A new method for calculating false vacuum decay rates on the lattice

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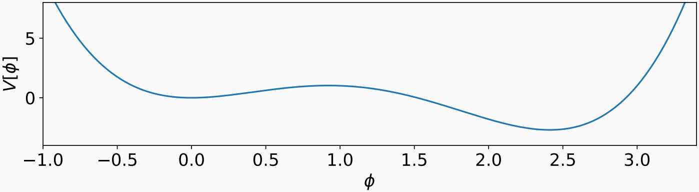
False Vacuum Decay

- Consider a field theory in which the potential has two local minima. e.g.

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu \phi(x))^2 + V[\phi(x)]$$



- The higher minimum gives a semi-stable “false vacuum” state.
- In the quantized theory, the false vacuum decays to the true vacuum.

Motivation

False vacuum decay is relevant to:

- The Standard Model (the vacuum may be unstable)
- The Schwinger mechanism (field decay through pair production)
- Cosmology (the original inflation model involved false vacuum decay)
- Condensed matter systems

Current Approaches

- The standard approach is the semi-classical approximation
- Recently, two lattice methods were proposed [1]
  - These methods were tested on 1D quantum mechanics with the hope of generalizing to field theory.
- Why a new approach?
  - Semi-classical approximation does not always work well.
  - The first lattice method struggles with small decay rates.
  - The second lattice method requires extrapolation to large volumes.
  - An new, independent lattice method could serve as a cross-check.

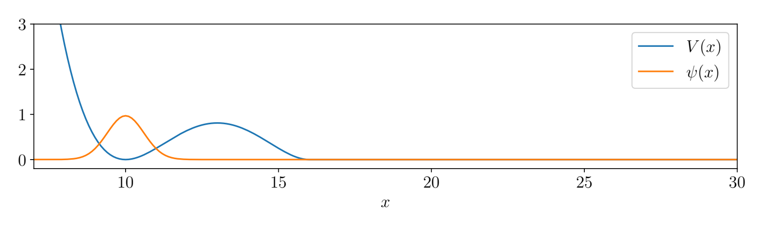
[1] Jiayu Shen, Patrick Draper, and Aida X. El-Khadra. Vacuum decay and Euclidean lattice Monte Carlo. Phys. Rev. D, 107(9):094506, 2023. doi: 10.1103/PhysRevD.107.094506.

Challenges of Using Monte Carlo for False Vacuum Decay

- **Sampling problem:** It is difficult to correctly sample configurations for Monte Carlo simulation.
- **Statistics problem:** Even with perfect sampling, the signal is exponentially suppressed.
- **Inverse problem:** To go from imaginary to real time, we need to do an inverse Laplace transformation.

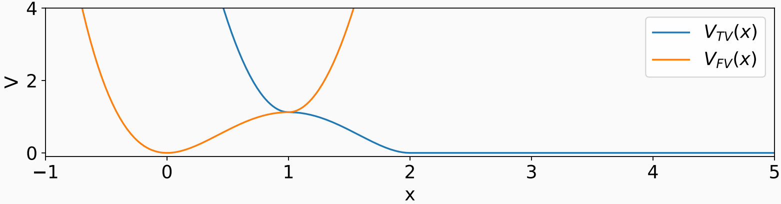
1D Quantum Mechanics Problem

- We will be dealing with a potential of the form shown below



Sampling Problem: Dynamic Constraints

- Consider the potentials below.
- The orange line is V_{FV} and is identical to V_{full} in the false vacuum.
- The blue line is V_{TV} and is identical to V_{full} in the true vacuum.

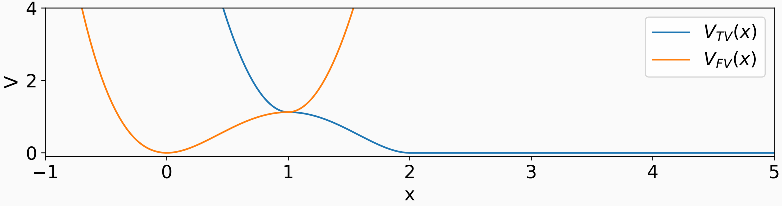


Sampling Problem: Dynamic Constraints

Procedure:

