η invariant of massive Wilson Dirac operator and the index

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Shoto Aoki(U. Tokyo), HF, Mikio Furuta (U. Tokyo), Shinichiroh Matsuo(Nagoya U.), Tetsuya Onogi(Osaka U.), and Satoshi Yamaguchi (Osaka U.), "The index of lattice Dirac operators and K-theory,"
arXiv:2407.17708 (appeared last Friday)

What is your favorite lattice Dirac operator

What is your favorite lattice Dirac operator when you want to understand the index theorem?

What is your favorite lattice Dirac operator when you want to understand the index theorem?

Overlap Dirac operator [Neuberger 1998] would be the most popular answer: realizes an “exact” chiral symmetry

$$S = \sum_x \bar{q}(x) D_{ov} q(x) \qquad q \to e^{i\alpha\gamma_5(1-aD_{ov})} q, \quad \bar{q} \to \bar{q} e^{i\alpha\gamma_5}.$$

through the Ginsparg-Wilson relation and reproduces the anomaly.

$$Dq\bar{q} \to \exp\left[2i\alpha \mathrm{Tr}(\gamma_5 + \gamma_5(1 - aD_{ov}))/2\right] Dq\bar{q}$$

The index is well-defined:

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$$

[Hasenfratz et al. 1998]

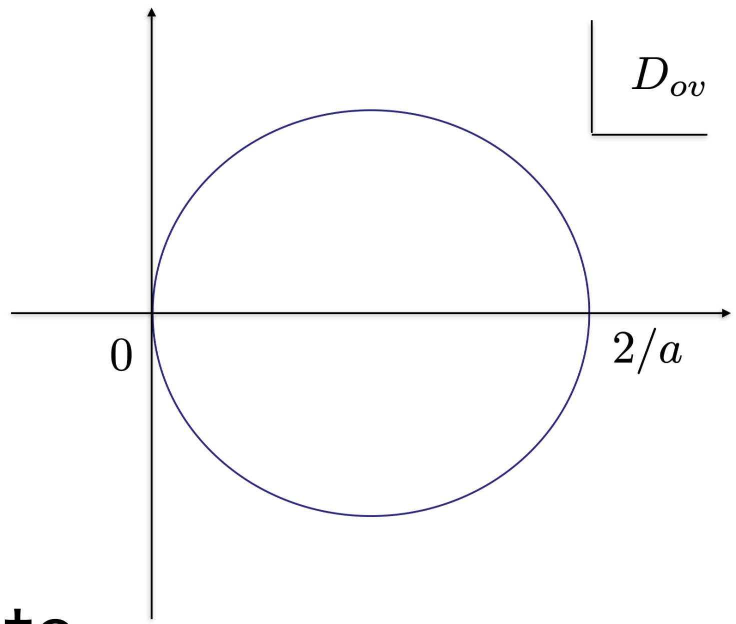
The overlap Dirac operator index

Overlap Dirac spectrum lies on a circle with radius 1/a
Complex eigenmodes form ± pairs of

$$\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$$

(therefore, no contribution to the trace).
The real 2/a (doubler poles) do not contribute.

a :lattice spacing



$$\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \mathop{\mathrm{Tr}}_{\text{zero-modes}} \gamma_5$$

But D_{ov} is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right) \qquad H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \underbrace{\mathrm{Tr}\frac{\gamma_5}{2}}_{=0} - \frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

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$$= -\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W)$$

What is this ???

η invariant of the massive Wilson Dirac operator

$$-\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W) = -\frac{1}{2}\sum_{\lambda_{H_W}} \mathrm{sgn}(\lambda_{H_W}) = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

This quantity is known as the Atiyah-Patodi-Singer η invariant (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

The Wilson Dirac operator and K-theory

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

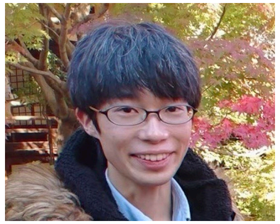
$$H_W = \gamma_5(D_W - M)$$
$$M = 1/a$$

In this talk, we try to show a deeper mathematical meaning of the right-hand side of the equality,

and try to convince you that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978…]

Phys-Math collaborators

Physicists



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Main theorem

[Aoki, F, Furuta,Matsuo, Onogi, Yamaguchi, arXiv:2407.17708]

For any $K^1(I, \partial I)$ group element defined by

$$[\mathcal{H}_{\mathrm{cont.}}, \gamma_5(D_{\mathrm{cont.}} + m)]$$

$[\cdots]$ means those with the same spectral flow between $m \in [-M, M]$

its lattice approximation at sufficiently small lattice spacings,

$$[\mathcal{H}_{\mathrm{lat.}}, \gamma_5(D_W + m)]$$

belongs to the same K^1 group element

= sufficient condition for having the same index.

