Fermionic anomalies on the lattice:
The Ginsparg-Wilson relation and its generalizations

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Based on: 2309.08542 | Michael Clancy, David Kaplan, HS

The Standard Model of Particle physics is an enormously successful theory.

But it has a rather embarrassing problem.

The problem is that the Standard Model is a chiral guage theory.

And we do not know how to do computer simulations of a chiral gauge theory.

- This is not a “technical” problem about algorithms or hardware.
- It means that we really do not how to define the Standard Model in a nonperturbative way.

There is no known nonperturbative lattice construction of a 4d nonabelian chiral gauge theory

For interesting recent work, see talks by
Sen, Tong, Cherman, Berkowitz, Fukaya, Honda, Onogi, Onada, Aoki, Kan, Catterall, Hasenfratz, Witzel, ...

Dirac fermion and chiral symmetry

- Recall that the massless Dirac fermion action in $d = 2k$,

$$S = \int d^4x \, \bar{\psi} \not{D} \psi$$

has both a vector and chiral $U(1)$ symmetry

$$\psi \xrightarrow{U(1)_V} e^{i\theta} \psi$$

$$\psi \xrightarrow{U(1)_\chi} e^{i\theta\gamma_5} \psi$$

- The chiral matrix γ^5 lets us define left and right handed Weyl fermions

$$\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi.$$

Naive attempts to get a Dirac fermion on the lattice leads to doublers

There are infinitely many ways to discretize the free fermion action.

Can you be clever with the discretization and avoid this problem?

A no-go theorem

- Consider the free Dirac fermion action on the lattice

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^{2k}} \bar{\Psi}_{-p} D(p) \Psi_p$$

- Nielsen and Ninomiya (1981) showed the following 4 conditions cannot all hold simultaneously for a Dirac fermion

1. $D(\mathbf{p})$ is a periodic, analytic function of $\mathbf{p}$ **(locality)**
2. $D(\mathbf{p}) \propto \gamma^\mu p_\mu$ for $a|p| \ll 1$ **(continuum limit)**
3. $D(\mathbf{p})$ is invertible everywhere except $\mathbf{p} = 0$ **(no doublers)**
4. $\{D(\mathbf{p}), \gamma^5\} = 0$ **(chirality)**

- So you cannot be clever, at least if you want all the above to be true.

To get a lattice theory free of doublers, we need to violate at least one of the assumptions of Nielsen-Ninomiya.

But which one?

Wilson's idea

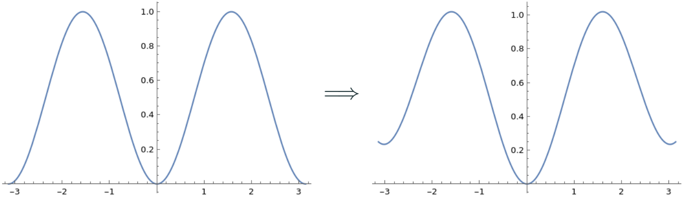
Here's an example of how to get rid of the doublers.

- Add a momentum-dependent mass term so that the doublers become heavy and decouple.

$$S = \int \bar{\psi} \left[\sum_{\mu} \gamma^{\mu} \sin(p_{\mu}) + mF(p) \right] \psi$$

- Choose $F(p)$ such that $F(0) = 0$ but $F(p) \sim 1$ at the corners of BZ.

$$F(p) = \sum_{\mu} 1 - \cos(p_{\mu})$$



- Doublers are gone, but so is the exact chiral symmetry
- The $U(1)$ chiral symmetry of the action is recovered only in the continuum limit

We can come up with many other such ideas to violate one of the assumptions of NN (and indeed people have).

But is there an “optimal” way to do this?

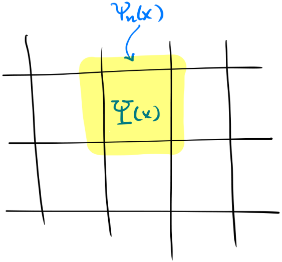
Ginsparg-Wilson relation

Observation:
The continuum theory has an exact chiral symmetry of the action, but the lattice theory does not.

Ginsparg and Wilson (1982) asked:
If you obtain a lattice theory by “RG blocking” a continuum theory, what happens to the chiral symmetry?

