Digitised Hamiltonian SU(2) Lattice Gauge Theories at Weak Couplings

Lattice 2024

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Introduction

Motivation

Hamiltonian Lattice Gauge Theories

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Pros

- no sign problems
- real time dynamics

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- Hamiltonian grows exponentially with system size

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Hamiltonian Lattice Gauge Theories

Pros

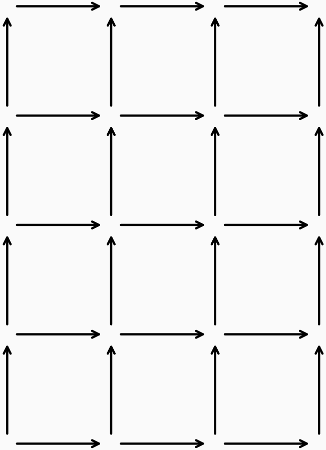
- no sign problems
- real time dynamics

Cons

- Hamiltonian grows exponentially with system size
- Mitigated by emerging algorithms/methods
  - Tensor Networks
  - Quantum Computing

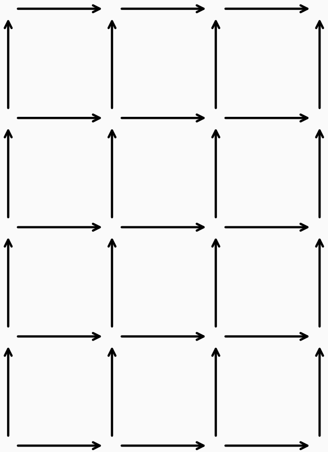
The Hamiltonian

The Hilbert Space



Positions of links labelled by $\mathbf{x}$, directions by k

The Hilbert Space



Positions of links labelled by $\mathbf{x}$, directions by k

Hamiltonian acts on wave functions:

$$\psi\,(\ldots, U_{\mathbf{x},k}, \ldots) \;:\; \mathrm{SU}(2)^{N_{\text{links}}} \;\to\; \mathbb{C}\,,$$

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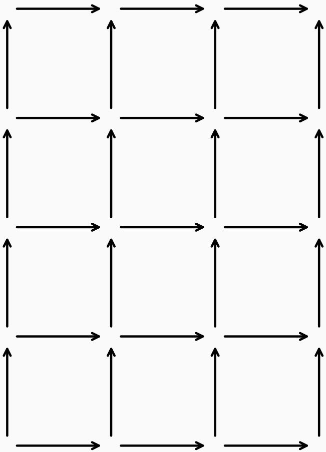

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Construction from single link basis functions

$$\hat{\phi}_n(U) \;:\; \mathrm{SU}(2) \to \;\mathbb{C}$$

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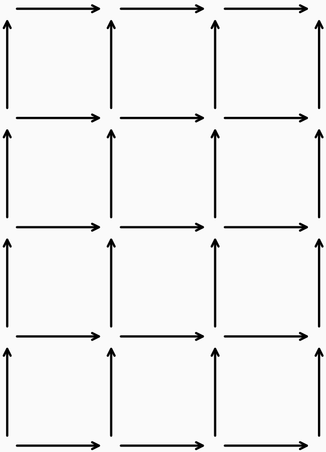

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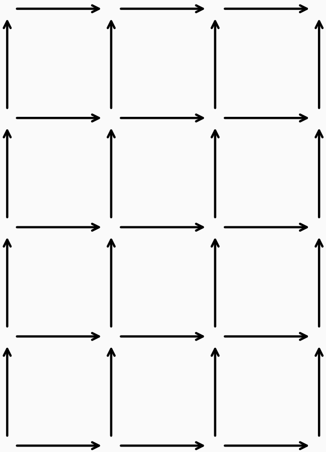
Basis for entire space:

$$|\ldots, n_{\mathbf{x},\mathbf{k}}, \ldots\rangle = \prod_{\mathbf{x},k} \hat{\phi}_{n_{\mathbf{x},\mathbf{k}}}(U_{\mathbf{x},k})$$