[Cf. Yamashita 2021, Kubota 2020]

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We revisit the lattice index theorem with a K-theoretic treatment of Wilson Dirac op.

2. Massless Dirac (K^0 group) vs. massive Dirac (K^1 group)

3. Main theorem

4. Comparison with the overlap Dirac index

5. Summary

Eigenvalues of continuum massive Dirac operator

$$H(m) = \gamma_5 (D_{\text{cont.}} + m)$$

For $D_{\text{cont.}}\phi = 0,$ $\quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\text{chirality}} m\phi.$

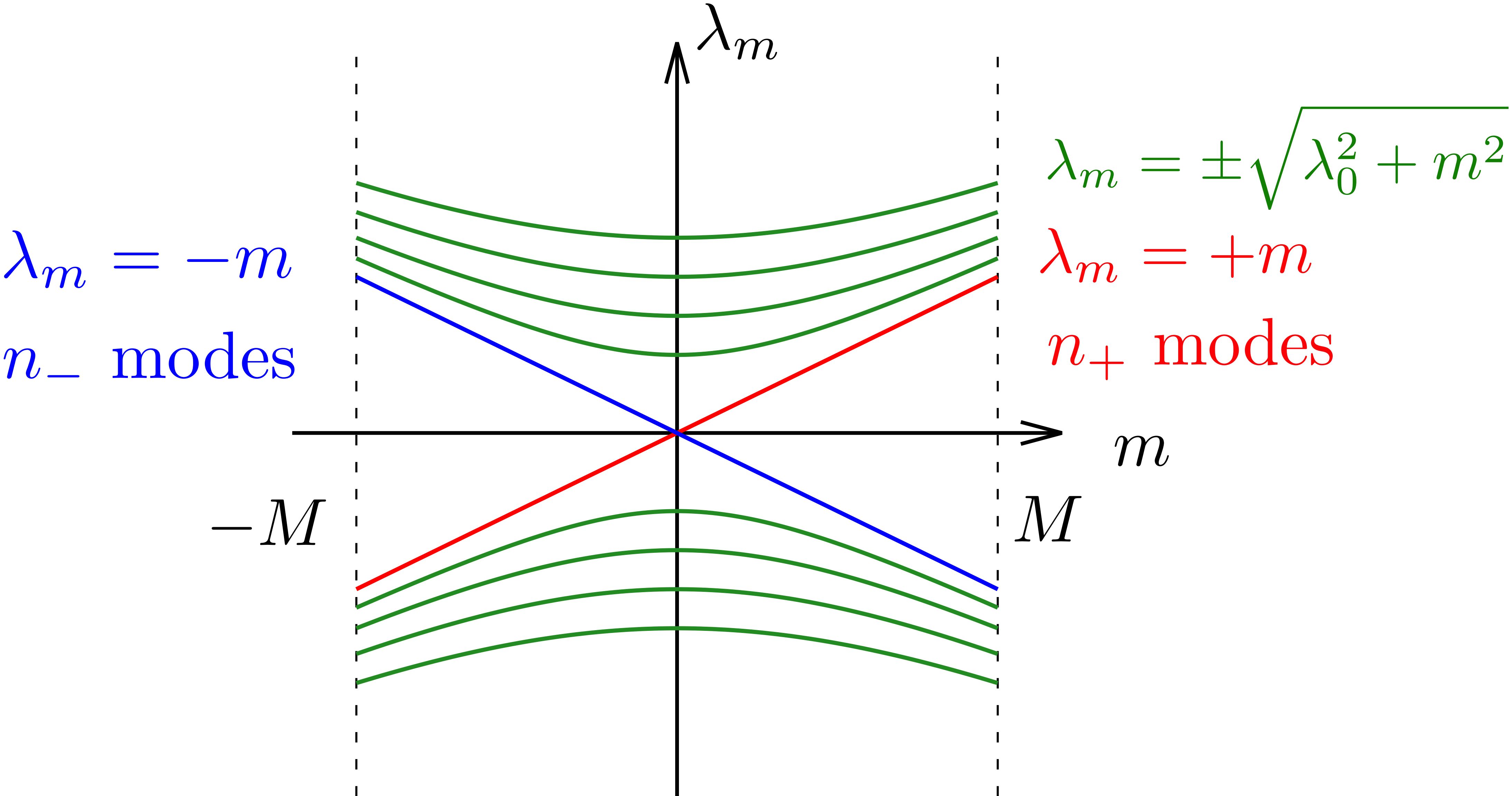
For $\quad D_{\text{cont.}}\phi \neq 0,$ $\quad \{H(m), D_{\text{cont.}}\} = 0.$

