With the support of the Erasmus+ Programme of the European Union

Supported by ECIU

University of Stavanger

Exact Space-Time Symmetry Conservation & Automatic Mesh Refinement for classical LFT

Alexander Rothkopf
Faculty of Science and Technology
Department of Mathematics and Physics
University of Stavanger

in collaboration with Jan Nordström (LiU) & Will Horowitz (UCT)

A.R., W. Horowitz and J. Nordström: **arXiv:2404.18676**
(see also JCP 498 (2024) 112652)

Norwegian Particle, Astroparticle & Cosmology Theory network

Space-time symmetry breaking on the lattice

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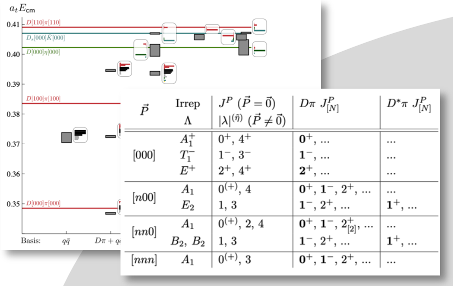
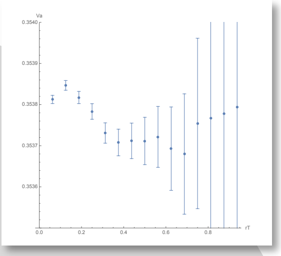
$$a \neq 0$$



$\vec{P}$	Irrep Λ	J^P ($\vec{P}=\vec{0}$) $\lvert\lambda\rvert^{(\tilde{\eta})}$ ($\vec{P}\neq\vec{0}$)	$D\pi$ $J^P_{[N]}$	$D^*\pi$ $J^P_{[N]}$
[000]	A_1^+	$0^+, 4^+$	$\mathbf{0}^+, \ldots$	...
	T_1^-	$1^-, 3^-$	$\mathbf{1}^-, \ldots$	...
	E^+	$2^+, 4^+$	$\mathbf{2}^+, \ldots$	...
[n00]	A_1	$0^{(+)}, 4$	$\mathbf{0}^+, \mathbf{1}^-, \mathbf{2}^+, \ldots$	...
	E_2	$1, 3$	$\mathbf{1}^-, \mathbf{2}^+, \ldots$	$\mathbf{1}^+, \ldots$
[nn0]	A_1	$0^{(+)}, 2, 4$	$\mathbf{0}^+, \mathbf{1}^-, \mathbf{2}^+_{[2]}, \ldots$	...
	B_2, B_2	$1, 3$	$\mathbf{1}^-, \mathbf{2}^+, \ldots$	$\mathbf{1}^+, \ldots$
[nnn]	A_1	$0^{(+)}, 3$	$\mathbf{0}^+, \mathbf{1}^-, \mathbf{2}^+, \ldots$	...

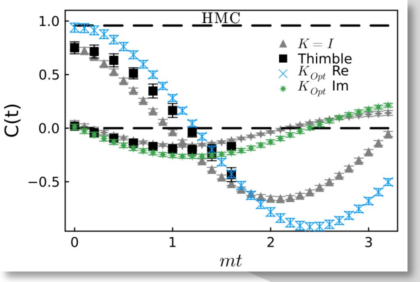
T=0 hadron spectroscopy

see e.g. HadSpec Collaboration JHEP 07 (2021) 123



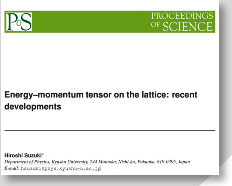
T>0 Spectral Functions

see e.g. R. Larsen, A.R. & HotQCD, PRD 109 (2024) 7, 074504



Machine Learning for Lattice QFT

see e.g. D. Alvestad, A.R., D. Sexty PRD 109 (2024) 3, L031502



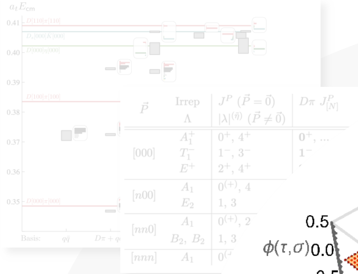
$T_{\mu\nu}^{\text{cont}}$ vs.

$$T_{\mu\nu}^{\text{latt}} = z_1 T_{\mu\nu}^{[6]} + z_2 T_{\mu\nu}^{[3]} + z_3 T_{\mu\nu}^{[1]}$$

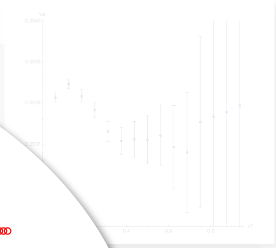
Energy Momentum Tensor

see e.g. H. Suzuki PoS LATTICE2016 (2017) 002 and references therein

Space-time symmetry breaking on the lattice

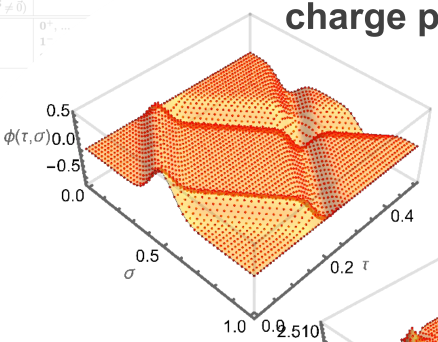


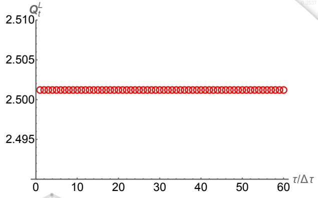
T=0 hadron spe…
see e.g. HadSpec Collabora…



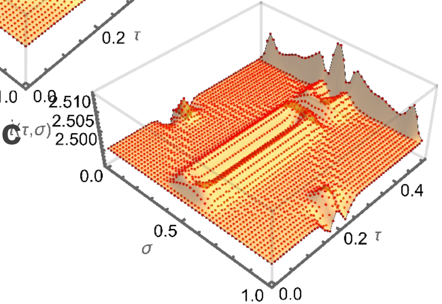
…al Functions
…A.R. & HotQCD, PRD …04

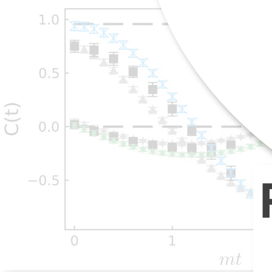
Exact Noether charge preservation

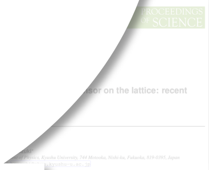




Automatic adaptive mesh







$T^{\rm cont}_{\mu\nu}$ vs.

$= z_1 T^{[6]}_{\mu\nu} + z_2 T^{[3]}_{\mu\nu} + z_3 T^{[1]}_{\mu\nu}$

Progress via a novel classical action for scalar fields (SCL)

Machine Learning for Lattice QFT
see e.g. D. Alvestad, A.R., D. Sexty PRD 109 (2024) 3, L031502

…gy Momentum Tensor
see e.g. H. Suzuki PoS LATTICE2016 (2017) 002 and references therein

Outline

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- Motivation – Space-time symmetry breaking on the lattice
- From the world-line formalism to a new action for classical fields (SCL)
- Summation-by-parts finite difference discretization
- Classical scalar wave propagation in (1+1)d as proof-of-principle
- Summary

Worldline Formalism in GR

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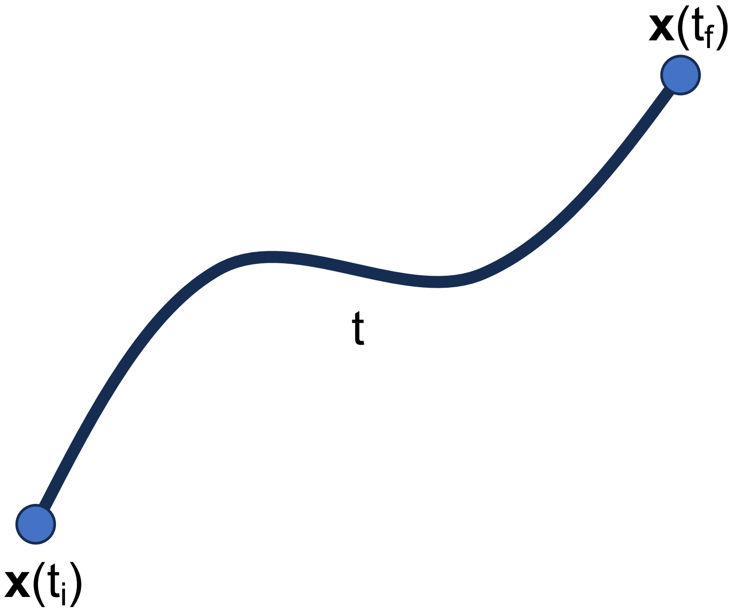
- Relativistic point particle motion: ”shortest path in given space-time” = geodesic



Worldline Formalism in GR

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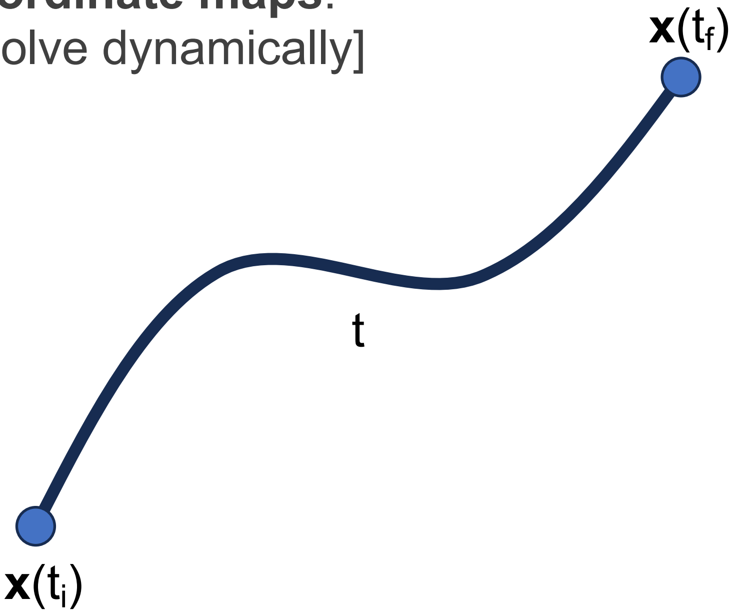
- Equal treatment of space & time as **dynamic coordinate maps**:
  from trajectory to world line [both t(ɣ) and x(ɣ) evolve dynamically]



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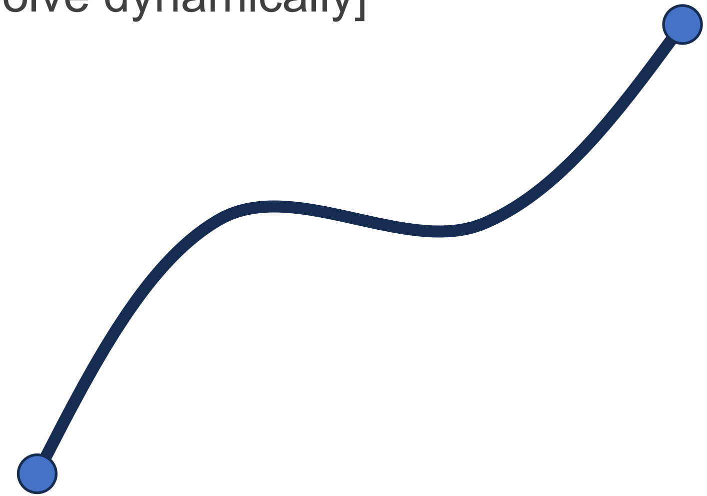

