1)} \frac{t^2}{\frac{N+1}{2N} g^2 + 2U}$$

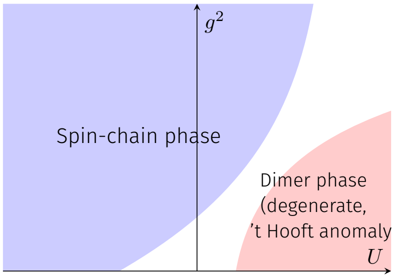
- When $N = 2$, $|J_{\perp}| = |J_z|$ (gapless) $\implies$ $\mathrm{SU}(2)$ symmetry $\leftrightarrow$ $\mathrm{SU}(2)_1$ WZW model.
- When $N > 2$, $|J_{\perp}| < |J_z|$ (gapped, Néel) $\implies$ $\mathrm{U}(1)$ symmetry $\leftrightarrow$ $\mathrm{U}(1)_N$ WZW model.

Strong coupling expansion - dimer phase

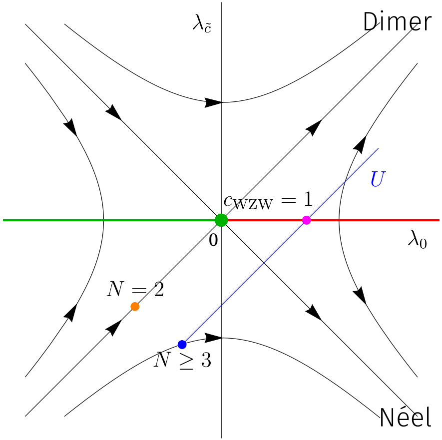
when $U/t \gg 1$, each site is forced to have one fermion $(N = 2)$



gapped, dimerized, doubly degenerate, expected from 't Hooft anomaly matching



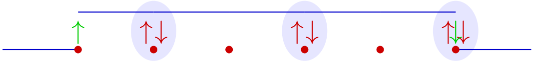
Lattice model in the continuum RG flow



Confinement in the strong coupling limit

Put two test quarks and pull them apart, see how the energy changes:

$-U/t \gg 1$: Raise links in-between to higher irreps, confined

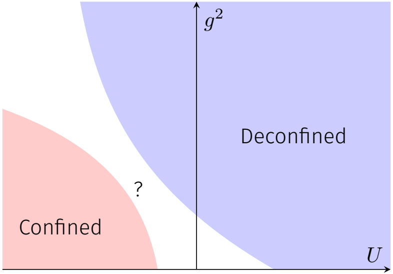


String tension: $$T_k = g^2 \frac{N+1}{2N} k(N-k).$$

$-U/t \gg 1$ or $g^2/t \gg 1$:

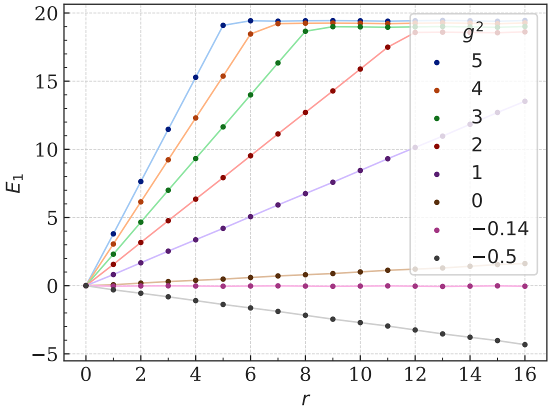
String tension: $$T_k = g^2 \frac{N+1}{2N} k(\lfloor \frac{N}{2} \rfloor - k).$$

$\implies$ Deconfined for $N = 2, 3$



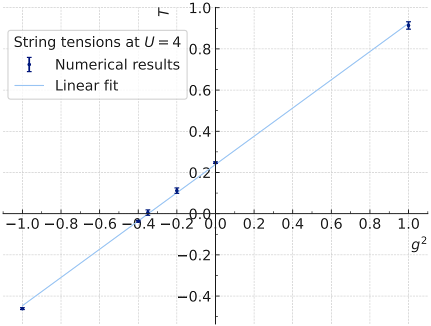
Confinement diagram for $N = 2, 3$

Confining phase



Energy as a function of the distance r between the test quarks at $N = 2$, $k = 1$ and $L = 20$ for $U = -10$.

