Loop-string-hadron approach to the SU(3) gauge invariant Hilbert space

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Motivation

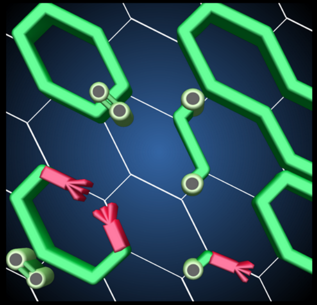
- Hamiltonian lattice gauge theory: Framework for quantum simulation and tensor networks
- Gauge symmetry → redundancy in description → multiple possible formulations possible/being considered for calculations
- For lattice QCD, a formulation must be adapted to *SU(3) gauge fields and 3+1 D*
- Gauge-invariant formulations are expected to offer advantages (but are not the only possibility)

SU(2) vertex

- Gauge singlets are associated with *vertices*, not links
- Trivalent (three legs) vertices: Minimal nontrivial setting for understanding gauge singlets
  - Natural in honeycomb discretization (2D)
  - Combine to make higher effective valency

SU(2) vertex

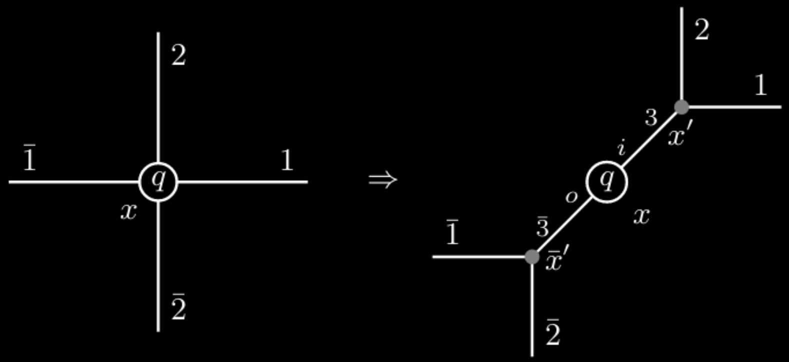
- Loop-string-hadron formulation of SU(2)
  - Derived from/based on Schwinger bosons (AKA prepotentials)
  - Elementary fields are strictly SU(2)-neutral
  - Local, with Abelian constraints
  - Developed for D=1+1, 2+1, 3+1, with or without staggered fermions
  - Hamiltonian is equivalent to Kogut-Susskind

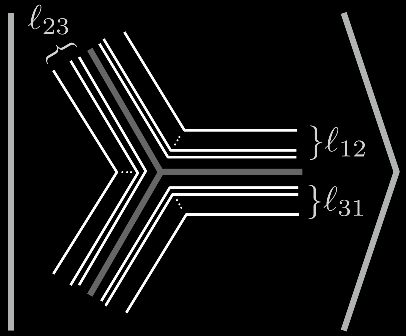


Raychowdhury, & JRS
Phys. Rev. D (2020)

SU(2) vertex

- Elementary building block of Yang-Mills theory is trivalent vertex
  - Four- and six-leg vertices achieved by "point splitting"
- Trivalent vertex is completely understood (orthonormal basis, and operator matrix elements)





$$|\ell_{12},\ell_{23},\ell_{31}\rangle \equiv \frac{(\mathcal{L}^{++}_{12})^{\ell_{12}}(\mathcal{L}^{++}_{23})^{\ell_{23}}(\mathcal{L}^{++}_{31})^{\ell_{31}}}{\sqrt{\ell_{12}!\ell_{23}!\ell_{31}!(\ell_{12}+\ell_{23}+\ell_{31}+1)!}}|0\rangle$$

$$\mathcal{L}^{++}_{12}|\ell_{12},\ell_{23},\ell_{31}\rangle = \sqrt{(\ell_{12}+1)(\ell_{12}+\ell_{23}+\ell_{31}+2)}\,|\ell_{12}+1,\ell_{23},\ell_{31}\rangle$$

$$\mathcal{L}^{+-}_{12}|\ell_{12},\ell_{23},\ell_{31}\rangle = -\sqrt{(\ell_{31}+1)\ell_{23}}\,|\ell_{12},\ell_{23}-1,\ell_{31}+1\rangle$$