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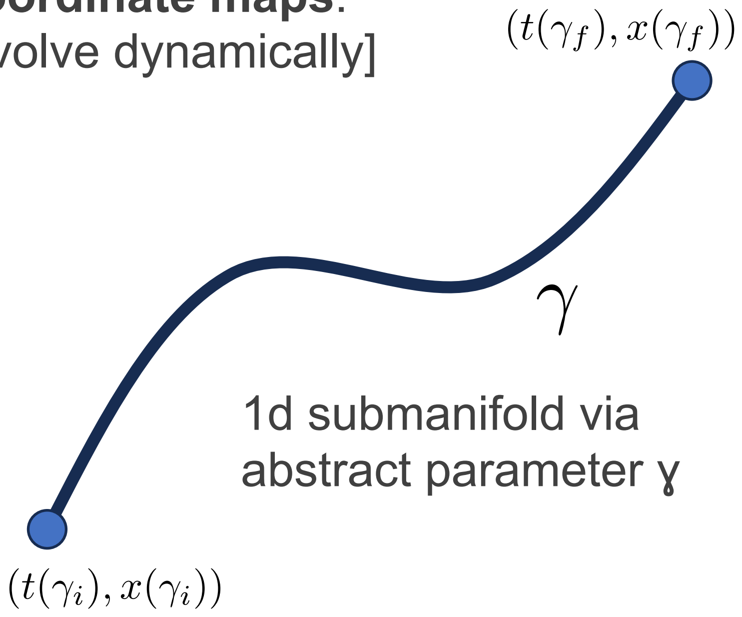
$$X^\mu = (t, \vec{x})^\mu$$



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$$S_{\rm geo} = \int d\gamma\, (-mc)\Big\{\sqrt{\Big(G_{00}+\frac{V(\vec{x})}{2mc^2}\Big)\frac{dX^0}{d\gamma}\frac{dX^0}{d\gamma}+G_{ii}\frac{dX^i}{d\gamma}\frac{dX^i}{d\gamma}}\Big\}$$

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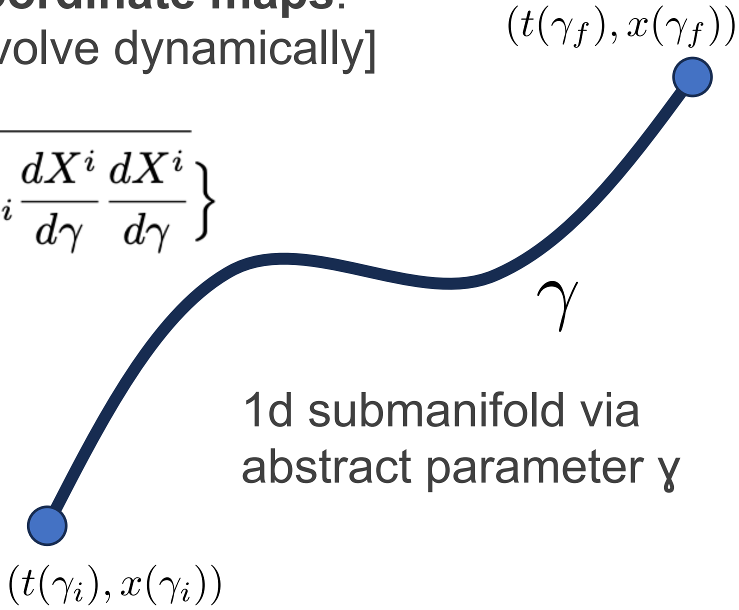

1d submanifold via abstract parameter ɣ

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$V(\vec{x})/2mc^2 \ll 1$

$\frac{d|\vec{x}|/d\gamma}{dt/d\gamma}/c = v/c \ll 1$

$$X^\mu = (t,\vec{x})^\mu$$

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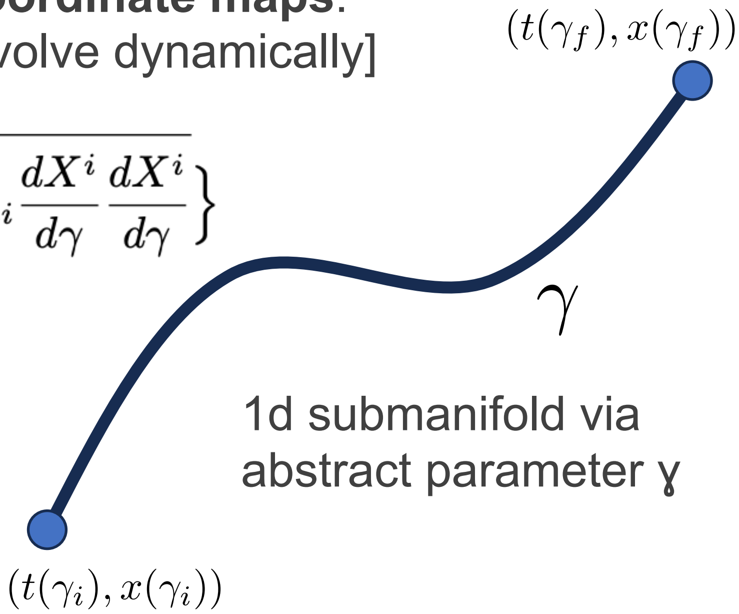

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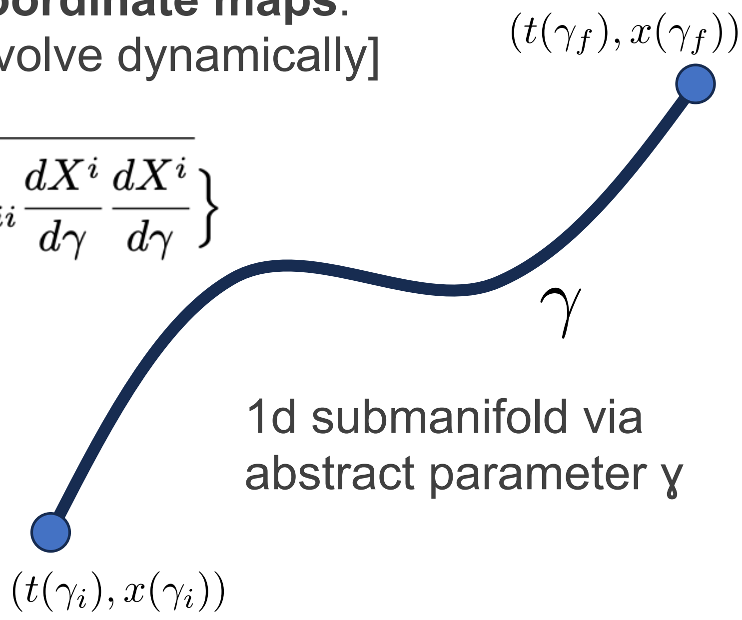

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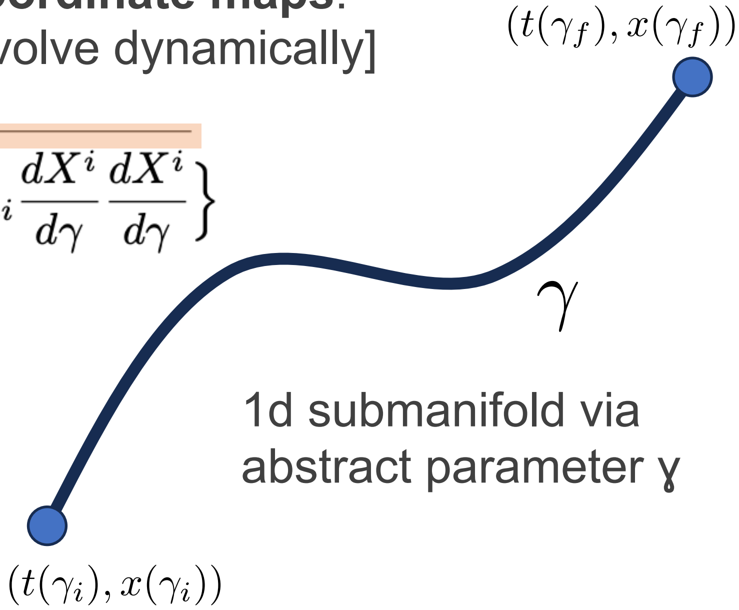

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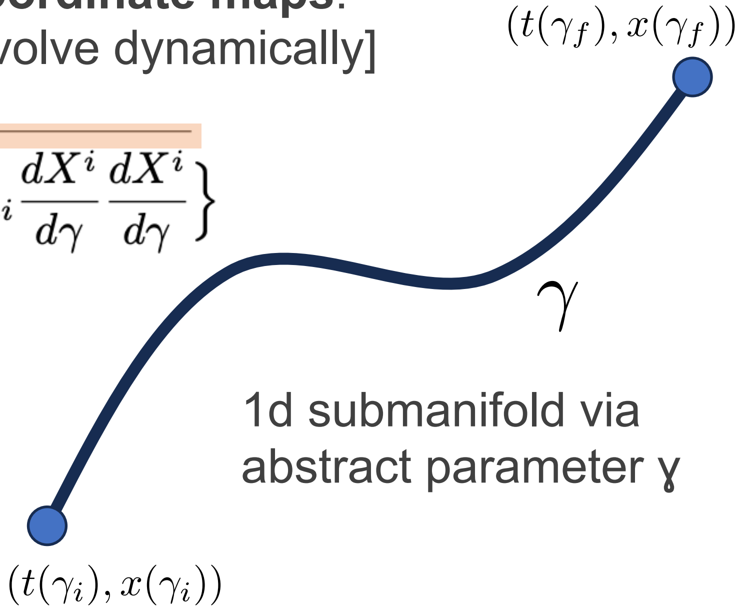

- mc denotes scale where motion through space and time becomes inseparable

Advantages of the worldline formalism

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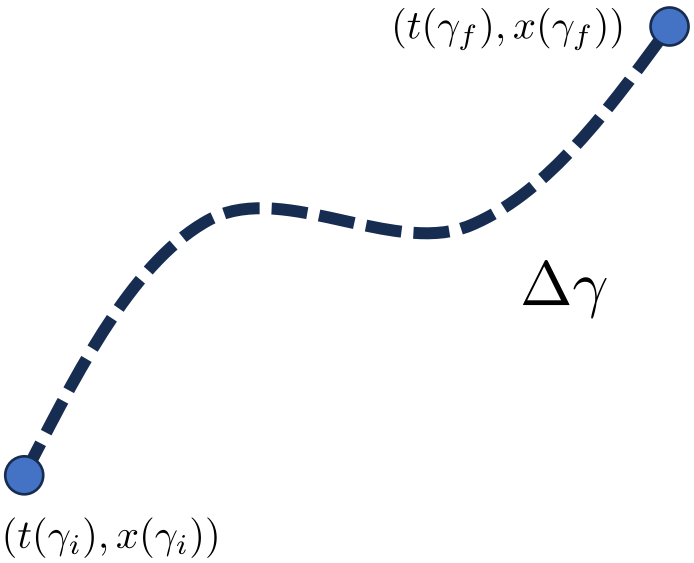
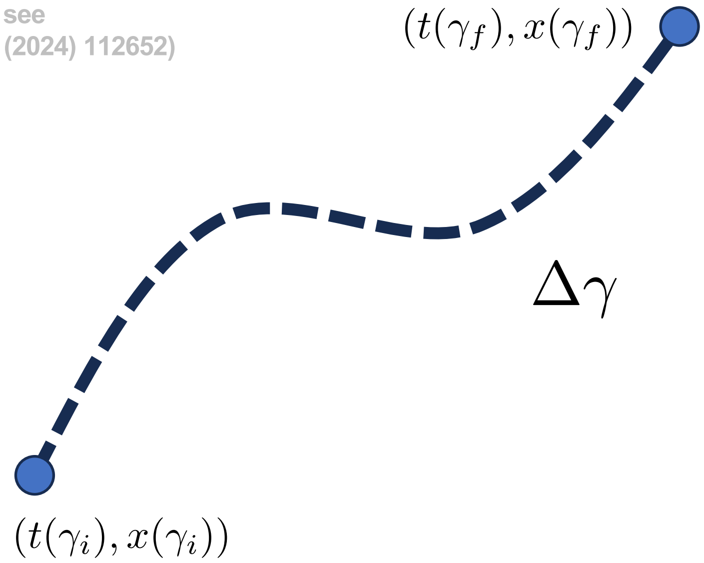
- Discretizing the action in γ leaves space-time coordinates $X^\mu = (t, \vec{x})^\mu$ continuous