String tensions at large U



$$T = 0.685(8)g^2 + 0.239(5)$$

- Strong coupling result: $T = 0.75g^2$
- Surprisingly, when $g^2 = 0$, $T > 0$. (In traditional theory, when $g^2 = 0$ the gauge field can be absorbed)
- In the qubit regularization, electric field term is generated by the hopping term in the RG sense:

$$H_{ij} = c_i^{\alpha\dagger} U_{ij}^{\alpha\beta} c_j^{\beta} + c_j^{\beta\dagger} (U_{ij}^{\alpha\beta})^{\dagger} c_i^{\alpha}$$

$$-\frac{1}{\beta}\log(\mathrm{tr}_f\, \mathrm{e}^{-\beta H_{ij}}) \begin{cases} \propto \mathbb{1} & : \text{traditional} \\ \propto L_{ij}^{a2} + R_{ij}^{a2} & : \text{qubit} \end{cases}$$

Marginal operator, level crossing and critical point

- $\mathrm{SU}(2)_1$ WZW has $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry
  Lowest 5 states: $(s_L, s_R) = (0,0)$ and $(\frac{1}{2}, \frac{1}{2})$
- On the lattice: chiral symmetry is broken
  $\lambda_c J_L \cdot J_R$ is allowed, can be tuned by U

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \xrightarrow{\text{broken}} \mathrm{SU}(2)_{\mathrm{diag}}$$

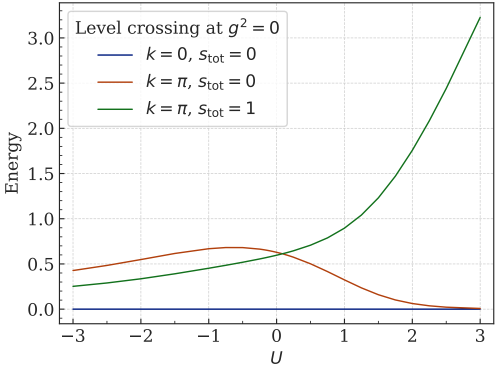
$$(s_L, s_R) = (\frac{1}{2}, \frac{1}{2}) \longrightarrow s_{\mathrm{tot}} = 1, 0$$

$$\langle J_L \cdot J_R \rangle = \frac{1}{2} \langle (J_L + J_R)^2 - J_L^2 - J_R^2 \rangle$$

$$= \frac{1}{2} (s_{\mathrm{tot}}(s_{\mathrm{tot}} + 1) - s_L(s_L + 1) - s_R(s_R + 1))$$

λ_c is marginal, β-function:

$$\frac{\mathrm{d}\lambda_c}{\mathrm{d}\ln\mu} = -\frac{1}{2\pi}\lambda_c^2$$

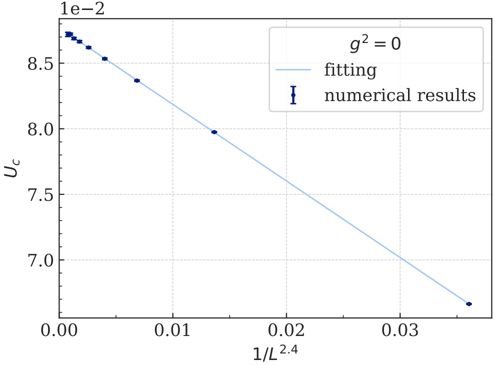


DMRG: ITensor[a]

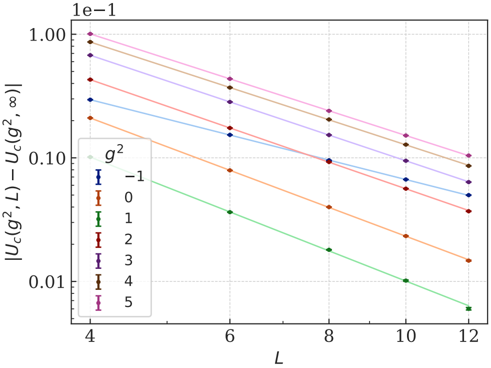
[a] M. Fishman et al., 2022, *SciPost Phys. Codebases*

Critical point extrapolation in L

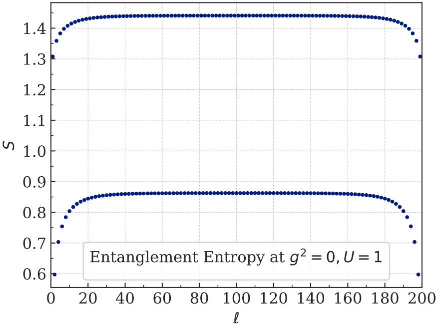
$$U_c(L) = U_c(\infty) + AL^{-a}$$

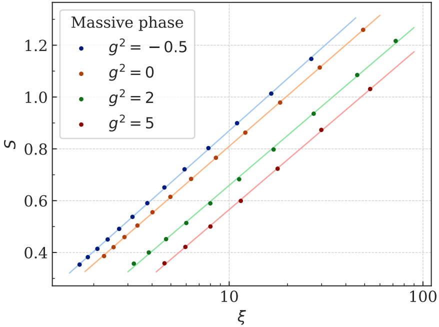


$$U_c(\infty) = -0.08769(3)$$



UV central charge





UV central charge via entanglement entropy

$$S = \frac{c_{\mathrm{UV}}}{6} \log \frac{\xi}{a} + \mathrm{const.}$$

ξ is correlation length.

$$c_{\mathrm{UV}} = 1.737(6), 1.693(4), 1.66(1), 1.66(1)$$