Effective mass-improvement of heavy valence Wilson quarks

P. Fritzsch, J. Heitger, S. Kuberski

Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

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THE PROBLEM

SymEFT sea fermion action (on-shell)

$$\mathcal{L}_{\mathrm{Sym,F}} = \bar{\psi} \not{D} \psi + \bar{\psi} M \psi + \rho \, \mathrm{Tr}[M] \, \bar{\psi}\psi$$
$$+ a\bar{\psi}\big[\sigma_0 \cdot \sigma_{\mu\nu} F_{\mu\nu} + \sigma_1 \cdot M^2 + \sigma_2 \cdot \mathrm{Tr}[M] \, M + \sigma_3 \cdot \mathrm{Tr}[M^2] + \sigma_4 \cdot (\mathrm{Tr}[M])^2 \big]\psi$$
$$+ \mathrm{O}(a^2)$$

Complicated renormalisation and improvement pattern:

- different renorm. for flavour singlet and non-singlet components of M
  $\Rightarrow$ finite renormalisation ρ at $O(1)$
- +1 mass-independent counterterm ($\sigma_0 \sim c_{\mathrm{sw}}$)
- +4 massive counterterms ($\sigma_{1,2,3,4}$)

- *6 EFT parameters to achieve $\mathrm{O}(a)$-improvement of massive fermion action*

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\begin{aligned}
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&+ \mathrm{O}(a^2)
\end{aligned}
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Let's focus on valence sector at $M = 0$.

SymEFT valence fermion action (on-shell)

$$\mathcal{L}_{\mathrm{Sym,val}} = \bar{\psi}_v \left[\not{D} + a\sigma_0 \sigma_{\mu\nu} F_{\mu\nu} \right] \psi_v + \bar{\psi}_v \psi_v \cdot m_{\mathrm{q}} [1 + b_{\mathrm{m}} a m_{\mathrm{q}}] + \mathrm{O}(a^2)$$

bare subtracted valence quark mass: $m_{\mathrm{q}} = m_0 - m_{\mathrm{cr}}$ where $m_{\mathrm{cr}} = m_{0,\mathrm{sea}}$, $m_0 = 1/(2\kappa) - 4$

- Only 1 improvement coefficient b_{m} $\Rightarrow \mathrm{O}(a)$-improved mass $\tilde{m}_{\mathrm{q}} = m_{\mathrm{q}}[1 + b_{\mathrm{m}} a m_{\mathrm{q}}]$
- b_{m} non-perturbatively determinable
  $N_{\mathrm{f}} = 0$: [hep-lat/0009021], $N_{\mathrm{f}} = 2$: [1004.3978], $N_{\mathrm{f}} = 3$: [1906.03445], ...

SymEFT valence fermion action (on-shell)

$$\mathcal{L}_{\mathrm{Sym,val}} = \bar{\psi}_v\left[\not{D} + a\sigma_0\sigma_{\mu\nu}F_{\mu\nu}\right]\psi_v + \bar{\psi}_v\psi_v \cdot m_\mathrm{q}[1 + b_\mathrm{m}am_\mathrm{q} + \xi_2(am_\mathrm{q})^2 + \xi_3(am_\mathrm{q})^3 + \xi_4(am_\mathrm{q})^4 + \ldots] + \mathrm{O}(a^2)$$

Basis operator $\bar{\psi}_v\psi_v$ reappears at higher orders in a!

- single relevant operator for quark mass $\Rightarrow \tilde{m}_\mathrm{q} = m_\mathrm{q}[1 + b_\mathrm{m}am_\mathrm{q} + \xi_2(am_\mathrm{q})^2 + \xi_3(am_\mathrm{q})^3 + \ldots]$
  (mass improvement of valence quark propagator)
- Theory & SymEFT parameters $\xi_i \in \{b_\mathrm{m}, \xi_2, \xi_3, \xi_4, \ldots\}$ defined using massless scheme.
  - This is a (convenient) choice!
  - Order by order determination in am_q
  - Efficiency depends on value of am_q.
  - Heavy quarks require $0.1 \lesssim am_\mathrm{q} \lesssim 0.5$ $\Rightarrow$ higher orders important!
- Non-pert. determination of b_m contaminated by higher orders until $am_\mathrm{q} \to 0$ („ambiguities“)