Kogut Susskind Hamiltonian[1]



$$\hat{H}_{\mathrm{KS}} = \frac{g^2}{2}\sum_{\mathbf{x},k,c}(\hat{L}^c_{\mathbf{x},k})^2 + \frac{2}{g^2}\sum_{\mathbf{x},j<k}\mathrm{Tr}\left[\mathbb{1} - \mathrm{Re}\left(\hat{P}_{\mathbf{x},jk}\right)\right]$$

[1] John Kogut and Leonard Susskind. “Hamiltonian formulation of Wilson’s lattice gauge theories”. In: *Phys. Rev. D* 11 (2 Jan. 1975), pp. 395–408. DOI: 10.1103/PhysRevD.11.395.

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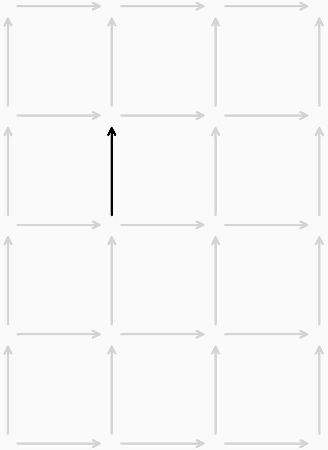

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Canonical Momentum Operators:

$$\hat{L}^c_{\mathbf{x},k}\,\psi = -\,\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\beta}\,\psi\left(\ldots, e^{-\,\mathrm{i}\,\beta\tau_c}\,U_{\mathbf{x},k},\ldots\right)|_{\beta=0}$$

and

$$\hat{R}^c_{\mathbf{x},k}\,\psi = -\,\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\beta}\,\psi\left(\ldots, U_{\mathbf{x},k}\,e^{\mathrm{i}\,\beta\tau_c},\ldots\right)|_{\beta=0}\,,$$

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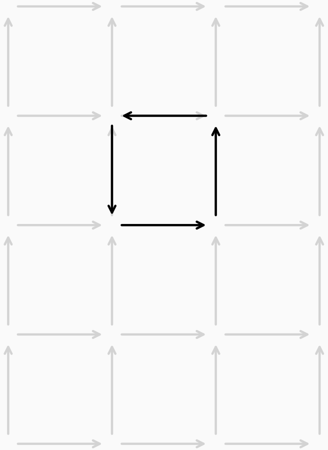

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Plaquette Operator:

$$\hat{P}_{\mathbf{x},ij} = \hat{U}_{\mathbf{x},i}\,\hat{U}_{\mathbf{x}+a\hat{\mathbf{i}},j}\,\hat{U}^\dagger_{\mathbf{x}+a\hat{\mathbf{j}},i}\,\hat{U}^\dagger_{\mathbf{x},j}$$

in terms of link operators

$$\hat{U}_{\mathbf{x},k}\,\psi = U_{\mathbf{x},k}\psi\left(\ldots, U_{\mathbf{x},k},\ldots\right)$$

and

$$\hat{U}^\dagger_{\mathbf{x},k}\,\psi = U^\dagger_{\mathbf{x},k}\psi\left(\ldots, U_{\mathbf{x},k},\ldots\right)\,.$$

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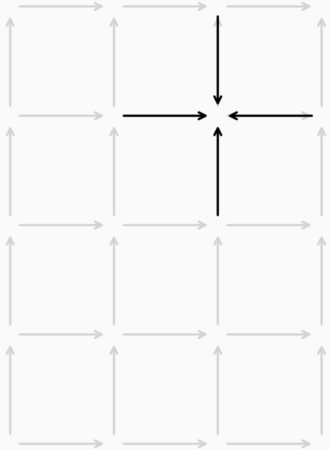