Advantages of the worldline formalism

University of Stavanger

- Discretizing the action in γ leaves space-time coordinates $X^\mu = (t, \vec{x})^\mu$ continuous
- Discretized world-line action invariant under infinitesimal coordinate transforms: Noether's theorem holds! (for a detailed study of point mechanics see A.R., J. Nordström, J.Comput.Phys. 498 (2024) 112652)



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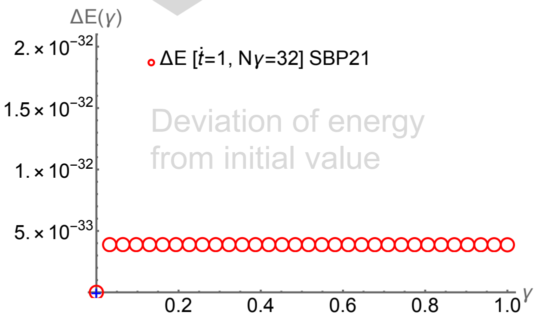
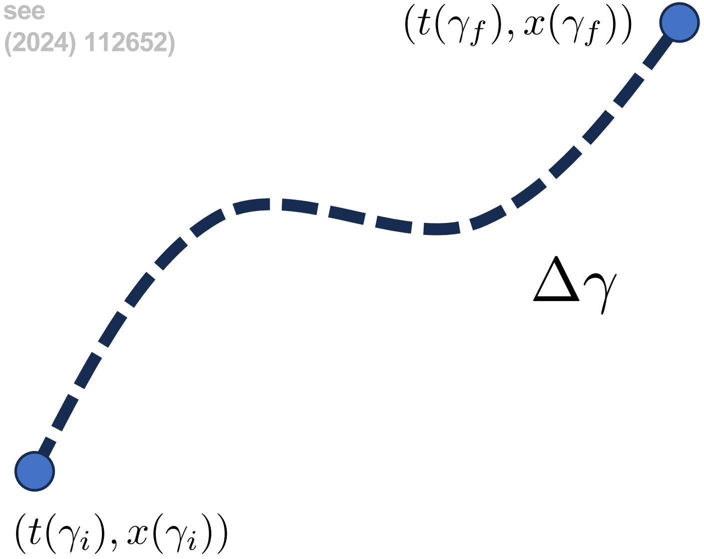


Energy of the system preserved *exactly* at its *continuum* value



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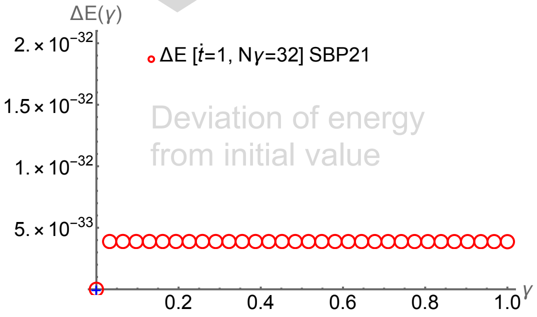

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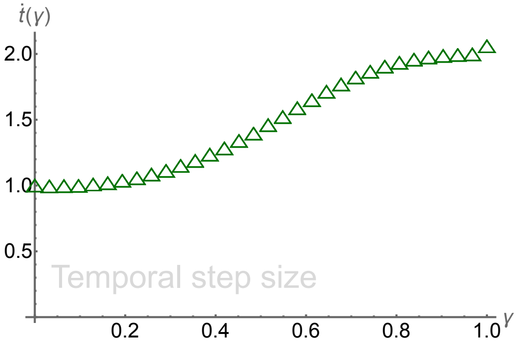
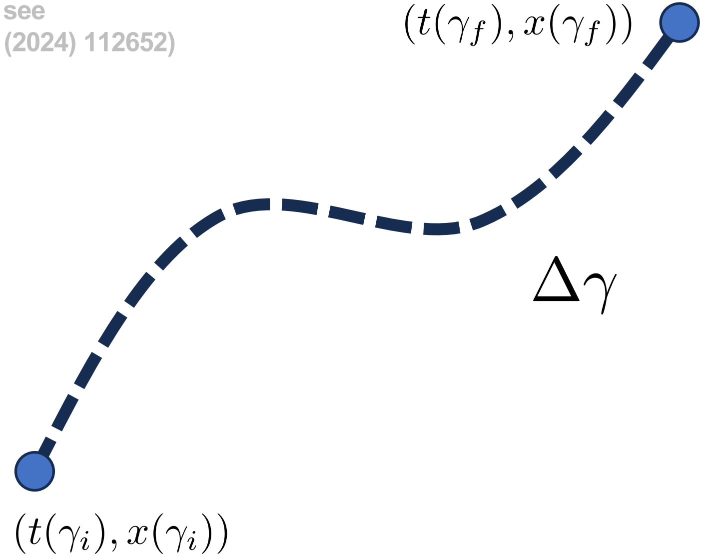
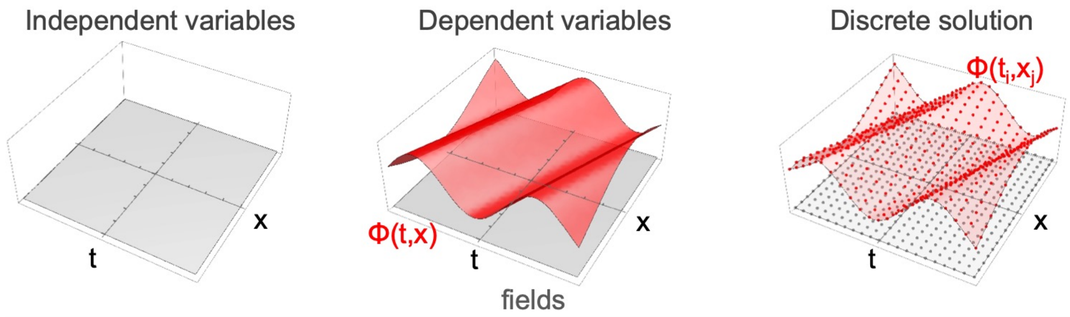


Resolution of the time grid *adapts* to dynamics of particle



A Field Theory Counterpart?

Conventional field theory



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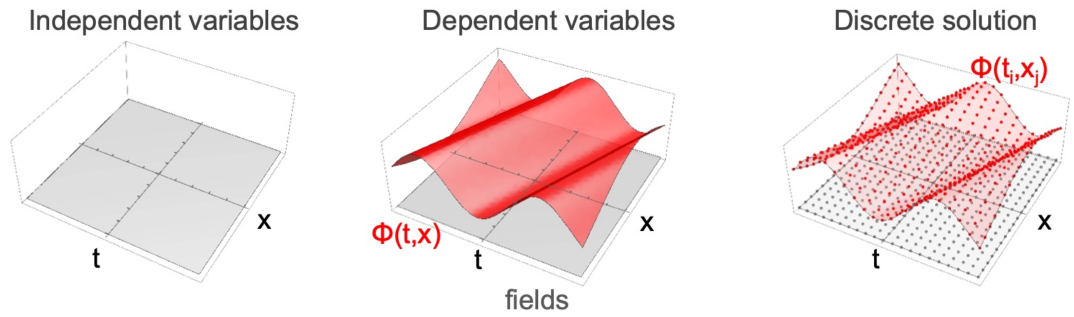


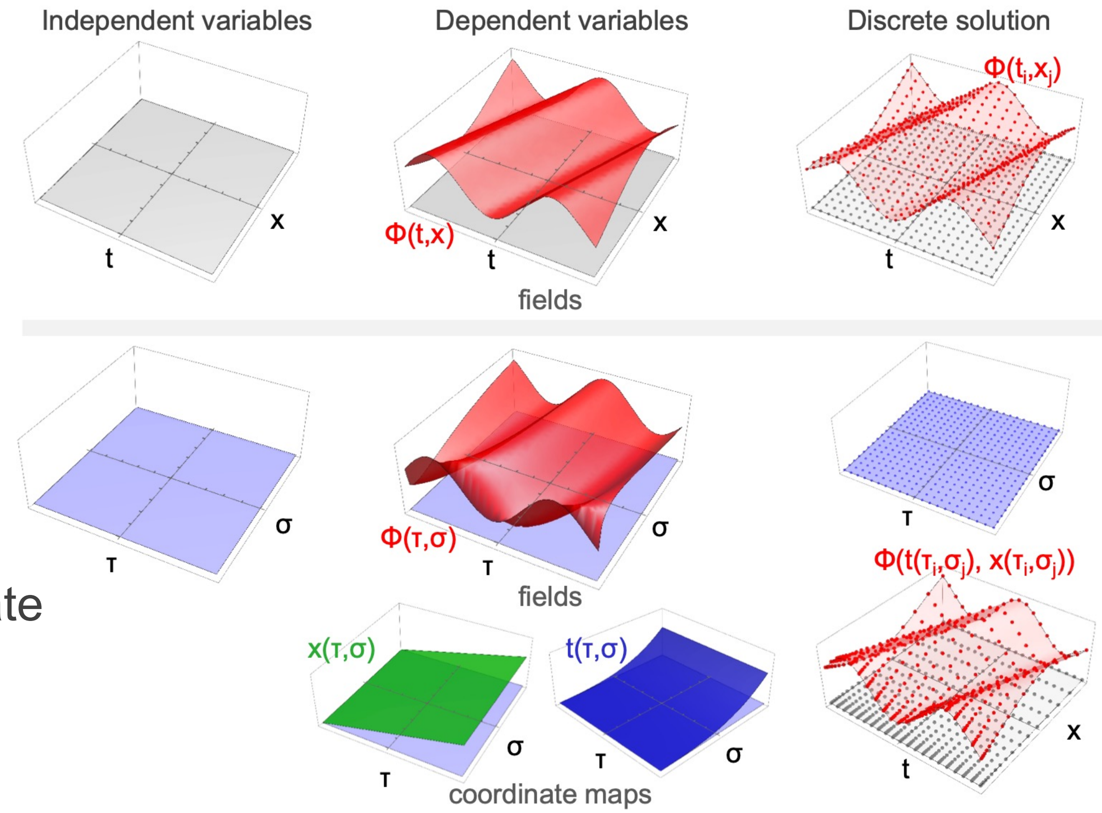
Spacetime symmetries broken by Δt and Δx

A Field Theory Counterpart?