Chiral symmetry breaking effects of Wilson fermions

Non-perturbative renormalisation & improvement

SymEFT valence fermion action (on-shell)

$$\mathcal{L}_{\mathrm{Sym,val}} = \bar{\psi}_v\big[\not{D} + a\sigma_0\sigma_{\mu\nu}F_{\mu\nu}\big]\psi_v + \bar{\psi}_v\psi_v \cdot m_{\mathrm{q}}[1 + b_{\mathrm{m}}(am_{\mathrm{q}})am_{\mathrm{q}}] + \mathrm{O}(a^2)$$

Let's turn interpretation around and define

$$b_{\mathrm{m}}(am_{\mathrm{q}}) \equiv b_{\mathrm{m}}(0) + \xi_2(am_{\mathrm{q}}) + \xi_3(am_{\mathrm{q}})^2 + \xi_4(am_{\mathrm{q}})^3 + \ldots$$

- Provides a massive valence scheme!
- Determining $b_{\mathrm{m}}(am_{\mathrm{q}})$ at relevant am_{q} cancels higher orders too.
- No need to determine $\xi_2, \xi_3, \xi_4, \ldots$
- Requires direct method (no fitting)

THE PROCEDURE

Axial ward identity

Well-established method in new shape

match renormalised & improved subtracted and current quark masses

subtracted quark mass (flavour i)

$$m_{i,\mathrm{R}}(\mu) = Z_{\mathrm{m}}(\mu)\,[\,1 + b_{\mathrm{m}} a m_{\mathrm{q},i}]\, m_{\mathrm{q},i} + \mathrm{O}(a^2)$$

current quark mass (flavour i, j)

$$m_{ij,\mathrm{R}}(\mu) = \frac{Z_{\mathrm{A}}}{Z_{\mathrm{P}}(\mu)} \frac{1 + b_{\mathrm{A}}\, a m_{\mathrm{q},ij}}{1 + b_{\mathrm{P}}\, a m_{\mathrm{q},ij}} \cdot m_{ij} + \mathrm{O}(a^2)\ ,$$

$$\tfrac{1}{2}[m_{i,\mathrm{R}}(\mu) + m_{j,\mathrm{R}}(\mu)] \equiv m_{ij,\mathrm{R}}(\mu)$$

$$m_{ij} = Z\,\{m_{\mathrm{q},ij} + a b_{\mathrm{m}} \tfrac{1}{2}(m_{\mathrm{q},i}^2 + m_{\mathrm{q},j}^2) - a(b_{\mathrm{A}} - b_{\mathrm{P}}) m_{\mathrm{q},ij}^2\} + \mathrm{O}(a^2)\ , \quad Z = \frac{Z_{\mathrm{m}}(a\mu) Z_{\mathrm{P}}(a\mu)}{Z_{\mathrm{A}}}$$

3 unknowns $Z,\, b_{\mathrm{m}},\, b_{\mathrm{AP}} = b_{\mathrm{A}} - b_{\mathrm{P}}$ $\Leftrightarrow$ 3 masses $0 = m_{\mathrm{q},1},\ m_{\mathrm{q},2}$ (free), $m_{\mathrm{q},3} = \frac{1}{2}(m_{\mathrm{q},1} + m_{\mathrm{q},2})$

Calculate estimators

$$b_{\mathrm{m}}(g_0^2, a\Delta) \equiv \frac{2\,(m_{12} - m_{33})}{(m_{22} - m_{11})\,a\Delta}\ ,$$

$$b_{\mathrm{AP}}(g_0^2, a\Delta) \equiv \frac{2m_{12} - m_{11} - m_{22}}{(m_{22} - m_{11})\,a\Delta}\ ,$$

$$Z(g_0^2, a\Delta) \equiv \frac{m_{22} - m_{11}}{2\Delta} + \left(b_{\mathrm{AP}}(g_0^2, a\Delta) - b_{\mathrm{m}}(g_0^2, a\Delta)\right)(am_{11} + am_{22})\ .$$