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Gauss Law for physical states:

$$\hat{G}^c_{\mathbf{x}} |\psi\rangle = \sum_k \left(\hat{L}^c_{\mathbf{x},k} + \hat{R}^c_{\mathbf{x}-a\hat{\mathbf{k}},k}\right) |\psi\rangle = 0$$

Enforced by adding

$$\hat{H}_{\mathrm{penalty}} = \kappa \sum_{\mathbf{x},c} \left(\hat{G}^c_{\mathbf{x}}\right)^2 \quad (1)$$

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Electric Basis Functions

Character / Clebsch-Gordon expansion:

- Eigenfunctions $|J, m_L, m_R\rangle$ of $\sum_c (\hat{L}_c)^2$, $\hat{L}_3$ and $\hat{R}_3$ known

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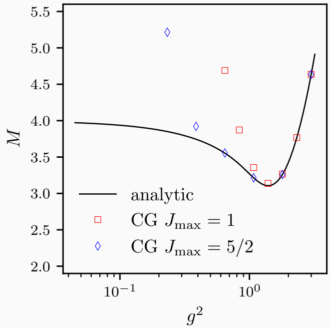

Mass gap M for a single plaquette as a function of g^2 using Clebsch-Gordon Operators

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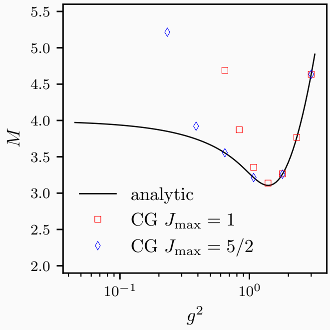
For $g^2 \to 0$ (Continuum limit):

$$\psi \to \prod_{\mathbf{x},jk} \delta\left(\mathbb{1} - P_{\mathbf{x},jk}\right)$$

$$\Rightarrow J_{\max} \to \infty$$



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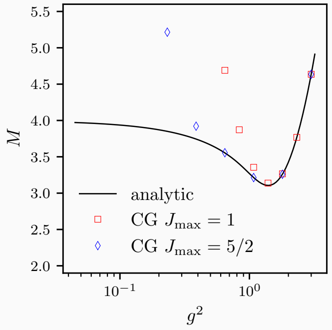
How to solve this:

1. Reformulate the KS-Hamiltonian s.t. the magnetic contributions become local
2. Choose a set of appropriate basis functions



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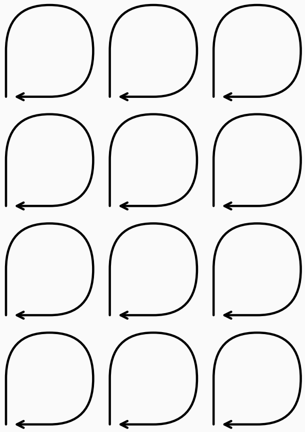
Magnetic Hamiltonian[2]



Plaquette links, labelled by $\mathbf{x}$

[2] Manu Mathur and Atul Rathor. *Exact duality and local dynamics in SU(N) lattice gauge theory*. 2023. arXiv: 2109.00992 [hep-lat]. URL: https://arxiv.org/abs/2109.00992.

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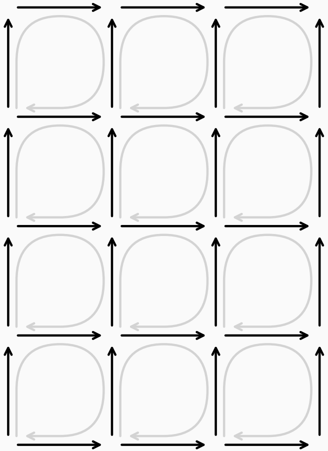