Irreducible Schwinger bosons

- SU(2): Arbitrary irrep *j* constructible by tensor-producting enough spin-1/2's → One doublet to construct all |*j*,*m*> states

$$a^\dagger = \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}$$

- In SU(3): Arbitrary irrep (P,Q) constructible by tensor products of one 3 and one 3* →

$$a^\dagger = \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{pmatrix}, \quad b^\dagger = \begin{pmatrix} b^{\dagger 1} \\ b^{\dagger 2} \\ b^{\dagger 3} \end{pmatrix}$$

- Ex: 3 $\quad |1,0\rangle_\alpha = a_\alpha^\dagger \, |\Omega\rangle$
- Ex: 3* $\quad |0,1\rangle^\beta = b^{\dagger\beta} \, |\Omega\rangle$

Irreducible Schwinger bosons

- Ex: 8 $\quad a^\dagger_\alpha b^{\dagger\beta}|\Omega\rangle \in (1,1)?$ $\quad$ *No!...* $3\times 3^* = 8\oplus 1$ $\quad a^\dagger_\alpha b^{\dagger\alpha}|\Omega\rangle \in (0,0)$

  To be *irreducible*, the rep should be *traceless*

$$|1,1\rangle^\beta_\alpha \equiv a^\dagger_\alpha b^{\dagger\beta}|\Omega\rangle - \frac{1}{3}\delta^\beta_\alpha a^\dagger\cdot b^\dagger|\Omega\rangle\,, \qquad a^\dagger\cdot b^\dagger \equiv a^\dagger_\gamma b^{\dagger\gamma}$$

- One can generalize solution to all states/irreps, but hopeless to work with directly

- Solution: "irreducible Schwinger bosons"

Anishetty, Mathur, & Raychowdhury, J. Math. Phys. 50, 053503 (2009)

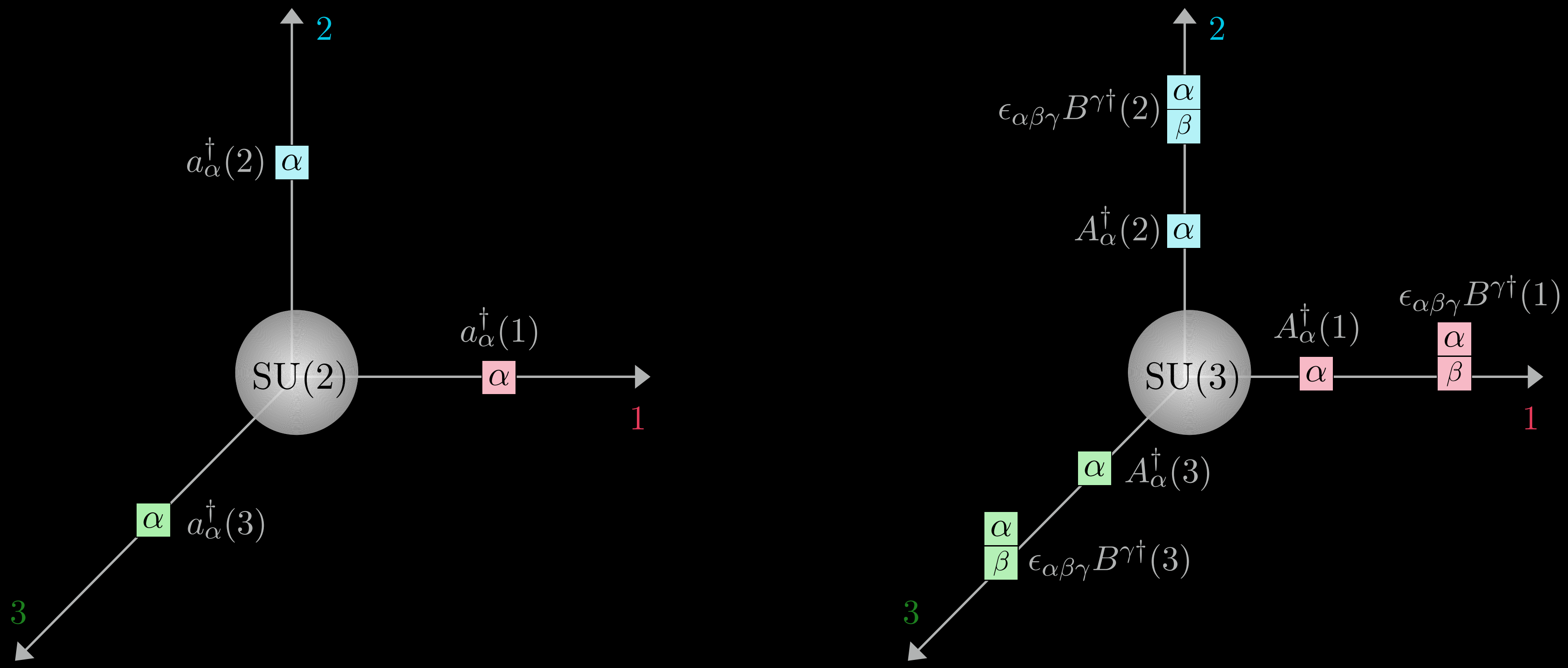
$$A^\dagger_\alpha \equiv a^\dagger_\alpha - \frac{1}{P+Q+1}(a^\dagger\cdot b^\dagger)b_\alpha,$$
$$B^{\dagger\alpha} \equiv b^{\dagger\alpha} - \frac{1}{P+Q+1}(a^\dagger\cdot b^\dagger)a^\alpha.$$