where

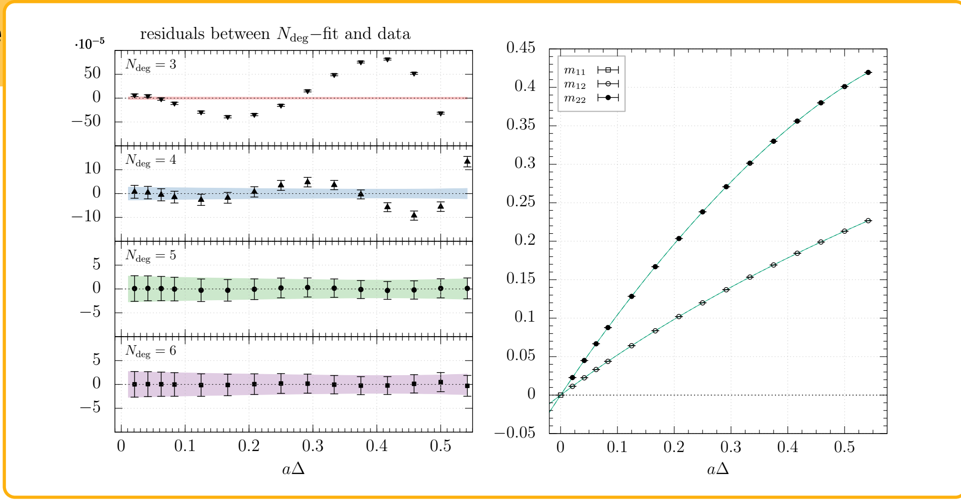
$$am_{12}(a\Delta) = am_{11} + \ N_1 a\Delta + \ N_2(a\Delta)^2 + \cdots + \ N_n(a\Delta)^n\ ,$$
$$am_{22}(a\Delta) = am_{11} + 2N_1 a\Delta + D_2(2a\Delta)^2 + \cdots + D_n(2a\Delta)^n\ .$$

to smoothly represent PCAC quark masses and estimators in terms of single param. $\Delta = \frac{1}{2}(m_{\mathrm{q},2} - m_{\mathrm{q},1})$.

Axial ward ide

Well-established method

Calculate estimators



$$b_{\text{m}}(g_0^2, a\Delta) \equiv$$

$$b_{\text{AP}}(g_0^2, a\Delta) \equiv$$

$$Z(g_0^2, a\Delta) \equiv$$

where

$$am_{12}(a\Delta) = am_{11} + N_1 a\Delta + N_2(a\Delta)^2 + \cdots + N_n(a\Delta)^n \,,$$

$$am_{22}(a\Delta) = am_{11} + 2N_1 a\Delta + D_2(2a\Delta)^2 + \cdots + D_n(2a\Delta)^n \,.$$

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SF simulations in $L_0 \approx 0.25\,\mathrm{fm}$ and $L_1 = 2L_0$

Light-quark line of constant physics (sea quark sector): $\bar{g}^2_{\mathrm{GF}}(\mu_0) = 3.949$, $L_0 m_{11} = 0$

LCP tuning very accurate, Z_A, Z_P known to high precision

$\frac{L_0}{a}$	β	κ_1	$\bar{g}^2_{\mathrm{GF}}(L_0)$	$L_0 m_{11}$	$Z_\mathrm{A}(g_0^2)$	$Z_\mathrm{P}(g_0^2, a/L_0)$	N_{cfg}	$\frac{\tau_{\mathrm{ms}}}{\mathrm{MD}}$
12	4.3030	0.1359947	3.9461(43)	−0.00024(33)	0.831798(50)	0.57835(32)	9669	8
16	4.4662	0.1355985	3.9475(58)	+0.00043(30)	0.838819(53)	0.56972(45)	5887	10
20	4.6017	0.1352848	3.9493(63)	+0.00100(20)	0.844883(25)	0.56502(53)	8478	10
24	4.7165	0.1350181	3.9492(62)	+0.00012(17)	0.849695(26)	0.56002(48)	7303	16
32	4.9000	0.1345991	3.9490(97)	+0.00562(15)	0.857020(22)	0.55390(70)	5014	20