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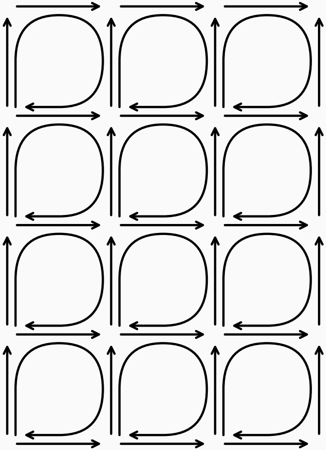

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$$\hat{H}_{\text{dual}} = g^2 \sum_{\mathbf{x},k} \text{Tr}\left[\left(\hat{\vec{L}}_{\mathbf{x},k} + \hat{\nabla}_k(U)E_{\mathbf{x}}\right)^2\right] + \frac{2}{g^2}\sum_{\mathbf{x}} \text{Tr}\left[\mathbb{1} - \text{Re}\,\hat{U}_{\mathbf{x}}\right]$$

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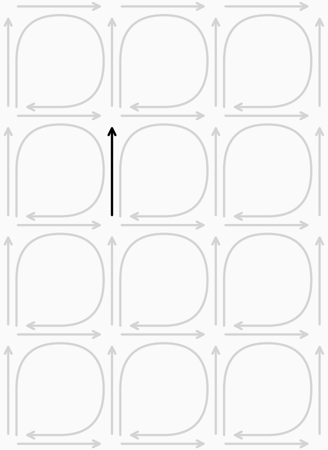

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$$\hat{\vec{L}} = \sum_c \hat{L}^c_{\mu,\mathbf{x}} \tau_c$$

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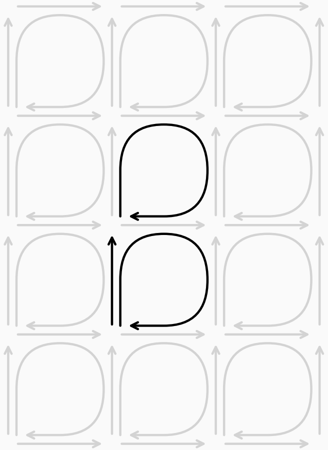

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With covariant derivatives

$$\nabla_1(U)E_{\mathbf{x}} = \hat{\vec{R}}_{\mathbf{x}} + \hat{U}^\dagger_{2,\mathbf{x}-\hat{e}_2}\hat{\vec{L}}_{\mathbf{x}-\hat{e}_2}\hat{U}_{2,\mathbf{x}-\hat{e}_2}$$

and

$$\nabla_2(U)E_{\mathbf{x}} = \hat{\vec{L}}_{\mathbf{x}} + \hat{U}^\dagger_{1,\mathbf{x}-\hat{e}_1}\hat{\vec{L}}_{\mathbf{x}-\hat{e}_1}\hat{U}_{1,\mathbf{x}-\hat{e}_1}\,.$$

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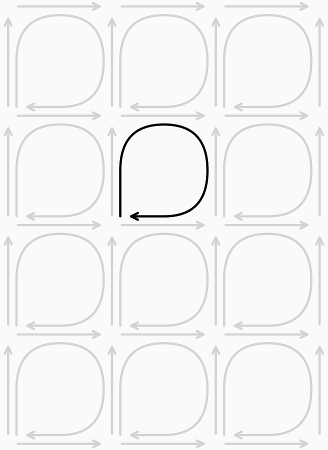

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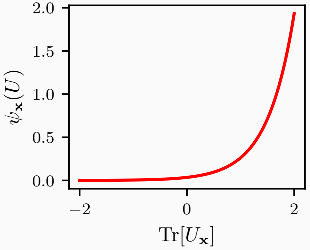
Plaquette link wave function expected to peak around $U_{\mathbf{x}} \approx \mathbb{1}$ for $g^2 \to 0$.