$$P\equiv a^\dagger\cdot a$$
$$Q\equiv b^\dagger\cdot b$$

With ISBs: $|1,1\rangle^\beta_\alpha \equiv A^\dagger_\alpha B^{\dagger\beta}|\Omega\rangle$

All irrep states have this 'monomial' form

Irreducible Schwinger bosons



SU(3) in 1+1

- Using ISBs, a direct generalization of 1+1 D follows
- Analytic understanding is on par with SU(2) theory

Kadam, Raychowdhury, & JRS
Phys. Rev. D (2023)

$$[A^\dagger(\underline{1}) \cdot B^\dagger(1)]^{n_P} \rightarrow$$

$$[B^\dagger(\underline{1}) \cdot A^\dagger(1)]^{n_Q} \rightarrow$$

$$|n_P, n_Q\rangle \propto |n_P, n_Q; 0, 0, 0\rangle \rightarrow$$

$$\psi^\dagger \cdot B^\dagger(1)\, |n_P, n_Q\rangle \propto |n_P, n_Q; 0, 0, 1\rangle \rightarrow$$

$$\psi^\dagger \cdot B^\dagger(\underline{1})\, |n_P, n_Q\rangle \propto |n_P, n_Q; 1, 0, 0\rangle \rightarrow$$

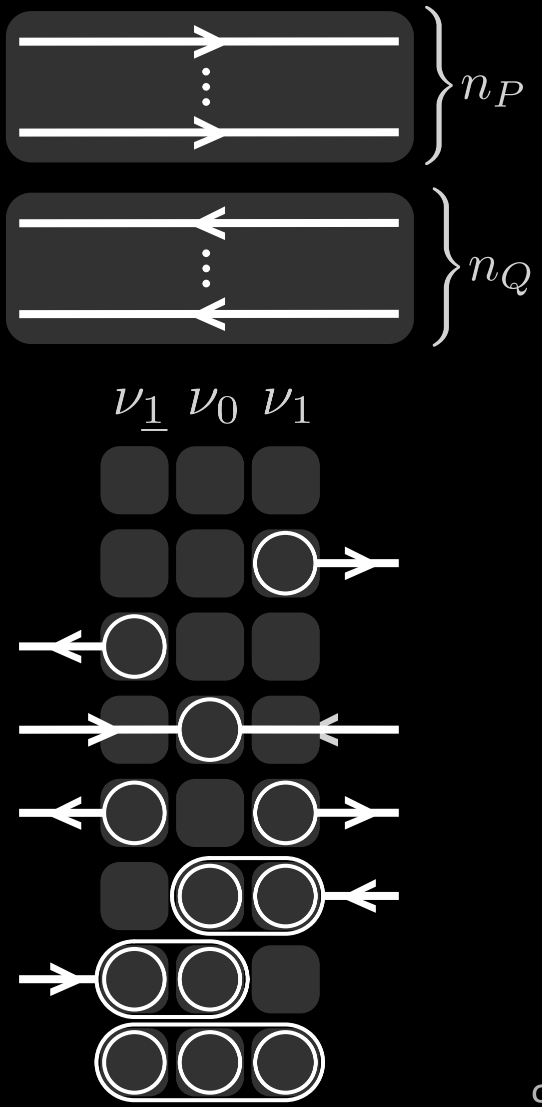
$$\psi^\dagger \cdot A^\dagger(\underline{1}) \wedge A^\dagger(1)\, |n_P, n_Q\rangle \propto |n_P, n_Q; 0, 1, 0\rangle \rightarrow$$