Ginsparg-Wilson relation

Start with a continuum theory and construct a lattice theory by RG blocking:



GW found that the lattice Dirac operator $\mathscr{D}$ satisfies

$$\{\mathsf{D}, \bar{\gamma}\} = 0 \rightarrow \boxed{\{\mathscr{D}, \bar{\gamma}\} = \mathsf{a}\mathscr{D}\bar{\gamma}\mathscr{D}}$$

This is the Ginsparg-Wilson relation.

An exact chiral symmetry on the lattice

- Lüscher (1998) showed that the GW relation in fact implies an exact symmetry of the action
- The exact symmetry is

$$\hat{\gamma}^5 = \gamma^5 \mathsf{V}, \quad \mathsf{V} = 1 - \mathsf{aD}$$
$$\delta\psi = \hat{\gamma}^5 \psi, \qquad \delta\bar{\psi} = \bar{\psi}\gamma^5$$

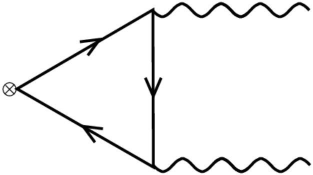
- The variation in the action is

$$\begin{aligned} \delta(\bar{\psi}\mathsf{D}\psi) &= (\delta\bar{\psi})\mathsf{D}\psi + \bar{\psi}\mathsf{D}(\delta\psi) \\ &= \bar{\psi}\left(\gamma^5\mathsf{D} + \mathsf{D}\gamma^5\mathsf{V}\right)\psi \\ &= \bar{\psi}\left(\{\mathsf{D}, \gamma^5\} - \mathsf{aD}\gamma^5\mathsf{D}\right)\psi = 0. \end{aligned}$$

- This means that any Dirac operator satisfying the GW relation automatically will not have additive mass renormalization (no fine tuning)

“No anomaly on the lattice”

- But the chiral symmetry supposed to be anomalous
- The free 4D Dirac fermion has a global chiral symmetry.
- However, the chiral symmetry is afflicted by a famous mixed $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ anomaly.



- If the lattice action is invariant, where does the anomaly come from?

Lore

There are no anomalies on the lattice.

"~~No~~ anomaly on the lattice"

- The exact symmetry is:

$$\psi \to \mathsf{U}\psi \approx (1 + \mathrm{i}\varepsilon\hat{\gamma}^5)\psi, \qquad \bar{\psi} \to \bar{\psi}\bar{\mathsf{U}} \approx \bar{\psi}(1 + \mathrm{i}\varepsilon\gamma^5)$$

- This implies that the measure transforms with the Jacobian

$$\begin{aligned} \mathrm{d}\psi\mathrm{d}\bar{\psi} \longrightarrow \quad & \mathrm{d}\psi\mathrm{d}\bar{\psi}\ \det(\mathsf{U}\bar{\mathsf{U}}) \\ &= \mathrm{d}\psi\mathrm{d}\bar{\psi}\ \exp(\mathrm{i}\varepsilon\mathrm{tr}\,(\hat{\gamma}^5)) \\ &= \mathrm{d}\psi\mathrm{d}\bar{\psi}\ \exp(-2\mathrm{i}\mathsf{a}\varepsilon\mathrm{tr}\,(\gamma^5\mathsf{D})) \\ &= \mathrm{d}\psi\mathrm{d}\bar{\psi}\,\exp(-\mathrm{i}\mathsf{a}\varepsilon\ \mathrm{index}\ \mathsf{D}) \end{aligned}$$

  using $\mathsf{V} = 1 - \mathsf{a}\mathsf{D}$.
- This is a subtle calculation in the continuum theory! [Fujikawa '79]
- On the lattice, it is almost trivial once you identify the correct modified symmetry [Lüscher '98].

For more on lattice anomalies, see talks by Cherman, Berkowitz, Catterall, Honda, Onogi, Onoda
Many other examples of lattice anomalies in recent years: [Catterall et al; Nguyen, HS; Sulejmanpasic, Gattringer; Shao, Seiberg, …]

Naive argument: doubling occurs because lattice cannot reproduce the anomaly.

But the GW relation implies the anomaly on the lattice. If so, it may allow for a solution to the doubling problem.

But is there actually a Dirac operator which actually satisfies it?