University of Stavanger

Conventional field theory

Field theory with dynamic coordinate maps



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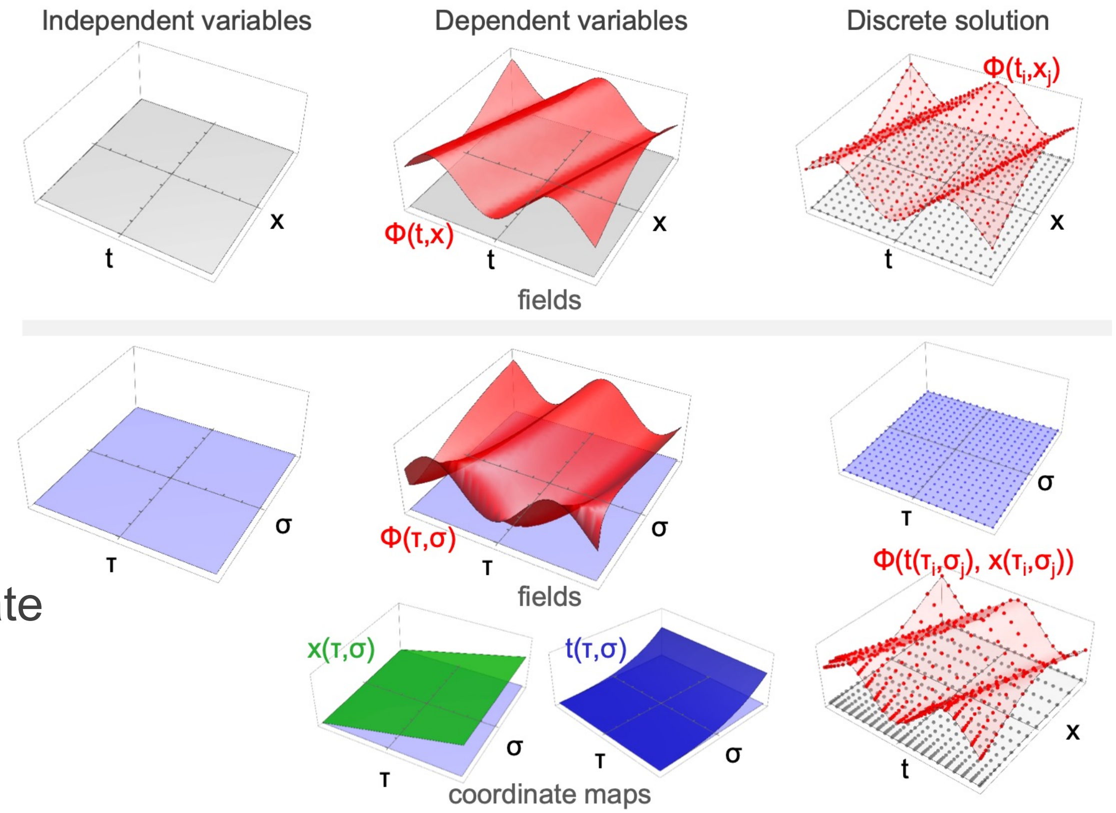

Spacetime symmetries broken by Δt and Δx

Spacetime symmetries unaffected by Δτ and Δσ?

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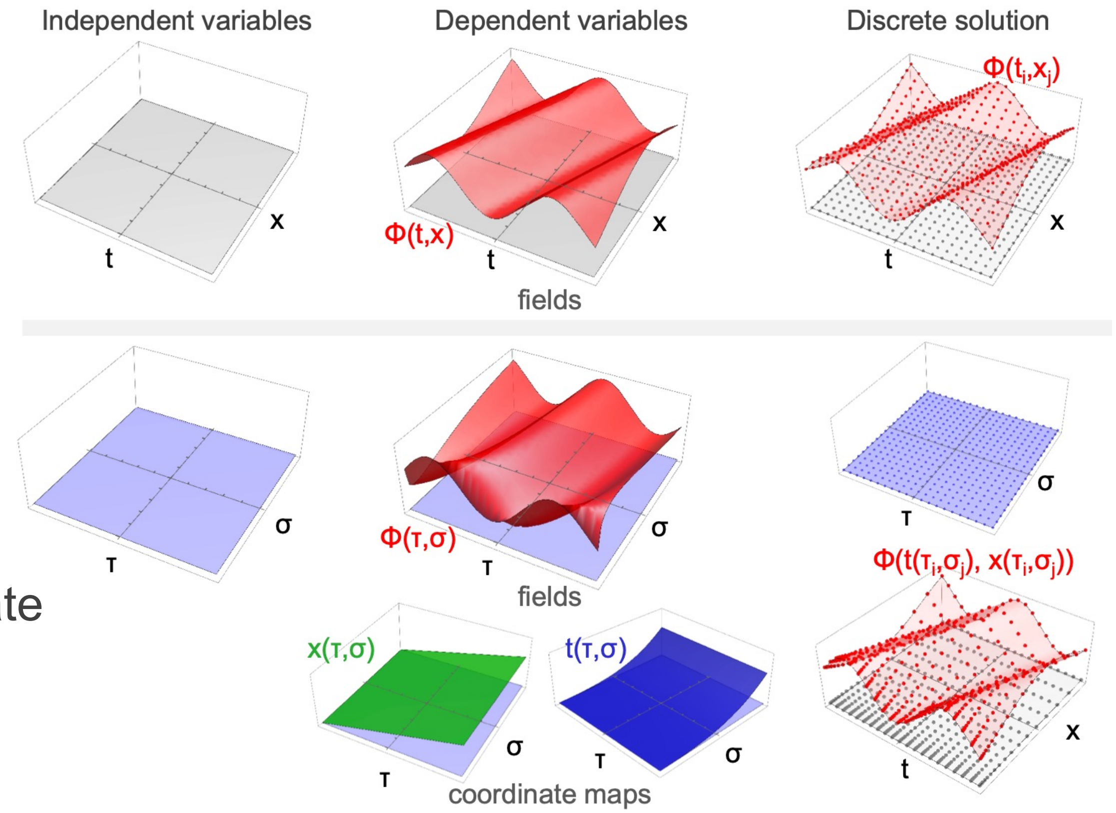

Spacetime symmetries broken by Δt and Δx

Spacetime symmetries unaffected by Δτ and Δσ?

YES!

A world ”volume” action for fields?

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- Starting point is the standard reparameterization invariant action

$$S = \int d^{(d+1)}X \sqrt{-\det[G]}\frac{1}{2}\Big(G^{\mu\nu}\partial_\mu\phi(X)\partial_\nu\phi(X) - V(\phi)\Big)$$

A world ”volume” action for fields?

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- Are we perhaps overlooking a constant term, just as in the non-relativistic action?

$$S = \int d^{(d+1)}X \sqrt{-\det[G]}\Big\{-T + \frac{1}{2}\Big(G^{\mu\nu}\partial_\mu\phi(X)\partial_\nu\phi(X) + V(\phi)\Big)\Big\}.$$

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- Consider as low energy limit of another more general action (κ = action density/T)

$$\mathcal{S}_{\rm BVP} \equiv \int d^{(d+1)}X \, \sqrt{-\det[G]}(-T)\Big\{1 - \frac{1}{2T}\Big(G^{\mu\nu}\partial_\mu\phi(X)\partial_\nu\phi(X) + V(\phi)\Big)\Big\} + \mathcal{O}(\kappa^2)$$

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$$= \int d^{(d+1)}X \sqrt{-\det[G]}(-T)\sqrt{1 - \frac{1}{T}\Big(G^{\mu\nu}\partial_\mu\phi(X)\partial_\nu\phi(X) + V(\phi)\Big)}$$

Towards the SCL action

- Crucial next step: elevate spacetime coordinates to dynamical coordinate maps

worldline: $t \to t(\gamma)$ here: $X^\mu \to X^\mu(\Sigma)$ $\quad \Sigma^a = (\tau, \vec{\sigma})^a = (\tau, \sigma_1, \ldots, \sigma_d)^a$

Towards the SCL action

University of Stavanger

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$$\mathcal{S}_{\rm BVP} = \int d^{(d+1)}\Sigma\, |\det[J]| \sqrt{-\det[G]}\,(-T) \times \sqrt{1 - \frac{1}{T}\Big(G^{\mu\nu}\partial_\mu\phi(X(\Sigma))\partial_\nu\phi(X(\Sigma)) + V(\phi)\Big)}$$

Towards the SCL action

University of Stavanger

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worldline: $t \to t(\gamma)$ here: $X^\mu \to X^\mu(\Sigma)$ $\quad \Sigma^a = (\tau, \vec{\sigma})^a = (\tau, \sigma_1, \ldots, \sigma_d)^a$

$$\mathcal{S}_{\rm BVP} = \int d^{(d+1)}\Sigma\, |\det[J]|\sqrt{-\det[G]}\,(-T)$$
$$\times \sqrt{1 - \frac{1}{T}\Big(G^{\mu\nu}\partial_\mu\phi(X(\Sigma))\partial_\nu\phi(X(\Sigma)) + V(\phi)\Big)}$$

- Can absorb the Jacobian into new *induced metric g* on the space of parameters

$$\sqrt{-\det[J]\det[G]\det[J]} = \sqrt{-\det[J^T]\det[G]\det[J]} = \sqrt{-\det[J^TGJ]} = \sqrt{-\det[g]}$$

Towards the SCL action

University of Stavanger

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- The scale T denotes where field and coordinate dynamics become inseparable

Summation-by-parts finite differences

University of Stavanger

- Derivation of Noether theorem or governing equations rely on integration by parts

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$$\int_{t_i}^{t_f} dt\, u(t)\, v(t) \approx \mathbf{u}^t\, \mathbb{H}\, \mathbf{v}$$

quadrature rule

⟷

$$\mathbb{D} = \mathbb{H}^{-1}\, \mathbb{Q}$$

finite difference stencil

$$\mathbb{Q} + \mathbb{Q}^t = \mathbb{E}_N - \mathbb{E}_0$$

$$= \mathrm{diag}[-1, 0, \ldots, 0, 1]$$

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$$\Delta t \begin{bmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix} \qquad \frac{1}{\Delta t} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$\mathbb{H}^{[2,1]}$ $\qquad$ $\mathbb{D}^{[2,1]}$

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for SBP operator as momentum operator for particle in a finite box see S.Kim, A.R. 2403.13558

A necessry alternative to the Wilson term

University of Stavanger

- Symmetric stencil leads to appearance of doubler modes when naïve SBP is used
- **Wilson term** trick **not applicable**: derivative acts on real-valued functions
- Modern approach in PDE community: weakly enforced boundary data

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Affine coordinate formulation

A. Rothkopf, J. Nordström, JCP 477 (2023) 111942

$$S = \int dt \big(\dot{x}(t)\dot{x}(t)\big) \quad x(0) = x_i$$

$$S \approx (\mathbb{D}\mathbf{x})^t \mathbb{H}\mathbb{D}\mathbf{x}$$

$$\bar{\mathbb{D}}\mathbf{x} = \mathbb{D}\mathbf{x} + \mathbb{H}^{-1}\mathbb{E}_0\big(\mathbf{x} - \mathbf{x}_i\big)$$

Modification acting on the path x itself

constant shift

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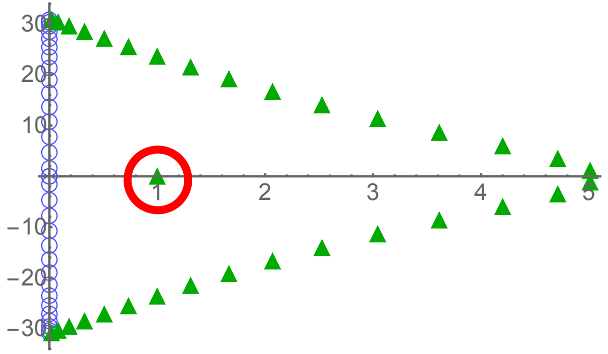

+ all zero modes are lifted

+ physical constant mode with correct IC as unit EV

The discretized action

University of Stavanger

- Due to mimetic nature of SBP operator simply replace derivatives by $\mathbb{D}$

$$\mathbb{E}_{\rm BVP}[\boldsymbol{X}_1^\mu, \bar{\mathbb{D}}_a^\mu \boldsymbol{X}_1^\mu, \boldsymbol{\phi}_1, \bar{\mathbb{D}}_a^\phi \boldsymbol{\phi}_1] = \frac{1}{2}\left\{\left(\frac{1}{T}V(\boldsymbol{\phi}_1) - 1\right) \circ \det[\boldsymbol{g}_1] + \frac{1}{T}(\bar{\mathbb{D}}_a^\phi \boldsymbol{\phi}_1) \circ (\bar{\mathbb{D}}_b^\phi \boldsymbol{\phi}_1) \circ \mathrm{adj}[\boldsymbol{g}_1]_{ab}\right\}^{\frac{1}{2}} \boldsymbol{h}$$