$0.0078 \leq a/\mathrm{fm} \leq 0.0208$ $\Rightarrow$ sufficiently fine for relativistic charm & bottom

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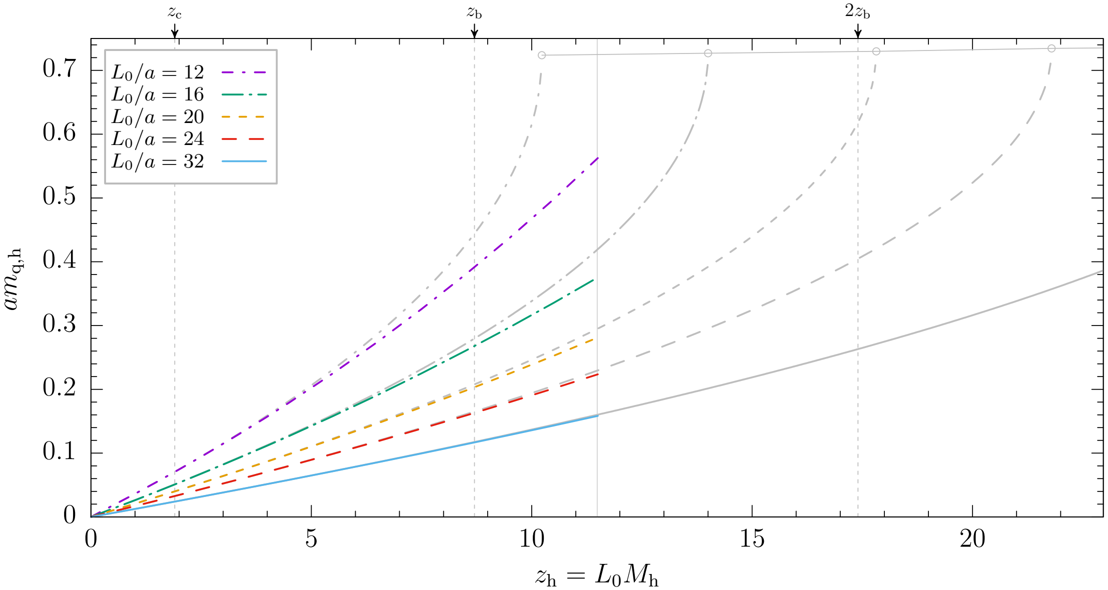
Heavy LCP (fix dimensionless RGI quark mass of heavy valence flavour)

$$z_{\mathrm{h}} = L\,h(\mu)\,m_{\mathrm{h,R}}(\mu)\,, \quad m_{\mathrm{h,R}}(\mu) = \lim_{a\to 0}\left[\frac{Z_{\mathrm{A}}(g_0^2)}{Z_{\mathrm{P}}(g_0^2, a\mu)} Z(g_0^2, a\Delta_{\mathrm{h}})\left\{1 + b_{\mathrm{m}}(g_0^2, a\Delta_{\mathrm{h}}) a m_{\mathrm{q,h}}\right\} m_{\mathrm{q,h}}\right]$$

– Pure theory relationship, no input from spectrum yet!

– Δ_{h} fixes scheme: $\Delta_{\mathrm{h}} = 0$ (massless), or $\Delta_{\mathrm{h}} = \Delta_{\mathrm{c}}, \Delta_{\mathrm{b}}$ (massive) from $z_{\mathrm{c}} = 1.9$, $z_{\mathrm{b}} = 8.7$

Bare mass parameters (massless vs massive scheme)