Ginsparg-Wilson relation

PHYSICAL REVIEW D VOLUME 25, NUMBER 10 15 MAY 1982

A remnant of chiral symmetry on the lattice

Paul H. Ginsparg* and Kenneth G. Wilson
Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853
(Received 20 December 1981)

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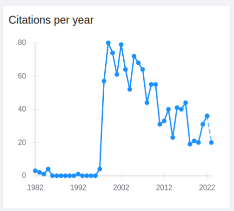

What happened in 1997?

Ginsparg-Wilson relation

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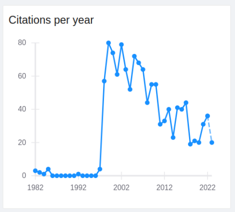

What happened in 1997?

Peter Hasenfratz discovered that “fixed point actions” satisfy this relation.
[Hasenfratz ’97; Hasenfratz, Laliena, Niedermayer ’98]

- “Overlap” operator satisfies this relation as well. [Neuberger ’98]
- Domain-wall fermions [Kaplan ’92] also satisfy this relation since they are essentially overlap in the $L \to \infty$ limit [Narayanan, Neuberger].
- GW relation implies an exact “modified” chiral symmetry on the lattice. [Lüscher ’98]

A solution to the GW relation

- Consider writing

$$\mathsf{D} = 2\frac{\mathsf{h}}{\mathsf{h}+1} \implies \mathsf{D}^{-1} = \frac{1}{2}(1+\mathsf{h}^{-1})$$

  where h is some operator. The GW relation implies

$$\{\gamma^5, \mathsf{D}\} = \mathsf{D}\gamma^5\mathsf{D} \implies \{\gamma^5, \mathsf{h}\} = 0$$

  Therefore, h can be any operator which satisfies the chiral symmetry.
- So choose $\mathsf{h} \sim \not{\mathsf{D}}/\mathsf{m} = \gamma^\mu \mathsf{D}_\mu/\mathsf{m}$, the continuum Dirac operator!
- Therefore this Dirac operator satisfies the GW relation (with $\mathsf{a} \sim \mathsf{m}^{-1}$)

$$\mathsf{D} = \frac{\not{\mathsf{D}}}{\not{\mathsf{D}}+\mathsf{m}}$$

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$$\mathsf{D} = \frac{\not{\mathsf{D}}}{\not{\mathsf{D}} + \mathsf{m}}$$

- But this is just a Pauli-Villars regulated Dirac fermion!
  A solution to the GW equation decades before lattice gauge theory was invented!

For more details, see Clancy, Kaplan, HS ’24

A solution to the GW relation: the overlap operator

- A "continuum" solution to the GW relation:

$$\mathsf{D} = \frac{\not{\mathsf{D}}}{\not{\mathsf{D}} + \mathsf{m}}$$

The overlap operator provides a nonperturbative lattice version of this construction.

$$\mathsf{D} = \frac{1}{2}(1 + \mathsf{V})$$

$$\mathsf{V} = \begin{cases} \frac{\not{\mathsf{D}} - \mathsf{m}}{\not{\mathsf{D}} + \mathsf{m}} & \text{PV} \\ \\ \mathsf{D}_\mathsf{W} / \sqrt{\mathsf{D}_\mathsf{W}^\dagger \mathsf{D}_\mathsf{W}} & \text{overlap} \end{cases}$$

- It can be easily shown that the overlap operator has no doublers

Ginsparg-Wilson for chiral symmetry: summary

- Continuum Dirac fermions in $\mathsf{d} = 2\mathsf{k}$ dimensions.

$$\mathsf{S} = \int \bar{\psi}(\mathsf{D} + \mathsf{m})\psi$$

- Chiral symmetry for $\mathsf{m} = 0$:

$$\psi \to \mathrm{e}^{\mathrm{i}\varepsilon\gamma^5}\psi, \quad \bar{\psi} \to \bar{\psi}\,\mathrm{e}^{\mathrm{i}\varepsilon\gamma^5}$$