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- **Discrete action** remains manifest **invariant under Poincare transformations**
- Integration by parts exactly mimicked: Noether current & charge as in continuum

$$\boldsymbol{q}^{\rm L} = \frac{\partial \mathbb{E}^{\rm L}_{\rm BVP}}{\partial(\mathbb{D}_0 \mathbf{X}^\mu)} \delta \mathbf{X}^\mu = \Big(\frac{\partial \mathbb{E}_{\rm BVP}}{\partial(\mathbb{D}_0 \mathbf{X}^\mu)} + \tilde{\boldsymbol{\lambda}}_\mu \circ \mathfrak{d}^0[0] + \tilde{\boldsymbol{\gamma}}_\mu \circ \mathfrak{d}^0[N_0]\Big)\delta \mathbf{X}^\mu,$$

$$\boldsymbol{Q}^{\rm L} = \Big(\mathbb{H}_\sigma \frac{\partial \mathbb{E}_{\rm BVP}}{\partial(\mathbb{D}_0 \mathbf{X}^\mu)} + (\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\lambda}}_\mu)\mathfrak{d}^0[0] + (\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\gamma}}_\mu)\mathfrak{d}^0[N_0]\Big)\delta \mathbf{X}^\mu.$$

Proof-of-principle in (1+1)d

University of Stavanger

- Scalar wave propagation is numerically challenging (stability, accuracy)

$$\mathcal{S}_{\rm BVP} = \int d\tau d\sigma\,(-T)\sqrt{\;-\det[g] + \frac{1}{T}\partial_a\phi(\Sigma)\partial_b\phi(\Sigma)\mathrm{adj}[g]_{ab}}$$

$$= \int d\tau d\sigma\,(-T)\Big\{c^2(\dot{t}x' - \dot{x}t')^2$$

$$+\frac{1}{T}\Big(\dot{\phi}^2(c^2(t')^2 - (x')^2) + 2\dot{\phi}\phi'(\dot{x}x' - c^2\dot{t}t') + (\phi')^2(c^2\dot{t}^2 - \dot{x}^2)\Big)\Big\}^{1/2}$$

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- Simplify by considering only time as dynamical mapping (trivial x[τ,σ]= σ)

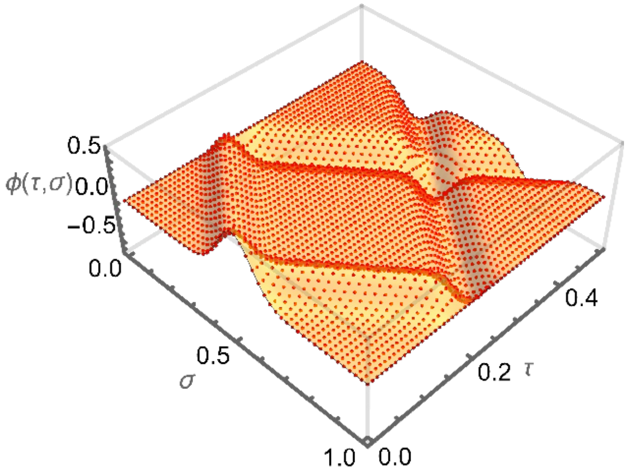
$$\mathcal{E}_{\rm BVP} \overset{x=\sigma}{\equiv} \int d\tau d\sigma \, \frac{1}{2}\Big\{(\dot{t})^2 + \frac{1}{T}\Big(\dot{\phi}^2((t')^2 - 1) - 2\dot{\phi}\phi'\dot{t}t' + (\phi')^2(\dot{t}^2)\Big)\Big\}$$

Classical wave propagation in (1+1)d

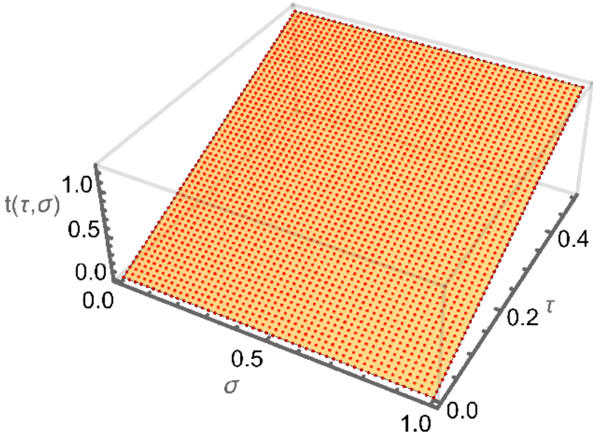
University of Stavanger

- Numerical search for critical point ($\phi_{cl}[\tau,\sigma]$, $t_{cl}[\tau,\sigma]$) of the classical action

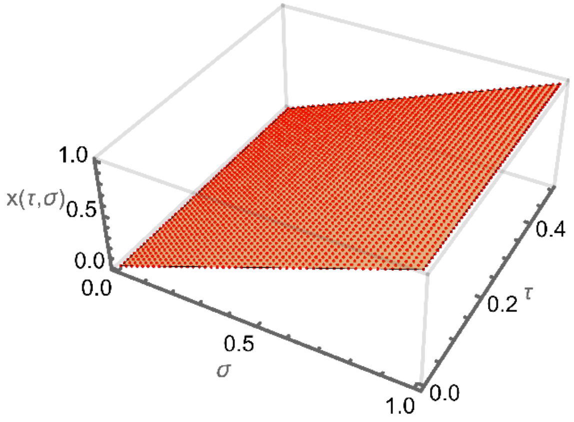
for more details on the IBVP formulation of the action see A.R., W.A. Horowitz, J. Nordström arXiv:2404.18676



field evolution $\phi[\tau,\sigma]$



temporal map $t[\tau,\sigma]$



trivial spatial map $x[\tau,\sigma]= \sigma$

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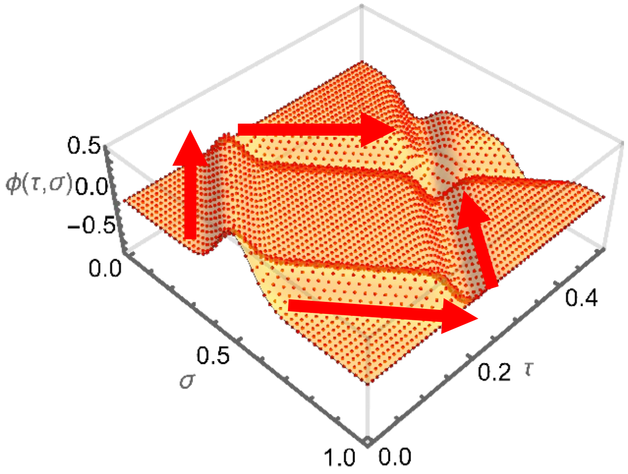
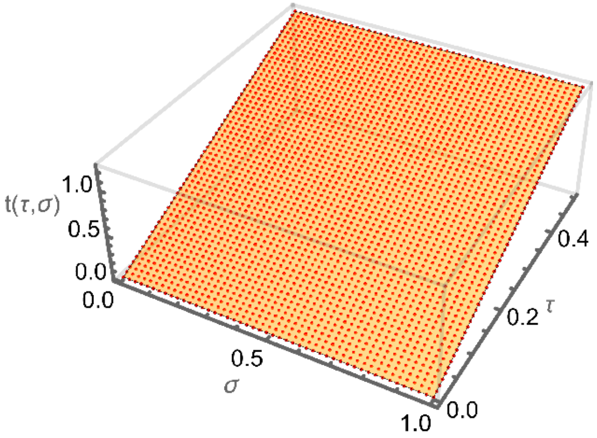


field evolution ϕ[τ,σ]



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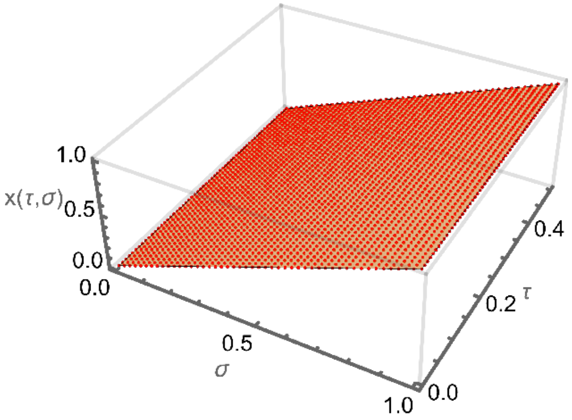

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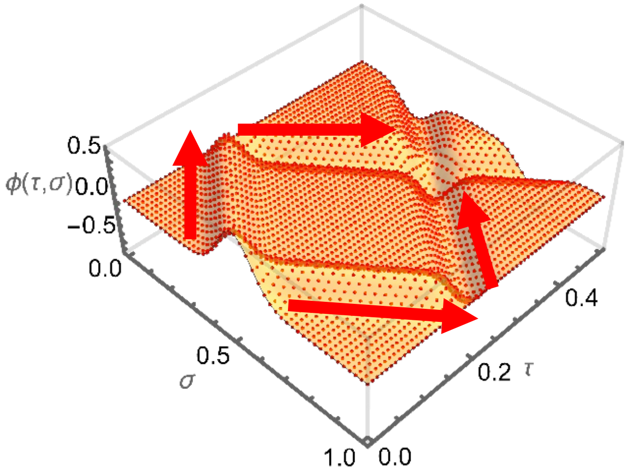

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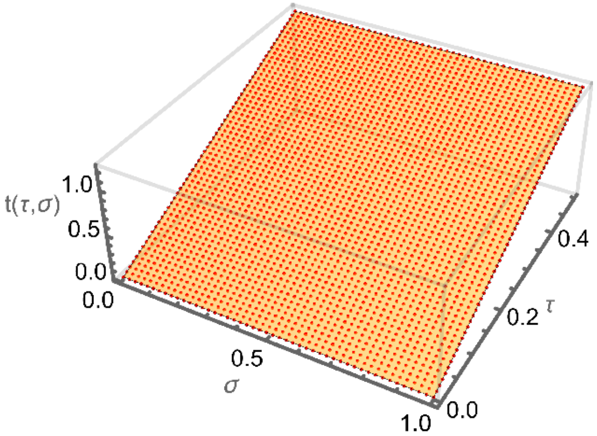

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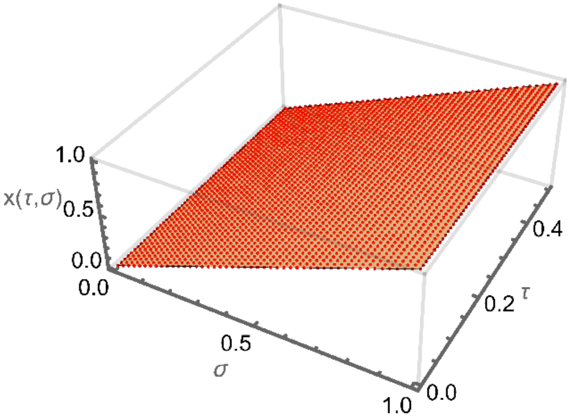