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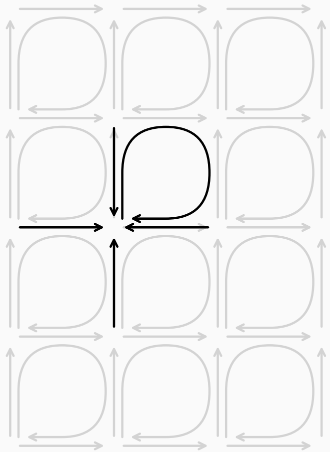

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Gauss Law:

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Discretisation

Plaquette Links Basis Functions

Parametrisation of SU(2):

$$U(\psi,\theta,\phi) = \cos(\psi)\mathbb{1} - 2\,\mathrm{i}\sin(\psi)\,\vec{n}(\theta,\phi)\cdot\vec{\tau}$$

where $$\vec{n}(\theta,\phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

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$$\Rightarrow \qquad \mathrm{Tr}\left[U\right] = 2\cos\psi$$

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$$B_{n,l,m} = C_n(\psi)\,Y_{l,m}(\theta,\phi)$$

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$$\int_{-\infty}^{\infty} \mathrm{d}u\, H_n(u) H_m(u)\,\mathrm{e}^{-u^2} \sim \delta_{n,m}$$

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$$C_n(\psi) \sim \underbrace{\frac{1}{\sin\psi}}_{\text{Correct for integration measure}} \qquad H_{2n+1}(u(\psi))\,\mathrm{e}^{-\frac{u^2(\psi)}{2}}$$

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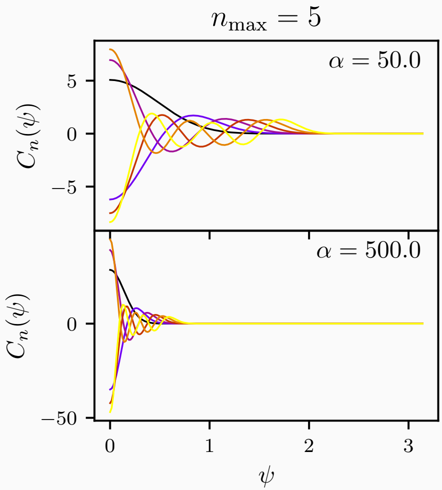
- α controls suppression away from $U = \mathbb{1}$
- only use odd Hermite polynomials to maintain orthogonality and avoid divergence for $\psi \to 0$

$$C_n(\psi) \sim \underbrace{\frac{1}{\sin\psi}}_{\text{Correct for integration measure}} \underbrace{\frac{1}{\sqrt{\pi^2-\psi^2}}}_{\text{correct for substitution}} H_{2n+1}(u(\psi))\,\mathrm{e}^{-\frac{u^2(\psi)}{2}}$$

Plaquette Link Radial Basis Functions



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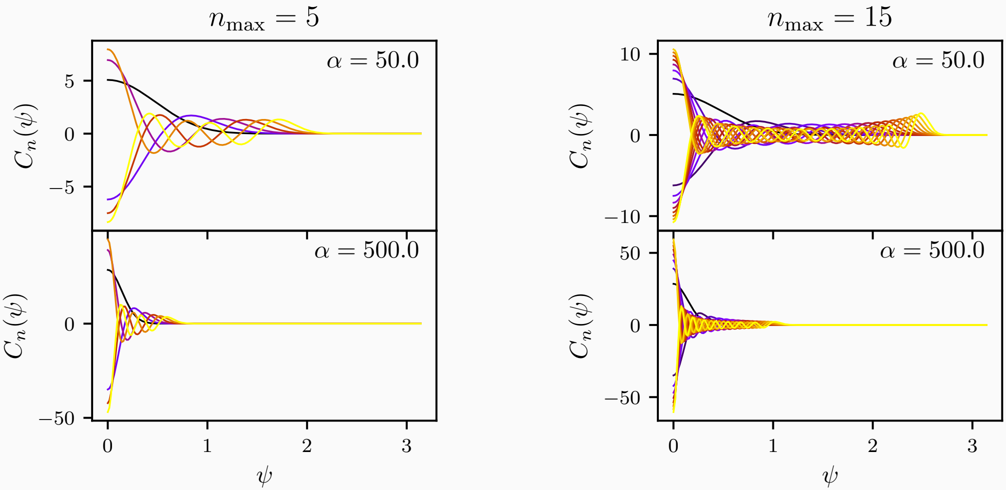