$$\psi^\dagger \cdot B^\dagger(\underline{1})\psi^\dagger \cdot B^\dagger(1)\, |n_P, n_Q\rangle \propto |n_P, n_Q; 1, 0, 1\rangle \rightarrow$$

$$\psi^\dagger \cdot \psi^\dagger \wedge A^\dagger(1)\, |n_P, n_Q\rangle \propto |n_P, n_Q; 0, 1, 1\rangle \rightarrow$$

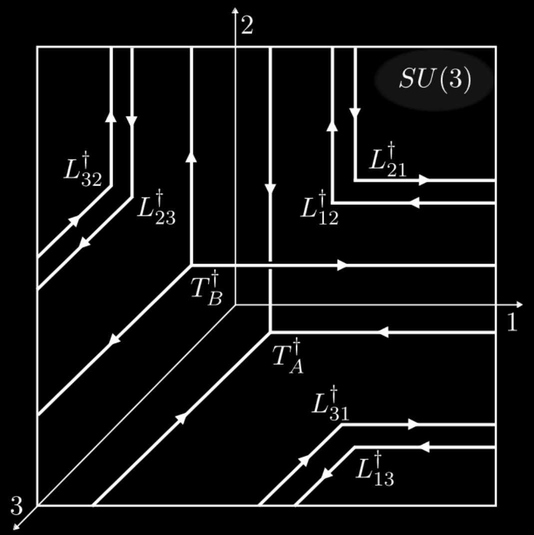
$$\psi^\dagger \cdot \psi^\dagger \wedge A^\dagger(\underline{1})\, |n_P, n_Q\rangle \propto |n_P, n_Q; 1, 1, 0\rangle \rightarrow$$

$$\psi^\dagger \cdot \psi^\dagger \wedge \psi^\dagger\, |n_P, n_Q\rangle \propto |n_P, n_Q; 1, 1, 1\rangle \rightarrow$$



Naive LSH basis

- Creation operators are constructed analogously
- SU(3) admits trilinear excitations



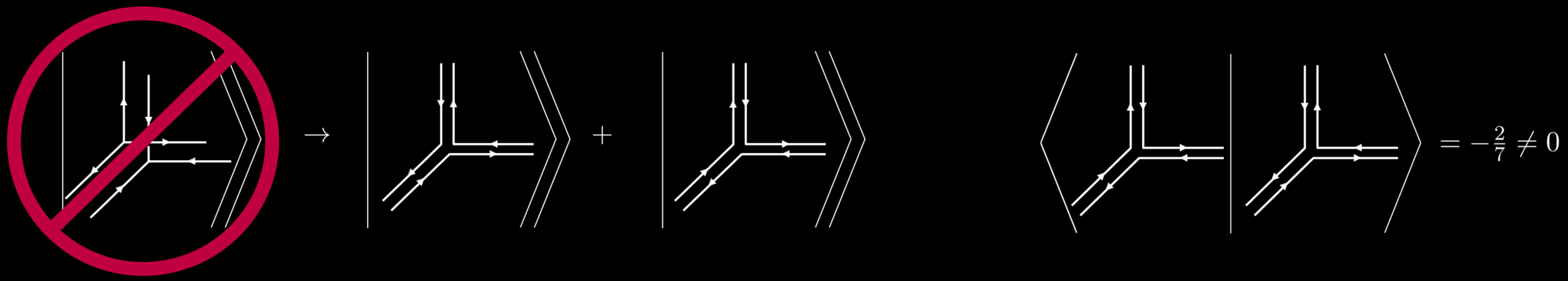
$$
\begin{aligned}
L_{12}^\dagger &= A_\alpha^\dagger(1) B^{\dagger\alpha}(2) & L_{21}^\dagger &= A_\alpha^\dagger(2) B^{\dagger\alpha}(1) \\
L_{23}^\dagger &= A_\alpha^\dagger(2) B^{\dagger\alpha}(3) & L_{32}^\dagger &= A_\alpha^\dagger(3) B^{\dagger\alpha}(2) \\
L_{31}^\dagger &= A_\alpha^\dagger(3) B^{\dagger\alpha}(1) & L_{13}^\dagger &= A_\alpha^\dagger(1) B^{\dagger\alpha}(3) \\
T_A^\dagger &= \epsilon^{\alpha\beta\gamma} A_\alpha^\dagger(1) A_\beta^\dagger(2) A_\gamma^\dagger(3) & T_B^\dagger &= \epsilon_{\alpha\beta\gamma} B^{\dagger\alpha}(1) B^{\dagger\beta}(2) B^{\dagger\gamma}(3)
\end{aligned}
$$