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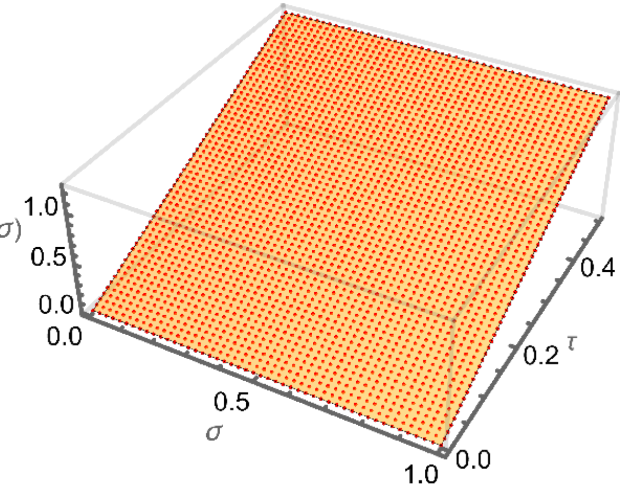
- Here T=10.000, choice to obtain effects on the coordinate maps on percent level

Automatic spacetime mesh refinement

University of Stavanger



- Temporal map automatically adapts resolution according to wave dynamics

Automatic spacetime mesh refinement

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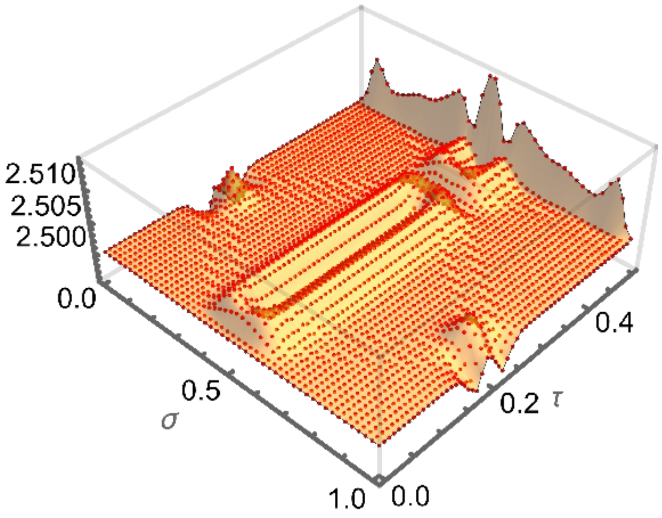

τ derivative of t[τ,σ]

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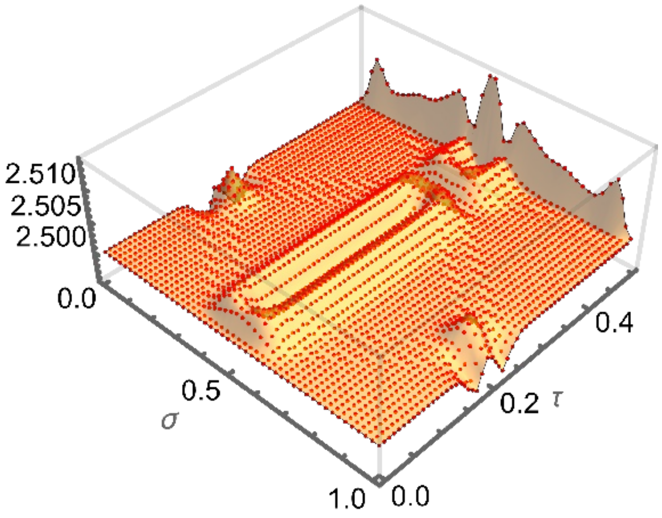

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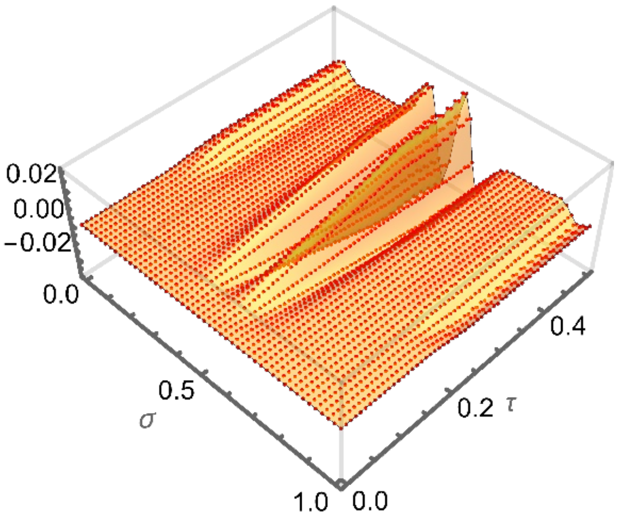

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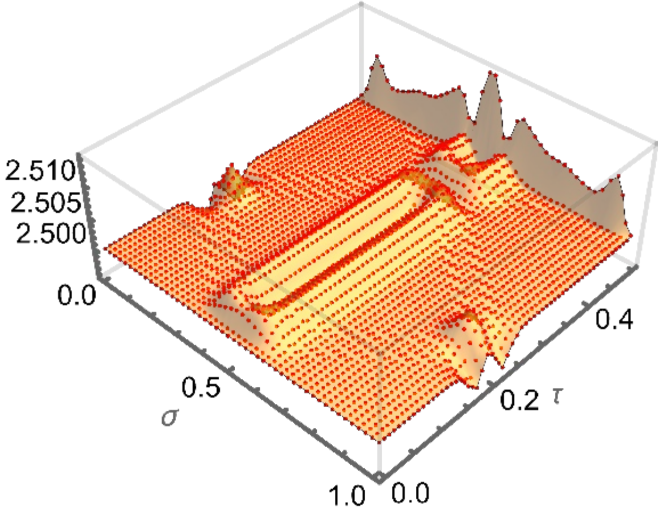

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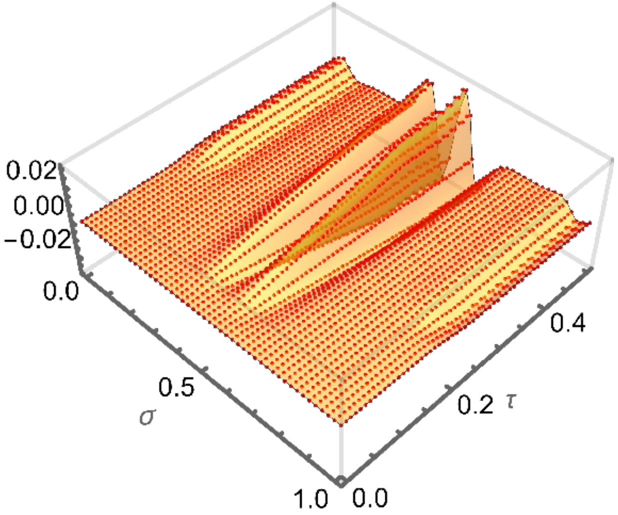

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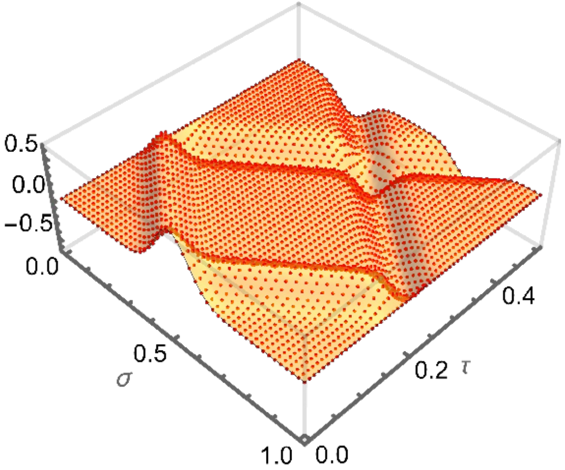

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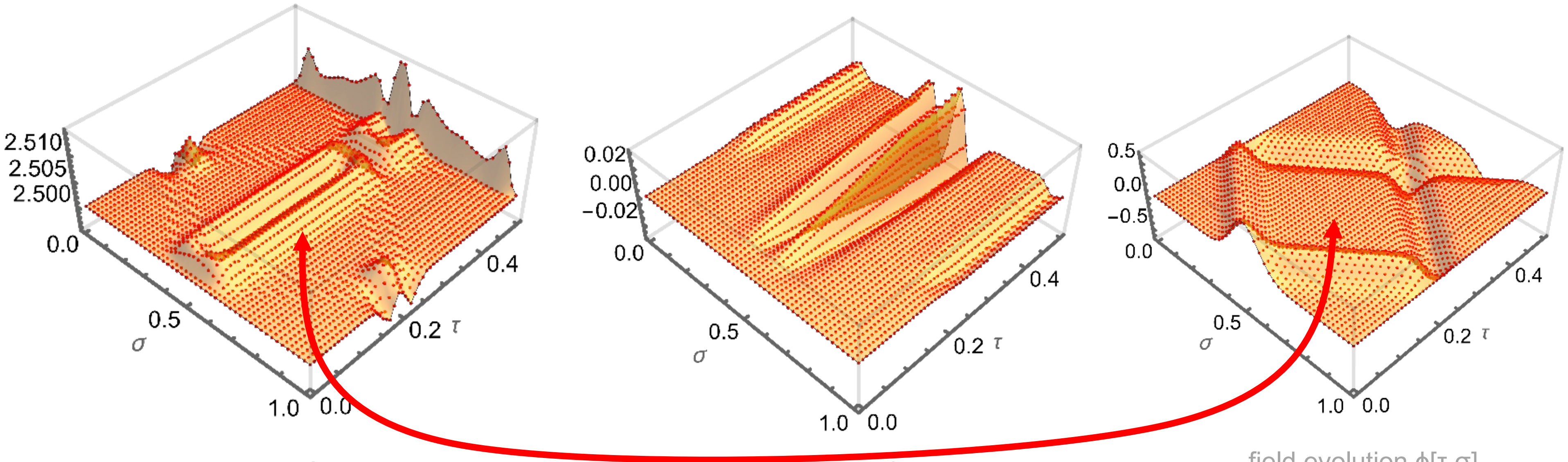

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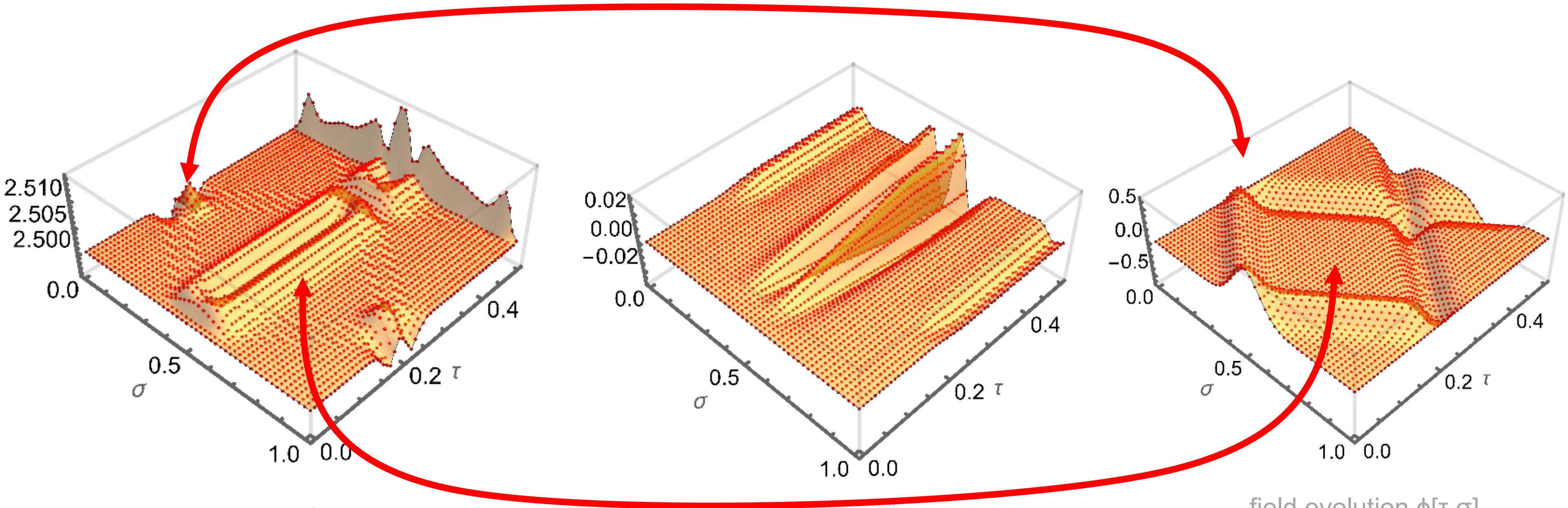