Plaquette Links

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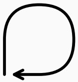
Helper Links

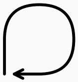
- No obvious constraint in terms of gauge link operators
  $\rightarrow$ Clebsch-Gordon operators
- Gauss Law suggests $J_{\max} \approx l_{\max}$

Results

Single Plaquette

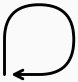


Single Plaquette



- Single degree of freedom
  $\Rightarrow$ exact diagonalization
- Gauss Law $\Rightarrow l_{\max} = 0$

Single Plaquette



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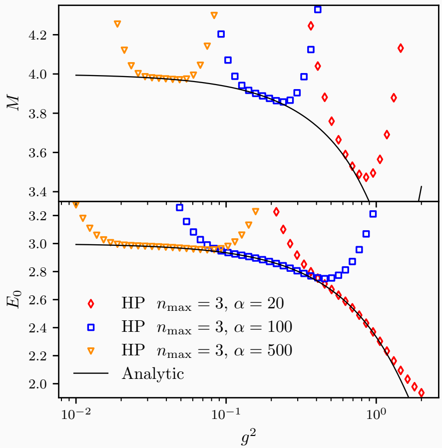

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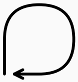

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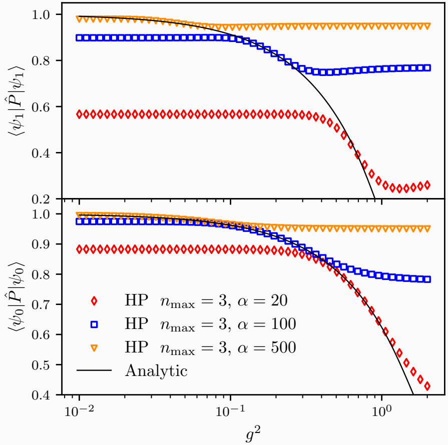
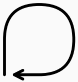
Plaquette Expectation Value



Single Plaquette



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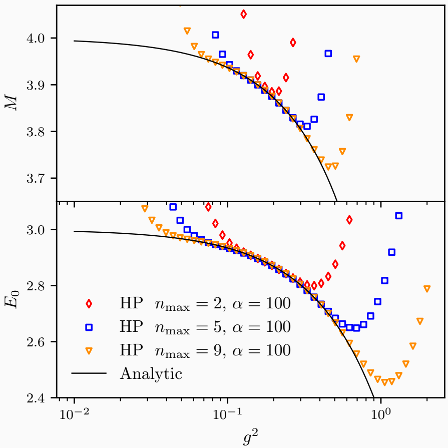
Convergence



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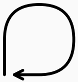

- Single degree of freedom
  $\Rightarrow$ exact diagonalization
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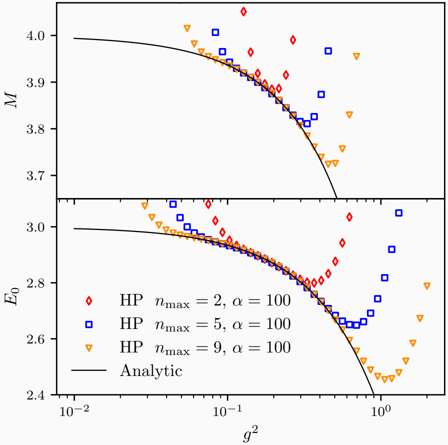
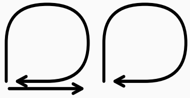
Takeaways:

- Good matching of ground and first excited state at appropriate α
- Convergence with $n_{\max} \to \infty$

Convergence



3×2 System, Open Boundaries



[3] N.E. Ligterink, N.R. Walet, and R.F. Bishop. "Toward a Many-Body Treatment of Hamiltonian Lattice SU(N) Gauge Theory". In: *Annals of Physics* 284 (2000). DOI: https://doi.org/10.1006/aphy.2000.6070.