The eigenvalues are paired: $\quad H(m)\phi_{\lambda_m} = \lambda_m \phi_{\lambda_m}$

$$H(m) D_{\text{cont.}}\phi_{\lambda_m} = -\lambda_m D_{\text{cont.}}\phi_{\lambda_m}$$

As $H(m)^2 = -D^2_{\text{cont.}} + m^2$, we can write them $\quad \lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$

Spectrum of $H(m) = \gamma_5(D_{\text{cont.}} + m)$



Spectral flow = Atiyah-Singer index = η invariant

n_+ = # of zero-crossing eigenvalues from - to +

n_- = # of zero-crossing eigenvalues from + to -

$$H(m) = \gamma_5(D_{\mathrm{cont.}} + m)$$

$n_+ - n_-$ =: spectral flow of $H(m)$ $\quad m \in [-M, M]$

Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H(m))$ jumps by two.

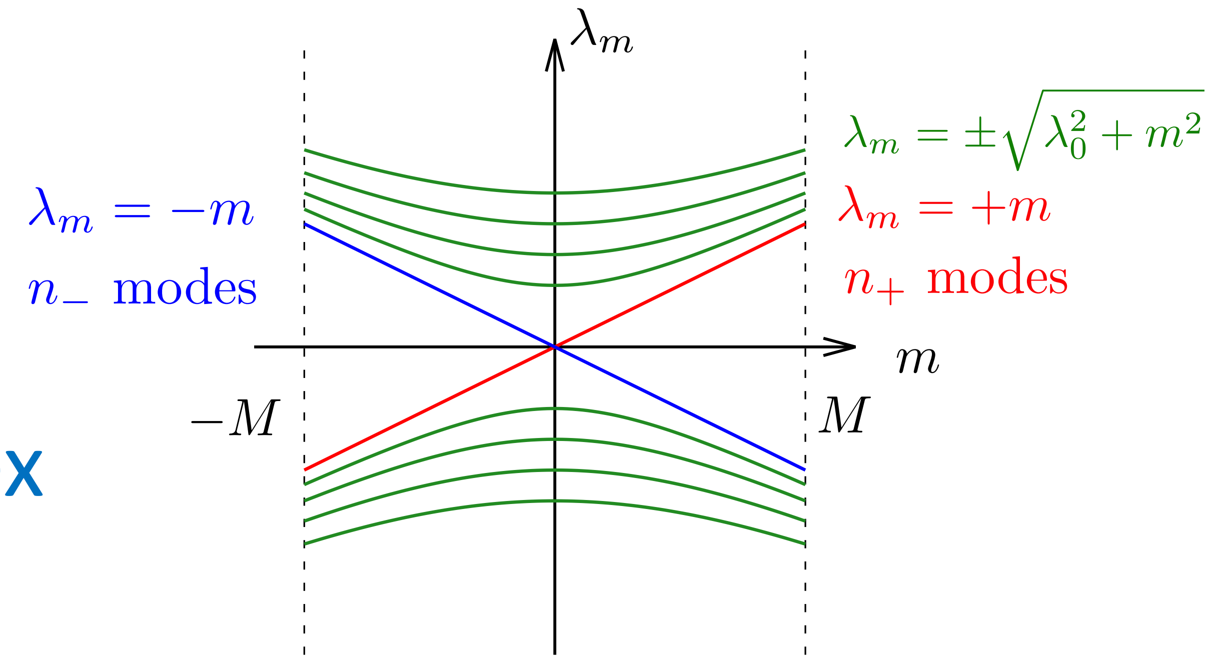
$$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