$$
|\ell_{12}\,\ell_{23}\,\ell_{31}; \ell_{21}\,\ell_{32}\,\ell_{13}; t\rangle\rangle \equiv L_{12}^{\dagger\,\ell_{12}} L_{23}^{\dagger\,\ell_{23}} L_{31}^{\dagger\,\ell_{31}} L_{21}^{\dagger\,\ell_{21}} L_{32}^{\dagger\,\ell_{32}} L_{13}^{\dagger\,\ell_{13}} \times \begin{cases} T_A^{\dagger\,t}\,|0\rangle\,, & t \geq 0 \\ T_B^{\dagger\,-t}\,|0\rangle\,, & t < 0 \end{cases}
$$

$$
\ell_{IJ} \in \{0,\ 1,\ 2,\ 3,\ \ldots\},
$$
$$
t \in \{0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots\}
$$

Naive LSH basis

- Problem: { l_{12}, l_{23}, l_{31}, l_{21}, l_{32}, l_{13}, t } not always "good" quantum numbers
- Interesting things happen in sector $\overrightarrow{p\,q} = (p_1, q_1, p_2, q_2, p_3, q_3) = (1,1,1,1,1,1)$



$$T_A^\dagger T_B^\dagger |0\rangle = |\ell_{12} = \ell_{23} = \ell_{31} = 1\rangle\rangle + |\ell_{21} = \ell_{32} = \ell_{13} = 1\rangle\rangle$$

- Irreps are insufficient to fully characterize a general state

Littlewood-Richardson coefficients

$$\lambda \otimes \mu = \bigoplus_{\nu} d^{\nu}_{\lambda,\mu}\, \nu$$

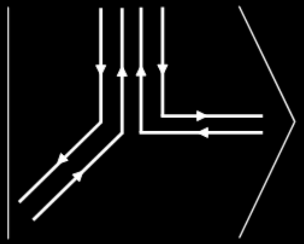
$$d^{(1,1)}_{(1,1),(1,1)} = 2$$

For SU(2), Littlewood-Richardson coefficients (LRCs) are either 0 or 1

For SU(3), LRCs can be larger than 1 → Extra, seventh d.o.f.

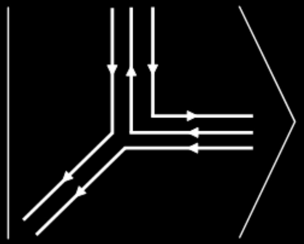
Nondegenerate states

- When the LRC is equal to one, there is just one LSH state, and no orthogonality problem
- Can sort such states into two inequivalent classes



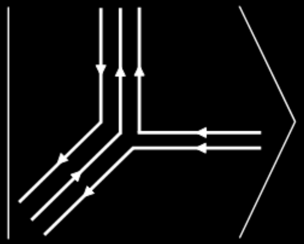
$= |\ell_{12} = \ell_{23} = \ell_{21} = \ell_{32} = 1\rangle$

Class I



$= |\ell_{12} = \ell_{23} = \ell_{21} = \ell_{13} = 1\rangle$

Class IIa



$= |\ell_{12} = \ell_{23} = \ell_{32} = \ell_{13} = 1\rangle$

Class IIb

Nondegenerate states

- Without overlap problem, it is known how to normalize the states in closed form