- GW relation:

$$\{\gamma^5, \mathsf{D}\} = \mathsf{a}\mathsf{D}\gamma^5\mathsf{D}$$

- Lüscher symmetry: $\psi \to \psi + \varepsilon\delta\psi$

$$\delta\psi = \hat{\gamma}^5\psi, \quad \delta\bar{\psi} = \bar{\psi}\gamma^5$$

- Exact anomaly from the noninvariance of the measure

$$\mathrm{d}\psi\mathrm{d}\bar{\psi} \to \mathrm{d}\psi\mathrm{d}\bar{\psi}\,\mathrm{e}^{-\mathrm{i}\varepsilon\,\mathrm{tr}\,\gamma_5\mathsf{D}} = \mathrm{d}\psi\mathrm{d}\bar{\psi}\,\mathrm{e}^{-\mathrm{i}\varepsilon(\mathsf{n}_+ - \mathsf{n}_-)}$$

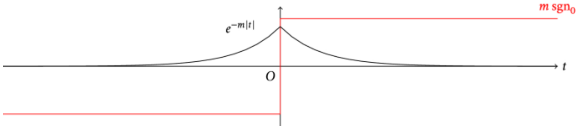
- GW relation implies
  - ✓ An exact symmetry of the action
  - ✓ Exact anomaly on the lattice (noninvariance of the measure)
  - ✓ No doublers
  - ✓ No additive mass renormlization

- We have a lattice theory which realizes a Dirac fermion with an 't Hooft anomly exactly on the lattice, which implies that the massless nature is robust.
- But there is another way of getting robust massless fermions: go to one higher dimension

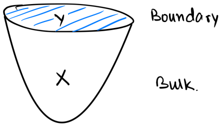
Bulk-boundary correspondence

$(\mathrm{D}+1)$-dim Bulk	$\leftrightarrow$	D-dim Boundary
SPT phases	$\leftrightarrow$	**t'Hooft anomalies**
Massive Dirac fermion	$\leftrightarrow$	Massless chiral fermion

- This is the essence of domain-wall fermions [Kaplan '92]



Generalizing GW relations?



Anomaly $\updownarrow$ SPT phase

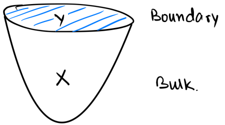
Ginsparg-Wilson relation $\updownarrow$ Domain-wall fermion

The traditional domain-wall fermions and GW relation correspond to a 5D topological insulator.

A domain-wall construction can be made for any SPT phase.

The GW relation seems to capture the anomaly of the boundary theory.

Generalizing GW relations?



Anomaly	Ginsparg-Wilson relation
$\updownarrow$	$\updownarrow$
SPT phase	Domain-wall fermion

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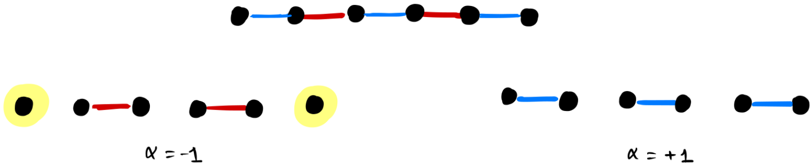
Are there GW relations for *all* fermionic SPT phases?

Let us look at a famous example in cond-mat:
Majorana fermions arising at the edge of the Kitaev Chain.

- A 1d chain of N Majorana fermions $\lambda_1, \ldots, \lambda_N$ (Kitaev, 2001):

$$H = \sum_{i \in \text{odd}} (1+\alpha)\, \lambda_i \lambda_{i+1} + \sum_{i \in \text{even}} (1-\alpha)\, \lambda_i \lambda_{i+1}$$

- α decides the strength of bond staggering



nontrivial SPT   trivial

- At $\alpha = 0$, this describes a massless 2-component Majorana fermion

0+1d Majorana fermion and a $\mathbb{Z}_8$ anomaly

- The edge modes of the 1+1D Kitaev chain are 0+1d Majorana fermions:

$$S = \int dt\, \chi \partial_t \chi$$

- This looks very simple. Is there an anomaly?

0+1d Majorana fermion and a $\mathbb{Z}_8$ anomaly

- The edge modes of the 1+1D Kitaev chain are 0+1d Majorana fermions:

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- This looks very simple. Is there an anomaly?
- **Global $\mathbb{Z}_8$ anomaly:** involving time-reversal and fermion parity symmetries [Fidkowski-Kitaev, Kapustin et al, Witten]
- A subtle anomaly from the point of view of continuum path integrals [Witten '16]
  - Need to compute the η-invariant on suitable unorientable manifold in one higher dimension ($\mathbb{RP}^2$)

Can we obtain this global anomaly in the GW formalism on the lattice?