$$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

Suspension isomorphism in K theory

Massless= counting index by points



Massive= counting index by lines

$$K^0(pt) \simeq K^1(I, \partial I)$$

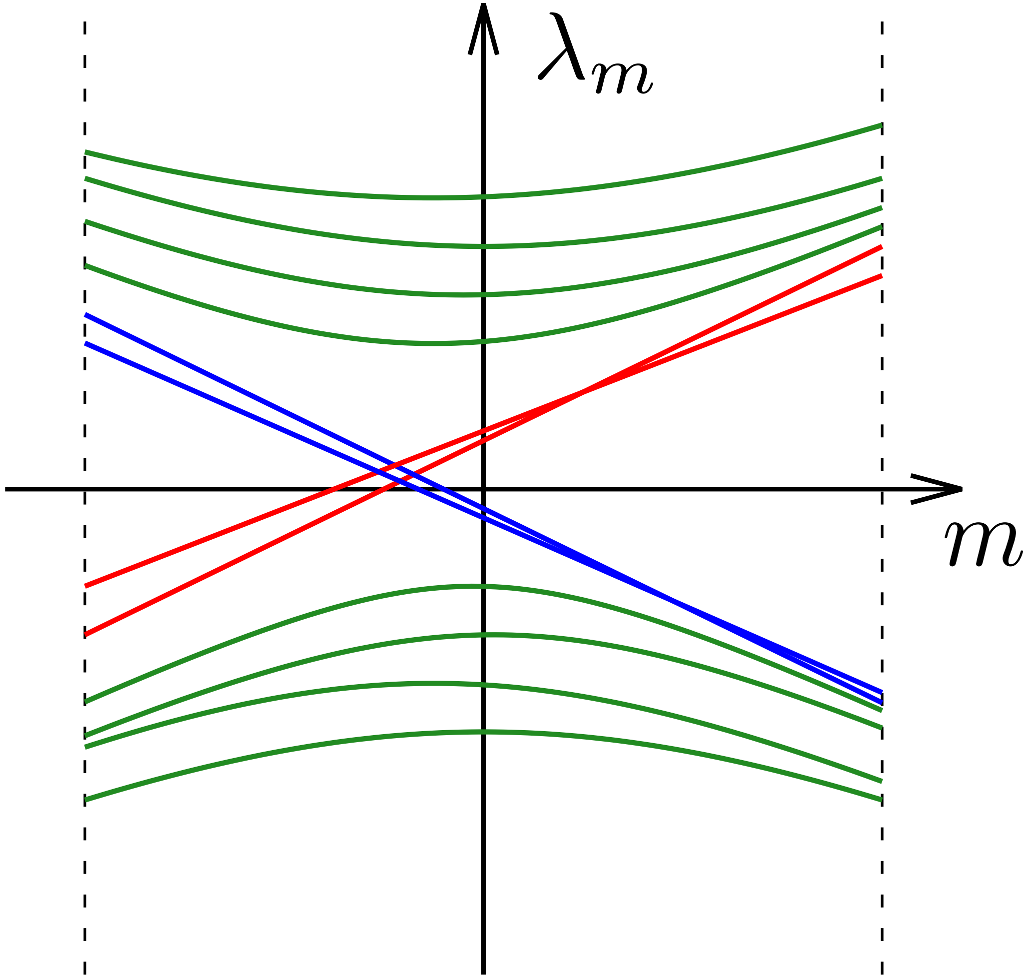
point — line=interval

With chirality operator — Without chirality operator

⇨ The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) still works.

Standard definition:
Where is m=0?
What are zero modes?



Eta invariant:
If m= ± M points are gapped, we can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki

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✓ 1. Introduction

We revisit the lattice index theorem with a K-theoretic treatment of Wilson Dirac op.

✓ 2. Massless Dirac (K^0 group) vs. massive Dirac (K^1 group)

Counting lines (massive, K^1) is easier than counting points (massless, K^0).

3. Main theorem

4. Comparison with the overlap Dirac index

5. Summary

Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: 2n-dimensional flat torus T^{2n}

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with gauge field A_i

D : Dirac operator on sections of E

$$D_{\text{cont.}} = \gamma_i(\partial_i + A_i)$$

Chirality (Z_2 grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

Wilson Dirac operator on a lattice

We regularize T^{2n} is by a square lattice with lattice spacing a

(The fiber is still continuous.)