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Noether Charge – Time Translations

University of Stavanger

- Due to mimetic SBP discretization: continuum expression with

A.R., W.A. Horowitz, J. Nordström arXiv:2404.18676

$$\boldsymbol{Q}_t^{\mathrm{L}} = \mathbb{H}_\sigma \underbrace{\left\{(\mathbb{D}_\tau \boldsymbol{t}_1) + \frac{1}{T}\Big((\mathbb{D}_\sigma \boldsymbol{\phi}_1)^2 \circ (\mathbb{D}_\tau \boldsymbol{t}_1) - (\mathbb{D}_\tau \boldsymbol{\phi}_1) \circ (\mathbb{D}_\sigma \boldsymbol{\phi}_1) \circ (\mathbb{D}_\sigma \boldsymbol{t}_1)\Big)\right\}}_{\boldsymbol{J}^0 \in \mathbb{R}^{N_\tau \times N_\sigma}}$$

$$+ \underbrace{\left\{(\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\lambda}}^t)\boldsymbol{\mathfrak{d}}^\tau[0] + (\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\gamma}}^t)\boldsymbol{\mathfrak{d}}^\tau[N_\tau]\right\}}_{\text{Lagr. mult. contrib.}},$$

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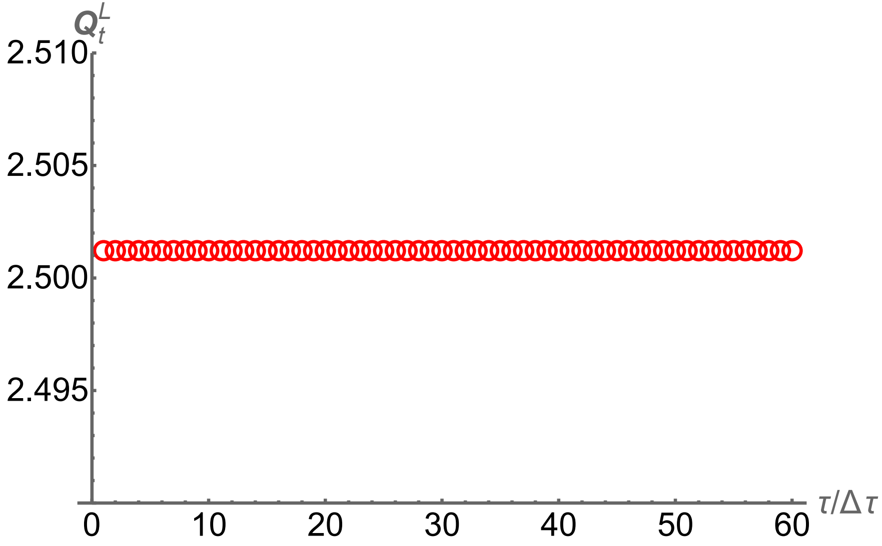
- Due to mimetic SBP discretization: continuum expression with
  A.R., W.A. Horowitz, J. Nordström arXiv:2404.18676

$$\boldsymbol{Q}_t^{\mathrm{L}} = \mathbb{H}_\sigma \underbrace{\left\{(\mathbb{D}_\tau \boldsymbol{t}_1) + \frac{1}{T}\Big((\mathbb{D}_\sigma \boldsymbol{\phi}_1)^2 \circ (\mathbb{D}_\tau \boldsymbol{t}_1) - (\mathbb{D}_\tau \boldsymbol{\phi}_1)\circ(\mathbb{D}_\sigma \boldsymbol{\phi}_1)\circ(\mathbb{D}_\sigma \boldsymbol{t}_1)\Big)\right\}}_{\boldsymbol{J}^0 \in \mathbb{R}^{N_\tau \times N_\sigma}}$$

$$+ \underbrace{\left\{(\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\lambda}}^t)\mathfrak{d}^\tau[0] + (\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\gamma}}^t)\mathfrak{d}^\tau[N_\tau]\right\}}_{\text{Lagr. mult. contrib.}},$$

- **Exact conservation** of the Noether charge associated with time translations!



Summary

University of Stavanger

- World-line formalism suggests dynamical coordinate maps essential ingredient
- SCL action incorporates **dynamical coordinate maps** with field d.o.f.s
- Discretization via **summation-by-parts** mimetic finite difference scheme
- Discretization of abstract parameter action **retains space-time symmetries**
- **Dynamical** emergence of **time-mesh** & **exact conservation** of Noether charge

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Thank you for your attention

Next steps

University of Stavanger

$$\mathcal{S}_{\rm BVP} = \int d^{(d+1)}\Sigma \left(-T\right) \sqrt{\left(\frac{1}{T}V(\phi) - 1\right)\det[g] + \frac{1}{T}\partial_a\phi(\Sigma)\partial_b\phi(\Sigma)\mathrm{adj}[g]_{ab}}.$$

- Formulate an initial value problem version of the SCL action
- Discretize the action and show that Noether’s theorem holds for Poincare group
- Demonstrate numerically feasibility of locating critical point of the action: classical field solution without the need to solve Euler-Lagrange equations

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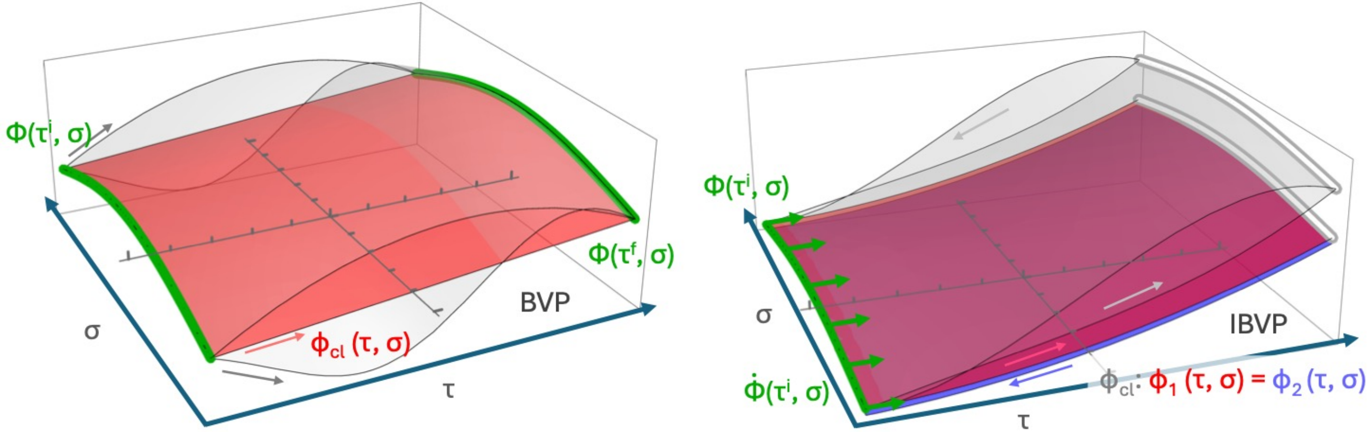
$$\mathcal{E}_{\rm BVP} = \int d^{(d+1)}\Sigma \; E_{\rm BVP} = \int d^{(d+1)}\Sigma \, \frac{1}{2}\Big\{\Big(\frac{1}{T}V(\phi)-1\Big)\det[g] + \frac{1}{T}\partial_a\phi(\Sigma)\partial_b\phi(\Sigma)\mathrm{adj}[g]_{ab}\Big\}$$

Classical Schwinger Keldysh (Galley)

University of Stavanger

- Doubling of all degrees of freedom by introducing forward and backward branches

see C. Galley PRL 110 (2013) 17, 174301 and A.R., J. Nordström JCP 477 (2023) 111942



$$\mathcal{E}_{\rm IBVP} = \int d^{(d+1)}\Sigma \left\{ E_{\rm BVP}[X_1, \partial_a X_1, \phi_1, \partial_a \phi_1] - E_{\rm BVP}[X_2, \partial_a X_2, \phi_2, \partial_a \phi_2] \right\}$$

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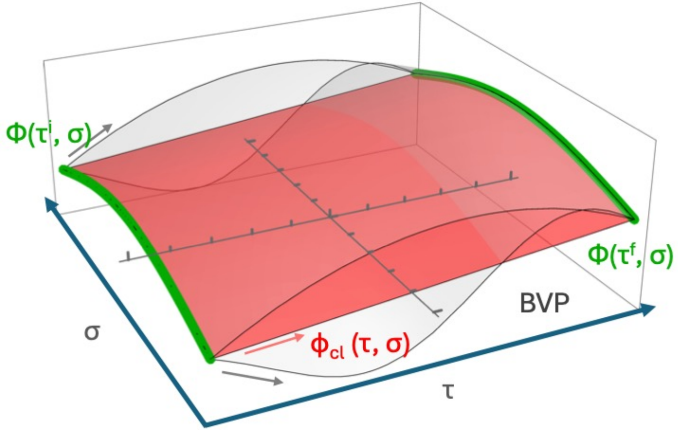

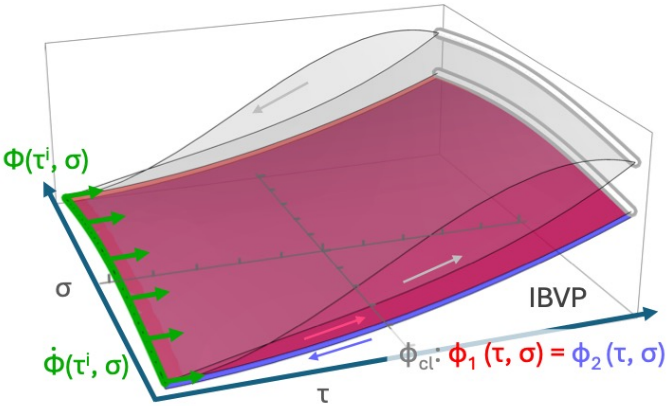

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connecting conditions

$$X_1^\mu(\tau = \tau^{\rm f}, \vec{\sigma}) = X_2^\mu(\tau = \tau^{\rm f}, \vec{\sigma}), \qquad \phi_1(\tau = \tau^{\rm f}, \vec{\sigma}) = \phi_2(\tau = \tau^{\rm f}, \vec{\sigma})$$

$$\partial_0 X_1^\mu|_{\tau=\tau^{\rm f}} = \partial_0 X_2^\mu|_{\tau=\tau^{\rm f}}, \qquad \partial_0 \phi_1|_{\tau=\tau^{\rm f}} = \partial_0 \phi_2|_{\tau=\tau^{\rm f}}.$$

Inclusion of initial & boundary conditions

University of Stavanger

- In contrast to implicit analytic treatment make explicit via Lagrange multipliers