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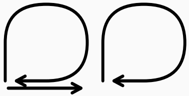

- Solved using ITensor MPS
- Simulations run at optimal α
- Compared to Max. Tree results [3]with CG operators

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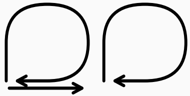

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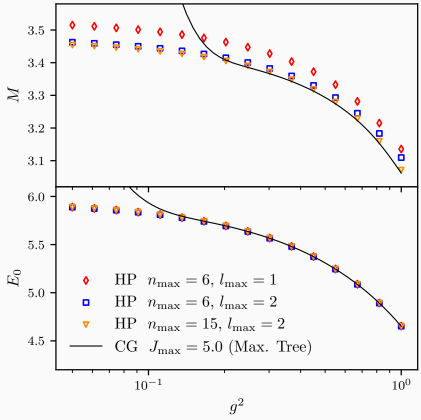

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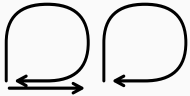

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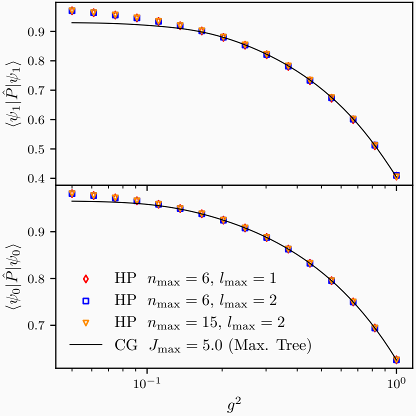

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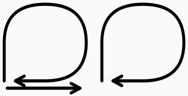

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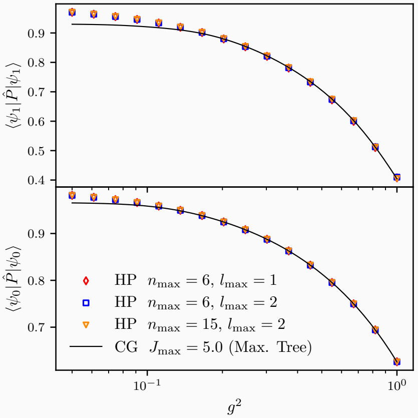
Takeaways:

- Convergence with $(n_{\max}, l_{\max}) \to \infty$
- Stable for $g^2 \to 0$

Plaquettes



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Outlook

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- bigger systems (and reference)
- add some fermions
- better algorithms
- more insightful observables
- eventually $\mathrm{SU}(3)$

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The End

Thanks for listening

The Question of Gauge Invariance

- Gauge Invariance $\Leftrightarrow [\hat{H}, \hat{G}^c_{\mathbf{x}}] = 0$
- $\hat{G}^c_{\mathbf{x}}$ only contains $\hat{L}^c_{\mathbf{x}} + \hat{R}^c_{\mathbf{x}}$
- It turns out that $\hat{L}^c_{\mathbf{x}} + \hat{R}^c_{\mathbf{x}} = \hat{\ell}^c$ where $\hat{\ell}^c$ are the orbital momentum operators only acting on $Y_{lm}(\theta, \phi)$
- This seems to lead to exact commutation relations for $[\hat{\ell}^c, \hat{L}^c]$, $[\hat{\ell}^c, \hat{R}^c]$ and $[\hat{\ell}^c, \sum_c (\hat{L}^c)^2]$
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- Clebsch Gordon Operators fullfill canonical commutation relations by construction
  $\Rightarrow$ Operators probably lead to a gauge invariant theory, but algebra is still under construction