We denote the bundle by E^a and

link variables :

$$U_k(\boldsymbol{x}) = P\exp\left[i\int_0^a A_k(\boldsymbol{x}')dl\right],$$

$$D_W = \sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2}\nabla_i^f\nabla_i^b\right]$$

Wilson term

$$a\nabla_i^f\psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

$$a\nabla_i^b\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

Note: In our paper, we consider "generalized link variables" to determine the gauge fields both in continuum and on a lattice simultaneously. But the standard Wilson line works, too.

Definition of $K^1(I, \partial I)$ group

Let us consider a Hilbert bundle with

Base space I = range of mass [-M, M]

boundary ∂I = $\pm$M points

Fiber space $\mathcal{H}$ = Hilbert space to which D acts

D_m : one-parameter family labeled by m.

We assume that $D_{\pm M}$ has no zero mode.

The group element is given by equivalence classes of the pairs:

$[(\mathcal{H}, D_m)]$ having the same spectral flow.

Note: K^1 group does NOT require any chirality operator.

Definition of $K^1(I, \partial I)$ group

Group operation: $[(\mathcal{H}^1, D^1_m)\} \pm \{(\mathcal{H}^2, D^2_m)] = [(\mathcal{H}^1 \oplus \mathcal{H}^2, \begin{pmatrix} D^1_m & \\ & \pm D^2_m \end{pmatrix})]$

Identity element: $[(\mathcal{H}, D_m)]|_{\mathrm{Spec.flow}=0}$

We compare $[(\mathcal{H}_{\mathrm{cont.}}, \gamma(D_{\mathrm{cont.}} + m))]$ and $[(\mathcal{H}_{\mathrm{lat.}}, \gamma(D_W + m))]$

taking their difference, and confirm if the lattice-continuum combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & f_a \\ f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

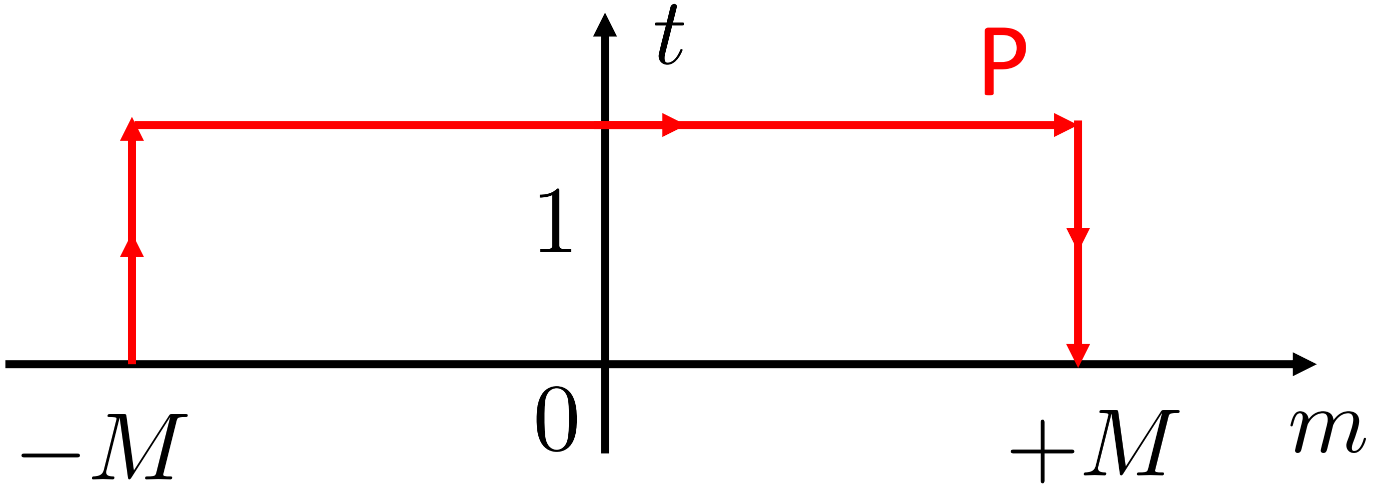
has Spectral flow =0 where f_a^* f_a are "mixing mass term" with some "nice" mathematical properties (see our paper for the details).

Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

on the path P :



Main theorem

There exists a finite lattice spacing a_0 such that for any $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

is invertible (having no zero mode) on the staple-shaped path P
[which is a sufficient condition for Spec.flow=0]

⇨ $\gamma(D_{\mathrm{cont.}} + m), \ \ \gamma(D_W + m)$ have the same spec.flow

⇨ $$\frac{1}{2}\eta(\gamma(D-M))^{\mathrm{PV\ reg.}} = \frac{1}{2}\eta(\gamma(D_W - M))$$

The continuum and lattice indices agree.

Proof (by contradiction)

Assume $$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

has zero mode(s) at arbitrarily small lattice spacing.

⇨ For a decreasing series of $\{a_j\}$

$$\begin{pmatrix} \gamma(D_{\text{cont.}} + m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j} + m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

Continuum limit

Multiplying $\begin{pmatrix} 1 & \\ & f_{a_j} \end{pmatrix}$ and taking the continuum limit

$$\begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m_\infty) & t_\infty \\ t_\infty & -\gamma(D_{\mathrm{cont.}} + m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

$$\hat{D}^2_\infty = D^2_{\mathrm{cont.}} + m^2_\infty + t^2_\infty$$

requires

$$m_\infty = t_\infty = 0.$$

$u_\infty, \quad v_\infty$ are

L^2_1 weakly convergent

=

L^2 strongly convergent

(Rellich's theorem)

Contradiction with $m^2 + t^2 > 0$ along the path P.

What not shown in this talk

Because of time limitation, we cannot explain the following details.

- The map f_a, f_a^* between lattice and continuum Hilbert spaces
- Convergence of $f_a f_a^* \to 1, \quad f_a^* f_a \to 1.$
- Convergence of $f_a^* D_W f_a \to D_{\mathrm{cont.}}$
- Elliptic estimate for the Wilson Dirac operator
- Relich theorem

Please see our paper [S. Aoki, HF, M. Furuta, S. Matsuo, T. Onogi, S. Yamaguchi, **arXiv:2407.17708**] or invite us to your (online) seminar.

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Wilson Dirac operator is equally good as D_{ov} to describe the index.

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

↑ By K theory for sufficiently small lattice spacings

↑ Suspension isomorphism

Wilson Dirac operator is equally good as D_{ov} to describe the index.

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

By K theory for sufficiently small lattice spacings

Suspension isomorphism

Or even better?

Application to the manifold with boundaries

Periodic b.c.

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind} D_{\mathrm{cont.}}$$

Dirichlet b.c. (Shamir domain-wall fermion) we can show

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = -\frac{1}{2}\eta(\gamma_5(D^{\mathrm{cont.}}_{DW})) = \mathrm{Ind}_{\mathrm{APS}} D^{\mathrm{cont.}}$$

[perturbative evidence F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]. (first equality)

[HF, plenary talk at Lat21]. (second equality)

But the overlap Dirac is missing because Ginsparg-Wilson relation is broken by the boundary [Luescher 2006].

Real Dirac operators and the mod-two index

For general complex Dirac operators,

$$K^1(I, \partial I) \Rightarrow -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D - M))$$

For real Dirac operators, for example, in SU(2) gauge theory in 5D (origin of Witten anomaly), we will be able to show

$$KO^1(I, \partial I) \Rightarrow -\frac{1}{2}\left[1 - \mathrm{sgn}\det\left(\frac{D_W - M}{D_W + M}\right)\right] = -\frac{1}{2}\left[1 - \mathrm{sgn}\det\left(\frac{D_{\mathrm{cont.}} - M}{D_{\mathrm{cont.}} + M}\right)\right]$$

$$= \mathrm{Ind}_{\mathrm{mod-two}} D_{\mathrm{cont.}}$$

But there is no overlap counterpart.

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Summary

$$\mathrm{Ind} D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

and that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978···]

Summary

$$\mathrm{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{cont.}} - M)) = \mathrm{Ind}D_{\mathrm{cont.}}$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

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Backup slides

Elliptic estimate

In continuum theory, For any $\phi \in \Gamma(E)$ and i,

a constant c exists such that

$$||D_i \phi||^2 \leq c(||\phi||^2 + ||D\phi||^2)$$

When a covariant derivative is large、 D is also large.