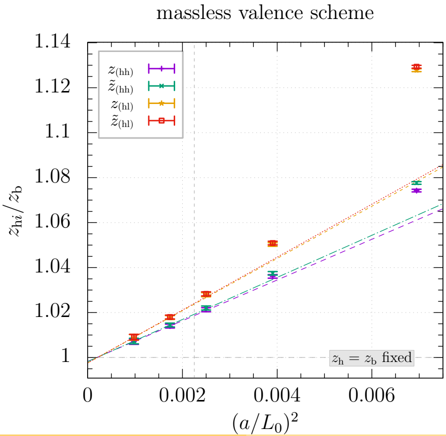


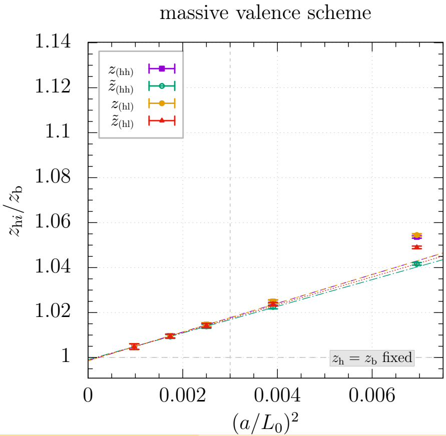
Tests

Test I, current quark masses (in L_0)

PCAC relation & b_{AP}

$$m_{ij,R}(\mu) = \frac{Z_A}{Z_P(\mu)}\,(1 + b_{AP}(a\Delta)\,am_{q,ij}) \cdot m_{ij}\ , \qquad z_{(hi)} = 2z_{hi} - z_{ii} = z_h + O(a^2)\ ,$$





Test II, effective masses (in $L_1 = 2L_0$)

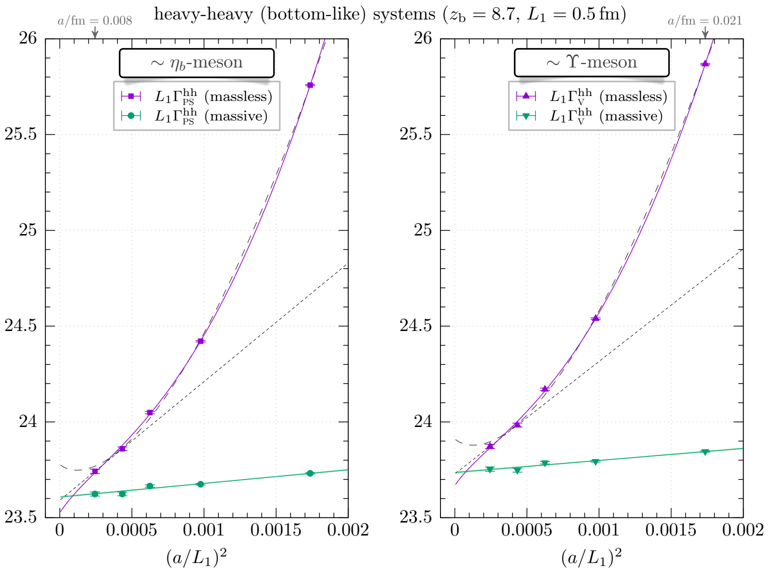
Bottomium system

massless scheme

- significant $\mathrm{O}(a^{n\geq 2})$ effects
- 10% effect at coarsest a/L
- 2-pt linear extrapolation ✓
- other extrapolations fail ✗

massive scheme

- pure $\mathrm{O}(a^2)$ scaling observed
- 0.6% effect at coarsest a/L
- a^2-effects significantly reduced



Test III, effective masses (in $L_1 = 2L_0$)