For more on the η invariant, see the next talk by Fukaya

GW for a 0+1d Majorana fermion

$$S = \int \mathrm{dt}\ \chi \partial_t \chi$$

- First, note that for a single Majorana fermion, we cannot even write down a mass term. So to proceed, we need 2 flavors

$$S = \int \mathrm{dt}\ \chi^{\mathsf{T}} \partial_t \chi + \mu \chi^{\mathsf{T}} \tau_2 \chi$$

- Reflection symmetry for the kinetic term

$$R\chi(t) = i\chi(-t)$$

such that $R^2 = -1$. But mass term breaks this symmetry $\implies$ anomaly!

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Generalized GW relations

We can generalize the GW relation and the overlap solution to discrete symmetries, any dimension, Dirac or Majorana fermions

[Clancy, Kaplan, HS '24]

Bietenholz, Nishimura '02 "guessed" the right GW relation for the parity anomaly in odd dimensions.
Kimura, Watanabe '23 also have a generalization to T and C symmetries

Exact Lüscher symmetry for Majorana fermions

- Note that now we have a path integral over a single variable $\int \mathscr{D}\chi$ with the kinetic term $\chi^{\mathsf{T}}\mathsf{D}\chi$
- So we do not have the same freedom as earlier, to define ψ and $\bar{\psi}$ independently
- Nevertheless, one can find a choice which works

$$\chi \to \mathsf{R}\sqrt{-\mathsf{V}}\chi$$

(but one has to be a bit careful with squareroots)
- Again, we find a Jacobian

$$\det(\mathsf{R}\sqrt{-\mathsf{V}}) = (-1)^{\nu_-/2}$$

where ν_- is the number of zero modes.
  - The anomaly depends on a mod-2 index of the Dirac operator in this case.
- So we find a $\mathbb{Z}_4$ global anomaly exactly on the lattice, simply by looking at the Jacobian!

Ginsparg-Wilson for Majorana fermions

	Dirac	Majorana
Action	$S = \int \bar{\psi}(D + m)\psi$	$S = \int dx\ \chi^T(\mathcal{K}\not{D} + \mu\mathcal{M})\chi$
Symmetry	Chiral symmetry	Time-reversal
Anomaly	Perturbative	Global
GW relation	$\{\gamma^5, D\} = aD\gamma^5 D$	$\{R, D\} = aD(\mathcal{M}^{-1}\mathcal{K})RD$
Lattice symmetry	$\delta\psi = \hat{\gamma}_5\psi, \quad \delta\bar{\psi} = \bar{\psi}\gamma^5$	$\chi \to R\sqrt{-V}\,\chi$
Jacobian	$e^{-i\varepsilon\ \mathrm{ind} D}$	$(-1)^{\frac{\nu}{2}}$
	index	mod-2 index

- GW relation implies
  - ✓ An exact symmetry of the action
  - ✓ Exact anomaly on the lattice (noninvariance of the measure)
  - ✓ No doublers
  - ✓ No additive mass renormlization

Generalized GW fermions

- The classic GW relation encodes how the anomalous chiral symmetry manifests on the lattice for Dirac fermions
  - Has almost all the nice features of the continuum theory that we want
- A Ginsparg-Wilson relation, and corresponding overlap operator, can written for many fermionic anomalies (including global anomalies)
  - Discrete symmetries such as time-reversal, reflection, chiral
  - Majorana and Dirac fermions
  - Any dimension

Some questions

- Generalized GW relations = **lattice bulk-boundary correspondence**
  - The bulk-boundary correspondence is used almost all recent attempts for nonabelian chiral lattice gauge theories (symmetric mass generation, single-wall)
- But crucially, we don't have a completely satisfactory Hamiltonian formulation yet [Creutz et al '02, Clancy '24].
  - There is a tension between locality, real time, unitarity and compactness
  - Needed to make direct contact with condensed-matter physics
  - Needed for quantum computing and tensor networks methods
- Can we reproduce on the Euclidean lattice all the anomalies captured by cobordism? Can the generalized GW relations be used to formulate a Dai-Freed theorem on the lattice?

For the Dai-Freed theorem, see Witten '16, Witten-Yonekura '20

We have made progress in using generalized GW relations to capture many **fermionic** (global) anomalies.

But some pieces are still missing.

Is this needed to solve the problem of nonabelian chiral gauge theories?

Thank you.