This property is nontrivial on a lattice.

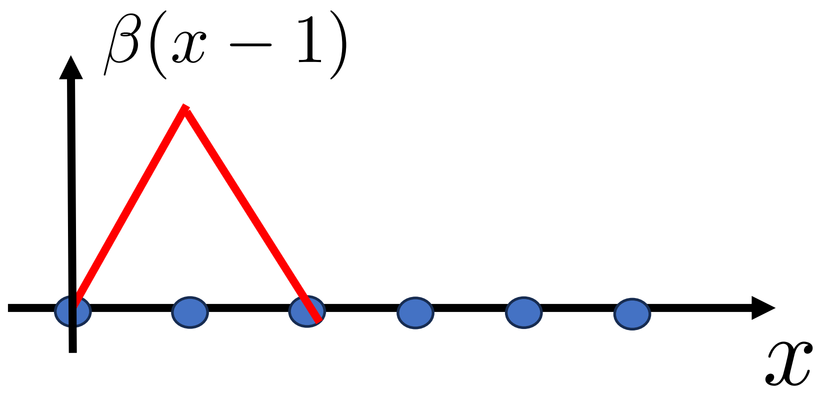
$$||\nabla_i^f \phi||^2 \leq c(||\phi||^2 + ||D_W \phi||^2)$$

Doubler modes have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important, too!

$$f_a$$

$$f_a : H^{\text{lat.}} \to H^{\text{cont.}}$$



From finite-dimensional vector bundle on a discrete lattice we need to make infinite-dimensional vector bundle on continuous x :

$$f_a \phi^{\text{lat.}}(x) = \sum_{l \in C_x} \beta(x-l) P(x-l) \phi^{\text{lat.}}(l)$$

C_x : a hyper cube containing x . l : lattice sites

$$P(x-l) = P \exp\left[i \int_l^x dx'^i A_i(x') \right]$$ Wilson line.

$\beta(x-l)$: linear partition of unity s.t.

$$\beta(0) = 1, \beta(1_\mu) = 0, \quad \sum_{l \in C_x} \beta_l(x) = 1.$$

$$f_a^*$$

$$f_a^* : H^{\mathrm{cont.}} \to H^{\mathrm{lat.}}$$

Is defined by

$$f_a^* \phi^{\mathrm{cont.}}(l) = \int_{y \in C_l} dy \beta(l-y) P(l-y) \phi^{\mathrm{cont.}}(y)$$

Note) $f_a^* f_a$ is not the identity but smeared to nearest-neighbor sites. (The gauge invariance is maintained by the Wilson lines.)

Continuum limit of f_a^* f_a

1. For arbitrary $\phi^{\text{lat.}}$

   $\lim_{a\to 0} f_a \phi^{\text{lat.}}$ weakly converges to a $\phi_0^{\text{cont.}} \in L_1^2$

   where L_1^2 is the square-integrable subspace of $H^{\text{cont.}}$

   to the first derivatives.

2. $\lim_{a\to 0} f_a \gamma(D_W + m)\phi^{\text{lat.}}$ weakly converges to

   $\gamma(D+m)\phi_0^{\text{cont.}} \in L^2$

3. There exists c s.t. $||f_a^* f_a \phi^{\text{lat.}} - \phi^{\text{lat.}}||_{L^2}^2 < ca^2 ||\phi^{\text{lat.}}||_{L_1^2}^2$

4. For any $\phi^{\text{cont.}} \in L_1^2$, $\lim_{a\to 0} f_a f_a^* \phi^{\text{cont.}}$

weakly converges to $\phi_0^{\text{cont.}} \in L_1^2$ and

$$\lim_{a\to 0} f_a f_a^* \phi_0^{\text{cont.}} = \phi_0^{\text{cont.}}$$

What are the weak convergence and strong convergence?

The sequence v_j weakly converges to v_∞

when for arbitrary w

$$\lim_{j\to\infty} \langle (v_j - v_\infty), w \rangle = 0.$$

Note) $\lim_{j\to\infty} (v_j - v_\infty)(x) \to \lim_{k\to\infty} e^{ikx}$ is weakly convergent.

Strong convergence means $\lim_{j\to\infty} ||v_j - v_\infty||^2 = 0.$

Rellich's theorem:

L_1^2 weak convergence = L^2 convergence