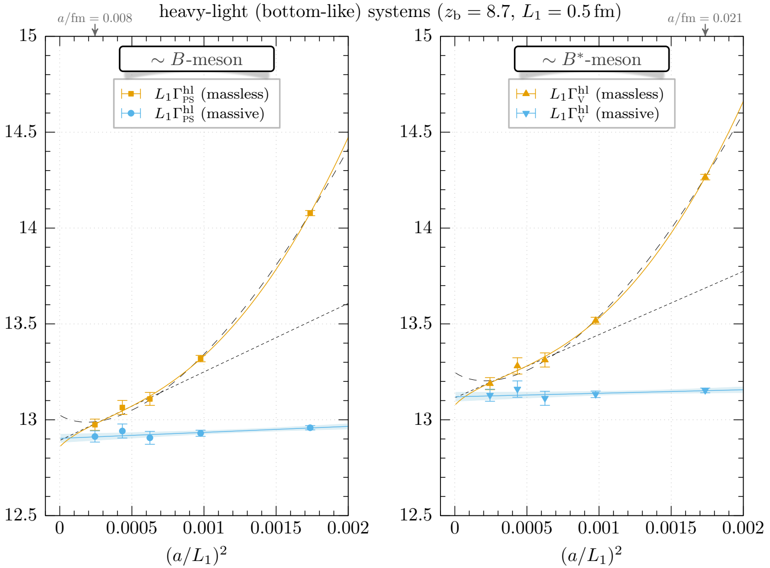
$B^{(*)}$ system

massless scheme

- significant $\mathrm{O}(a^{n\geq 2})$ effects
- 9% effect at coarsest a/L
- 3-pt linear extrapolation ✓
- other extrapolations fail ✗

massive scheme

- pure $\mathrm{O}(a^2)$ scaling observed
- a^2-effects significantly reduced



Test IV, effective masses (in $L_1 = 2L_0$)

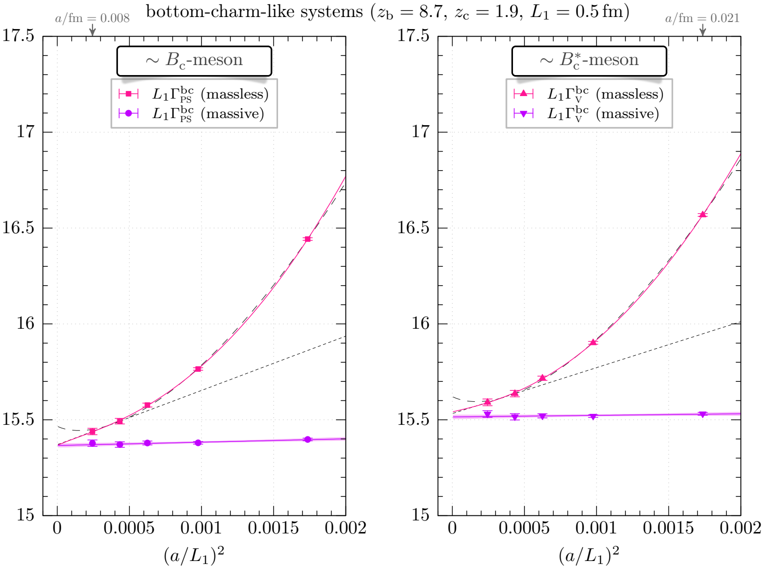
$B_c^{(*)}$ system

massless scheme

- significant $\mathrm{O}(a^{n\geq 2})$ effects
- 7% effect at coarsest a/L
- 2-pt linear extrapolation ✓
- other extrapolations fail ✗

massive scheme

- pure $\mathrm{O}(a^2)$ scaling observed
- a^2-effects significantly reduced



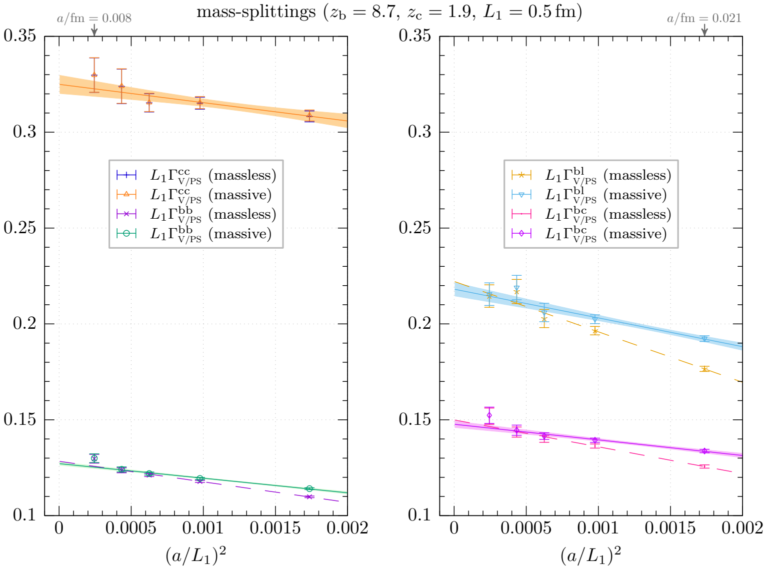
$$\Gamma^{ij}_{\mathrm{V/PS}} = m^{\mathrm{eff}}_{\mathrm{VV}} - m^{\mathrm{eff}}_{\mathrm{PP}}$$

for flavour combinations

$$ij \in \{\mathrm{cc}, \mathrm{bb}, \mathrm{bc}, \mathrm{bl}\}$$

massless vs massive scheme

- no problematic extrapolations in general
- cont. limits agree very well
- reduced cutoff effects for massive case