- Start with sum over all states.
- Evolve with $H_{\text{FV}} \equiv K + V_{\text{FV}}$ (projects false vacuum "ground state")
- Evolve with $H_{\text{full}} \equiv K + V_{\text{full}}$ (allows transitions)
- Evolve with $H_{\text{TV}} \equiv K + V_{\text{TV}}$ (projects configurations that transition)
- Evolve with $H_{\text{full}} \equiv K + V_{\text{full}}$ (allows transitions)

$$\text{Tr}\left[e^{-H_{\text{full}}t}e^{-H_{\text{TV}}t_{\text{TV}}}e^{-H_{\text{full}}t}e^{-H_{\text{FV}}t_{\text{FV}}}\right]$$



Sampling Problem: Ratios

- To get the decay rate, we need the ratio

$$\frac{\mathrm{Tr}\left[e^{-H_{\text{full}}t}e^{-H_{\text{TV}}t_{\text{TV}}}e^{-H_{\text{full}}t}e^{-H_{\text{FV}}t_{\text{FV}}}\right]}{\mathrm{Tr}\left[e^{-H_{\text{full}}t}e^{-H_{\text{FV}}t_{\text{TV}}}e^{-H_{\text{full}}t}e^{-H_{\text{FV}}t_{\text{FV}}}\right]}$$

- It is very difficult to directly calculate this ratio.
- However, we can split this ratio into sub-ratios:

$$\frac{\mathrm{Tr}\left[\cdots e^{-H_{\text{TV}}t_{\text{TV}}}\cdots\right]}{\mathrm{Tr}\left[\cdots e^{-H_{1}t_{\text{TV}}}\cdots\right]}\cdot\frac{\mathrm{Tr}\left[\cdots e^{-H_{1}t_{\text{TV}}}\cdots\right]}{\mathrm{Tr}\left[\cdots e^{-H_{2}t_{\text{TV}}}\cdots\right]}\cdots\frac{\mathrm{Tr}\left[\cdots e^{-H_{n}t_{\text{TV}}}\cdots\right]}{\mathrm{Tr}\left[\cdots e^{-H_{\text{FV}}t_{\text{TV}}}\cdots\right]}$$

- The intermediate Hamiltonians H_i transition smoothly between H_{TV} and H_{FV}.
- This solves **both** the sampling and statistics problems.

Inverse Problem: Fermi's Golden Rule

- We can relate the ratio

$$\frac{\mathrm{Tr}\left[e^{-H_{\mathrm{full}}t}e^{-H_{\mathrm{TV}}t_{\mathrm{TV}}}e^{-H_{\mathrm{full}}t}e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}\right]}{\mathrm{Tr}\left[e^{-H_{\mathrm{full}}t}e^{-H_{\mathrm{FV}}t_{\mathrm{TV}}}e^{-H_{\mathrm{full}}t}e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}\right]}$$

  to the decay rate using Fermi's Golden Rule (FGR).
- To use FGR, we need information on the energy spectrum.
- Finding the full spectrum is equivalent to the inverse problem.
- However, for FGR, we only need the spectrum at the energy of the false vacuum.
- This allows us to try simple ansatz to approximate the spectrum.
- This procedure works when the false vacuum energy is not too big.

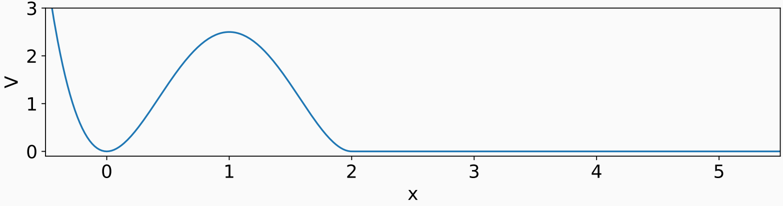
Results

We use the same action as Shen et al. [2023]

$$S = \int dt \left[\frac{1}{2}(\partial_t x)^2 + \begin{cases} \frac{1}{2} m^2 x^2 - \eta x^3 + \frac{\eta^2}{2m^2} x^4 & x < \frac{m^2}{\eta} \\ 0 & x > \frac{m^2}{\eta} \text{ (true vacuum)} \end{cases} \right]$$

$$\Longrightarrow$$

$$S = \beta \int dt \left[\frac{1}{2}(\partial_t x)^2 + \begin{cases} \frac{1}{2} x^2 - \frac{1}{2} x^3 + \frac{1}{8} x^4 & x < 2 \\ 0 & x > 2 \text{ (true vacuum)} \end{cases} \right]$$



Results

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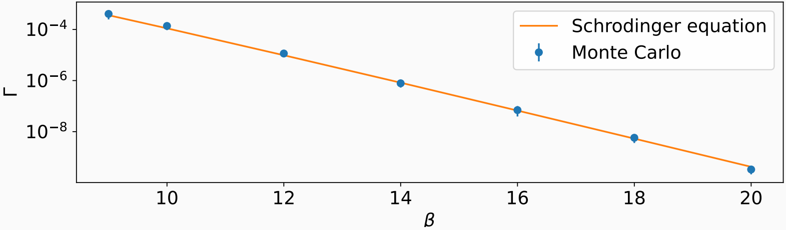

References

Jiayu Shen, Patrick Draper, and Aida X. El-Khadra. Vacuum decay and Euclidean lattice Monte Carlo. *Phys. Rev. D*, 107(9):094506, 2023. doi: 10.1103/PhysRevD.107.094506.