Class I:

$$\langle\langle \ell_{IJ}, \ell_{JK}, \ell_{JI}, \ell_{KJ}, t | \ell_{IJ}, \ell_{JK}, \ell_{JI}, \ell_{KJ}, t \rangle\rangle$$
$$= \tfrac{1}{2}(\ell_{IJ} + \ell_{JI} + \ell_{JK} + \ell_{KJ} + |t| + 2)\ell_{IJ}!\,\ell_{JK}!\,\ell_{JI}!\,\ell_{KJ}!\,|t|!\,(\ell_{IJ} + \ell_{KJ} + |t| + 1)!\,(\ell_{JK} + \ell_{JI} + |t| + 1)!\,.$$

Classes IIa:

$$\langle\langle \ell_{IJ}, \ell_{JK}, \ell_{JI}, \ell_{IK}, t | \ell_{IJ}, \ell_{JK}, \ell_{JI}, \ell_{IK}, t \rangle\rangle$$
$$= \frac{1}{2} \frac{(\ell_{IJ}+\ell_{JK}+\ell_{JI}+\ell_{IK}+|t|+2)\,\ell_{IJ}!\,\ell_{JK}!\,\ell_{JI}!\,\ell_{IK}!\,|t|!\,(\ell_{IJ}+\ell_{IK}+|t|+1)!\,(\ell_{JK}+\ell_{JI}+|t|+1)!\,\binom{\ell_{IJ}+\ell_{JK}+\ell_{JI}+\ell_{IK}+|t|+1}{\ell_{IK}}}{\binom{\ell_{IJ}+\ell_{JI}+\ell_{IK}+|t|+1}{\ell_{IK}}}$$

Degenerate subspaces

- When LRC > 1, multiple LSH states exist in a sector, and fail to be orthogonal (overlap matrices)
- Counting LSH states provides a way to evaluate SU(3) LRCs
- Normalization becomes much harder (still maybe possible)
- Orthogonal basis is even less obvious

$\overrightarrow{p\,q} = (1,1,1,1,1,1)$:

$$\begin{pmatrix} \langle\langle 1\,1\,1; 0\,0\,0; 0| \\ \langle\langle 0\,0\,0; 1\,1\,1; 0| \end{pmatrix} \Big(|1\,1\,1; 0\,0\,0; 0\rangle\rangle \,,\, |0\,0\,0; 1\,1\,1; 0\rangle\rangle \Big) = \begin{pmatrix} \frac{56}{3} & \frac{-16}{3} \\ \frac{-16}{3} & \frac{56}{3} \end{pmatrix}$$

Orthogonalization

- Gram-Schmidt always possible, but
  - No insight into seventh d.o.f.
  - Not analytically solvable
- Alternate solution: Define a “seventh Casimir” operator
  - Should commutes with (p_i,q_i)
  - Hermitian with nondegenerate spectrum → Eigenbasis is orthogonal
  - One choice:

$$C_T \equiv (T_A T_B)^\dagger T_A T_B .$$

$$\mathrm{Spec}_{111111}(C_T) = \left\{0, \frac{80}{3}\right\},$$

$$\begin{pmatrix} |\phi_1\rangle\rangle_{111111} \\ |\phi_2\rangle\rangle_{111111} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |1\,1\,1;0\,0\,0;0\rangle\rangle \\ |0\,0\,0;1\,1\,1;0\rangle\rangle \end{pmatrix}.$$

Status

- Gram-Schmidt bases are the cheapest to generate
- Choice of seventh Casimir is not final
- Ideally: We find a seventh Casimir whose eigenstates can be constructed analytically
  - Ex: Some “ladder” operator applied to a reference state, like with SU(2) irreps
- Looking at matrix elements of LSH operators in the different bases

Summary

- SU(3) gauge invariant basis can be constructed by direct analogy with SU(2)
- One does not need any Clebsch-Gordon coefficients
- For certain choices of irreps, the states are on par with SU(2) theory
- Subtleties arise for other choices of irreps
  - Basis is linearly independent, but not orthogonal
  - These states are the main obstacle to putting SU(3) completely on par with SU(2)

Acknowledgments

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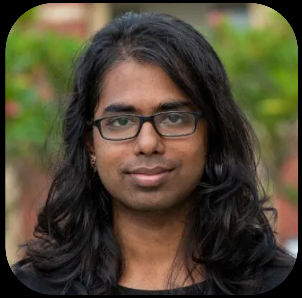

Aahiri Naskar (BITS-Pilani)

LSH talk by I. Raychowdhury (15:55 Monday)
LSH talk by Emil Mathew (**15:35 TODAY!**)

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