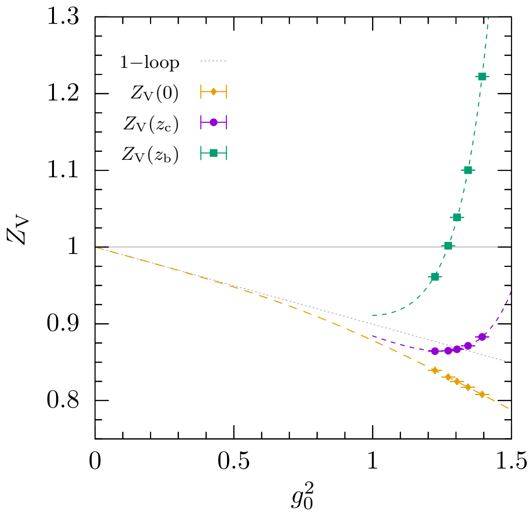


Test VI, vector renormalisation factor

$$Z_\mathrm{V} = \frac{f_1}{f_\mathrm{V}(x_0)}\Big|_{x_0=T/2}$$

flavour-diagonal determination

- $Z_\mathrm{V}(0)$: massless scheme
- $Z_\mathrm{V}(z_\mathrm{c})$: massive charm scheme
- $Z_\mathrm{V}(z_\mathrm{b})$: massive bottom scheme



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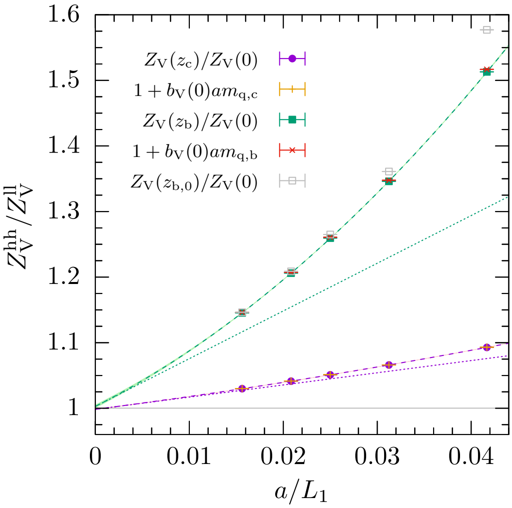
check $Z_{\mathrm{V}}(z_{\mathrm{h}})/Z_{\mathrm{V}}(0) \xrightarrow{a\to 0} 1$

- fit ansatz: $c_0 + c_1(a/L) + c_2(a/L)^2$
- $Z_{\mathrm{V}}(z_{\mathrm{c}})/Z_{\mathrm{V}}(0) \xrightarrow{a\to 0} 0.9984(2)$
- $Z_{\mathrm{V}}(z_{\mathrm{b}})/Z_{\mathrm{V}}(0) \xrightarrow{a\to 0} 1.0029(23)$



Conclusions

Massive improvement of valence Wilson quarks

- Renormalisation & improvement problem solved for heavy valence Wilson quarks.
  Improve the quark (fermion action) at the physical mass value, not in chiral limit!
- Employed method provides good separation of $b_{\rm m}(a\Delta)$ and $b_{\rm AP}(a\Delta)$.
- Cancels dominant mass-dependent cutoff effects.
- Gives excellent $\mathrm{O}(a^2)$ scaling behaviour.
- $b_{\rm A} - b_{\rm P}$ approximation limits applicability range.

ToDo:

- Better understanding of limitations (coarse lattice spacing, . . .).
- Extend procedure to heavy valence quarks on $M_{\rm sea} \neq 0$.
- Extend procedure to heavy sea quarks.