Appendix: Fermi's Golden Rule

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- One way to calculate decay rates is Fermi's golden rule:

$$\Gamma = \sum_m 2\pi\delta(E_m - E_{\mathrm{FV}})\left|\langle\phi_m|V|\varphi_{\mathrm{FV}}\rangle\right|^2$$

$$= 2\pi\langle\varphi_{\mathrm{FV}}|V\delta(H_{\mathrm{sep}} - E_{\mathrm{FV}})V|\varphi_{\mathrm{FV}}\rangle.$$

- H_{sep} is a Hamiltonian such that the FV and TV are separated.
- V is defined so that $H = H_{\mathrm{sep}} + V$.
- ϕ_m are eigenstates of H_{sep} in the TV.
- φ_{FV} is the lowest eigenstate of H_{sep} in the FV.

Appendix: Fermi's Golden Rule

- We can show that

$$\frac{\langle\psi|e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}e^{-Ht}e^{-H_{\mathrm{TV}}t_{\mathrm{TV}}}e^{-Ht}e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}|\psi\rangle}{\langle\psi|e^{-H_{\mathrm{FV}}(2t_{\mathrm{FV}}+2t+t_{\mathrm{TV}})}|\psi\rangle}$$

$$= \int_0^t dt_1 \int_0^t dt_2 \langle \mathrm{FV}|V^\dagger e^{-(H_{\mathrm{TV}}-E_{\mathrm{FV}})(t_{\mathrm{TV}}+t_1+t_2)}V|\mathrm{FV}\rangle$$

- We want to calculate

$$\langle\varphi_{\mathrm{FV}}|V\delta(H_{\mathrm{sep}}-E_{\mathrm{FV}})V|\varphi_{\mathrm{FV}}\rangle.$$

- Define

$$\rho(E) \equiv \int_0^t dt_1 \int_0^t dt_2 \langle \mathrm{FV}|V^\dagger \delta(H_{\mathrm{TV}}-E) e^{-(H_{\mathrm{TV}}-E_{\mathrm{FV}})(t_{\mathrm{TV}}+t_1+t_2)}V|\mathrm{FV}\rangle$$

- Then

$$\begin{aligned}\rho(E_{\mathrm{FV}}) &= \int_0^t dt_1 \int_0^t dt_2 \langle \mathrm{FV}|V^\dagger \delta(H_{\mathrm{TV}}-E_{\mathrm{FV}})V|\mathrm{FV}\rangle \\ &= t^2 \langle \mathrm{FV}|V^\dagger \delta(H_{\mathrm{TV}}-E_{\mathrm{FV}})V|\mathrm{FV}\rangle\end{aligned}$$

Appendix: Fermi's Golden Rule

- We want

$$\rho(E) = \frac{\langle\psi|e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}e^{-Ht}\delta(H_{\mathrm{TV}} - E)e^{-H_{\mathrm{TV}}t_{\mathrm{TV}}}e^{-Ht}e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}|\psi\rangle}{\langle\psi|e^{-H_{\mathrm{FV}}(2t_{\mathrm{FV}}+2t+t_{\mathrm{TV}})}|\psi\rangle}.$$

- But we can only calculate

$$\int dE\rho(E) = \frac{\langle\psi|e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}e^{-Ht}e^{-H_{\mathrm{TV}}t_{\mathrm{TV}}}e^{-Ht}e^{-H_{\mathrm{FV}}t_{\mathrm{FV}}}|\psi\rangle}{\langle\psi|e^{-H_{\mathrm{FV}}(2t_{\mathrm{FV}}+2t+t_{\mathrm{TV}})}|\psi\rangle}.$$

- But we can also calculate the moments

$$\int dE\ E\rho(E) \text{ and } \int dE\ E^2\rho(E).$$

- If we assume $\rho(E)$ has a Gaussian distribution, we can calculate it.