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$$+ \int \prod_{a=1}^{d} d\Sigma_a \Big\{ \lambda_\mu \big(X_1^\mu(\tau^{\rm i}, \vec\sigma) - X^\mu_{\rm IC} \big) + \lambda_\phi \big(\phi_1(\tau^{\rm i}, \vec\sigma) - \phi_{\rm IC} \big)$$

$$+ \tilde\lambda_\mu \big(\partial_0 X_1^\mu(\tau^{\rm i}, \vec\sigma) - \dot X^\mu_{\rm IC} \big) + \tilde\lambda_\phi \big(\partial_0 \phi_1(\tau^{\rm i}, \vec\sigma) - \dot\phi_{\rm IC} \big)$$

$$+ \gamma_\mu \big(X_1^\mu(\tau^{\rm f}, \vec\sigma) - X_2^\mu(\tau^{\rm f}, \vec\sigma) \big) + \gamma_\phi \big(\phi_1(\tau^{\rm f}, \vec\sigma) - \phi_2(\tau^{\rm f}, \vec\sigma) \big)$$

$$+ \tilde\gamma_\mu \big(\partial_0 X_2^\mu(\tau^{\rm f}, \vec\sigma) - \partial_0 X_2^\mu(\tau^{\rm f}, \vec\sigma) \big) + \tilde\gamma_\phi \big(\partial_0 \phi_1(\tau^{\rm f}, \vec\sigma) - \partial_0 \phi_2(\tau^{\rm f}, \vec\sigma) \big) \Big\}$$

$$+ \sum_{j=1}^{d} \int \prod_{\substack{a=0 \\ a \neq j}}^{d} d\Sigma_a \Big\{ \kappa^j_\mu \big(X_1^\mu(\sigma^{\rm i}_j) - X^\mu_{\rm sBCL}(\sigma^{\rm i}_j) \big) + \xi^j_\mu \big(X_2^\mu(\sigma^{\rm i}_j) - X^\mu_{\rm sBCL}(\sigma^{\rm i}_j) \big)$$

$$+ \tilde\kappa^j_\mu \big(X_1^\mu(\sigma^{\rm f}_j) - X^\mu_{\rm sBCR}(\sigma^{\rm f}_j) \big) + \tilde\xi^j_\mu \big(X_2^\mu(\sigma^{\rm f}_j) - X^\mu_{\rm sBCR}(\sigma^{\rm f}_j) \big)$$

$$+ \kappa^j_\phi \big(\phi_1(\sigma^{\rm i}_j) - \phi_{\rm sBCR}(\sigma^{\rm i}_j) \big) + \xi^j_\phi \big(\phi_2(\sigma^{\rm i}_j) - \phi_{\rm sBCR}(\sigma^{\rm i}_j) \big)$$

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Inclusion of initial & boundary conditions

University of Stavanger

- In contrast to implicit analytic treatment make explicit via Lagrange multipliers

see also A.R., J. Nordström JCP 511 (2024) 113138

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Forward & backward branch Lagrangian

$$\begin{aligned}
&+\int \prod_{a=1}^{d} d\Sigma_a \Big\{ \lambda_\mu\big(X_1^\mu(\tau^{\rm i},\vec\sigma) - X^\mu_{\rm IC}\big) + \lambda_\phi\big(\phi_1(\tau^{\rm i},\vec\sigma) - \phi_{\rm IC}\big)\\
&+\tilde\lambda_\mu\big(\partial_0 X_1^\mu(\tau^{\rm i},\vec\sigma) - \dot X^\mu_{\rm IC}\big) + \tilde\lambda_\phi\big(\partial_0\phi_1(\tau^{\rm i},\vec\sigma) - \dot\phi_{\rm IC}\big)\\
&+\gamma_\mu\big(X_1^\mu(\tau^{\rm f},\vec\sigma) - X_2^\mu(\tau^{\rm f},\vec\sigma)\big) + \gamma_\phi\big(\phi_1(\tau^{\rm f},\vec\sigma) - \phi_2(\tau^{\rm f},\vec\sigma)\big)\\
&+\tilde\gamma_\mu\big(\partial_0 X_2^\mu(\tau^{\rm f},\vec\sigma) - \partial_0 X_2^\mu(\tau^{\rm f},\vec\sigma)\big) + \tilde\gamma_\phi\big(\partial_0\phi_1(\tau^{\rm f},\vec\sigma) - \partial_0\phi_2(\tau^{\rm f},\vec\sigma)\big)\Big\}\\
&+\sum_{j=1}^{d}\int \prod_{\substack{a=0\\ a\neq j}}^{d} d\Sigma_a \Big\{ \kappa^j_\mu\big(X_1^\mu(\sigma^{\rm i}_j) - X^\mu_{\rm sBCL}(\sigma^{\rm i}_j)\big) + \xi^j_\mu\big(X_2^\mu(\sigma^{\rm i}_j) - X^\mu_{\rm sBCL}(\sigma^{\rm i}_j)\big)\\
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Forward & backward branch Lagrangian

Initial conditions for coordinate maps and fields

Connecting conditions for maps and fields from calssical Schwinger-Keldysh

Spatial boundary conditions for the coordinate maps and fields

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Forward & backward branch Lagrangian

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Initial conditions for coordinate maps and fields

$$+\gamma_\mu\big(X_1^\mu(\tau^{\rm f},\vec\sigma)-X_2^\mu(\tau^{\rm f},\vec\sigma)\big)+\gamma_\phi\big(\phi_1(\tau^{\rm f},\vec\sigma)-\phi_2(\tau^{\rm f},\vec\sigma)\big)$$
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Connecting conditions for maps and fields from calssical Schwinger-Keldysh

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$$+\kappa^j_\phi\big(\phi_1(\sigma_j^{\rm i})-\phi_{\rm sBCR}(\sigma_j^{\rm i})\big)+\xi^j_\phi\big(\phi_2(\sigma_j^{\rm i})-\phi_{\rm sBCR}(\sigma_j^{\rm i})\big)$$
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Spatial boundary conditions for the coordinate maps and fields

$$=\int d^{(d+1)}\Sigma\,\Big\{E^{\rm L}_{\rm BVP}[X_1,\partial_aX_1,\phi_1,\partial_a\phi_1]-E^{\rm L}_{\rm BVP}[X_2,\partial_aX_2,\phi_2,\partial_a\phi_2]\Big\}$$

Redefined Lagrangians including Lagrange mult.

Discretized IBVP action

University of Stavanger

- Introduce forward and backward branch (classical Schwinger-Keldysh)

$$\begin{aligned}\mathbb{E}^{\rm L}_{\rm IBVP} =& \frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_1)^2 - 1\big) \\ &- 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1) \circ (\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1) \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1) \circ (\mathbb{D}^t_\sigma \boldsymbol{t}_1) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1)^2 \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2\Big)\Big\}^T \boldsymbol{h} \\ &- \frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_2)^2 - 1\big) \\ &- 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2) \circ (\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2) \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2) \circ (\mathbb{D}^t_\sigma \boldsymbol{t}_2) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2)^2 \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2\Big)\Big\}^T \boldsymbol{h}\end{aligned}$$

Discretized IBVP action

University of Stavanger

- Introduce forward and backward branch (classical Schwinger-Keldysh)

$$\mathbb{E}^{\rm L}_{\rm IBVP} = \frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_1)^2 - 1\big) - 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1) \circ (\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1) \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1) \circ (\mathbb{D}^t_\sigma \boldsymbol{t}_1) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1)^2 \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2\Big)\Big\}^T \boldsymbol{h}$$
$$-\frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_2)^2 - 1\big) - 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2) \circ (\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2) \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2) \circ (\mathbb{D}^t_\sigma \boldsymbol{t}_2) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2)^2 \circ (\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2\Big)\Big\}^T \boldsymbol{h}$$

- Enforce initial (temporal), Dirichlet boundary (spatial) and connecting conditions

$$+(\boldsymbol{\lambda}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[\boldsymbol{t}_1] - \boldsymbol{t}_{\rm IC}\big) + (\boldsymbol{\lambda}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[\boldsymbol{\phi}_1] - \boldsymbol{\phi}_{\rm IC}\big)$$
$$+(\tilde{\boldsymbol{\lambda}}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[(\mathbb{D}_\tau \boldsymbol{t}_1)] - \dot{\boldsymbol{t}}_{\rm IC}\big) + (\tilde{\boldsymbol{\lambda}}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_1)] - \dot{\boldsymbol{\phi}}_{\rm IC}\big)$$
$$+(\boldsymbol{\gamma}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[\boldsymbol{t}_1] - \mathbb{P}^{N_\tau}_\tau[\boldsymbol{t}_2]\big) + (\boldsymbol{\gamma}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[\boldsymbol{\phi}_1] - \mathbb{P}^{N_\tau}_\tau[\boldsymbol{\phi}_2]\big)$$

$$+(\tilde{\boldsymbol{\gamma}}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{t}_1)] - \mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{t}_2)]\big)$$
$$+(\tilde{\boldsymbol{\gamma}}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_1)] - \mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_2)]\big)$$
$$+(\boldsymbol{\kappa}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^0_\sigma[\boldsymbol{\phi}_1] - \boldsymbol{0}\big) + (\tilde{\boldsymbol{\kappa}}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^{N_\sigma}_\sigma[\boldsymbol{\phi}_1] - \boldsymbol{0}\big)$$
$$+(\boldsymbol{\xi}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^0_\sigma[\boldsymbol{\phi}_2] - \boldsymbol{0}\big) + (\tilde{\boldsymbol{\xi}}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^{N_\sigma}_\sigma[\boldsymbol{\phi}_2] - \boldsymbol{0}\big).$$

Discretized IBVP action

University of Stavanger

- Introduce forward and backward branch (classical Schwinger-Keldysh)

$$
\begin{aligned}
\mathbb{E}^{\rm L}_{\rm IBVP} =& \frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_1)^2 - 1\big) \\
&\quad - 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1)\circ(\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1)\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)\circ(\mathbb{D}^t_\sigma \boldsymbol{t}_1) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1)^2\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2\Big)\Big\}^T \boldsymbol{h} \\
&-\frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_2)^2 - 1\big) \\
&\quad - 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2)\circ(\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2)\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)\circ(\mathbb{D}^t_\sigma \boldsymbol{t}_2) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2)^2\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2\Big)\Big\}^T \boldsymbol{h}
\end{aligned}
$$

- Enforce initial (temporal), Dirichlet boundary (spatial) and connecting conditions

$$
\begin{aligned}
&+(\boldsymbol{\lambda}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[\boldsymbol{t}_1] - \boldsymbol{t}_{\rm IC}\big) + (\boldsymbol{\lambda}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[\boldsymbol{\phi}_1] - \boldsymbol{\phi}_{\rm IC}\big) \\
&+(\tilde{\boldsymbol{\lambda}}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[(\mathbb{D}_\tau \boldsymbol{t}_1)] - \dot{\boldsymbol{t}}_{\rm IC}\big) + (\tilde{\boldsymbol{\lambda}}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_1)] - \dot{\boldsymbol{\phi}}_{\rm IC}\big) \\
&+(\boldsymbol{\gamma}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[\boldsymbol{t}_1] - \mathbb{P}^{N_\tau}_\tau[\boldsymbol{t}_2]\big) + (\boldsymbol{\gamma}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[\boldsymbol{\phi}_1] - \mathbb{P}^{N_\tau}_\tau[\boldsymbol{\phi}_2]\big)
\end{aligned}
$$

$$
\begin{aligned}
&+(\tilde{\boldsymbol{\gamma}}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{t}_1)] - \mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{t}_2)]\big) \\
&+(\tilde{\boldsymbol{\gamma}}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_1)] - \mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_2)]\big) \\
&+(\boldsymbol{\kappa}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^0_\sigma[\boldsymbol{\phi}_1] - \boldsymbol{0}\big) + (\tilde{\boldsymbol{\kappa}}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^{N_\sigma}_\sigma[\boldsymbol{\phi}_1] - \boldsymbol{0}\big) \\
&+(\boldsymbol{\xi}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^0_\sigma[\boldsymbol{\phi}_2] - \boldsymbol{0}\big) + (\tilde{\boldsymbol{\xi}}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^{N_\sigma}_\sigma[\boldsymbol{\phi}_2] - \boldsymbol{0}\big).
\end{aligned}
$$

- Locate extremum via numerical optimization (Interior Point Optimization)