The constraint potential for fermionic order parameters

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- Spontaneous symmetry breaking
- Bosonic order parameter
- Fermionic order parameter
- Summary

Motivation

- QCD in the chiral limit exhibits genuine phase transition.
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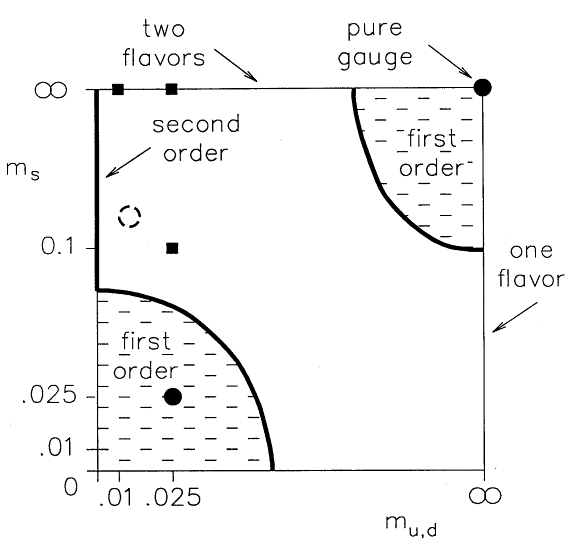
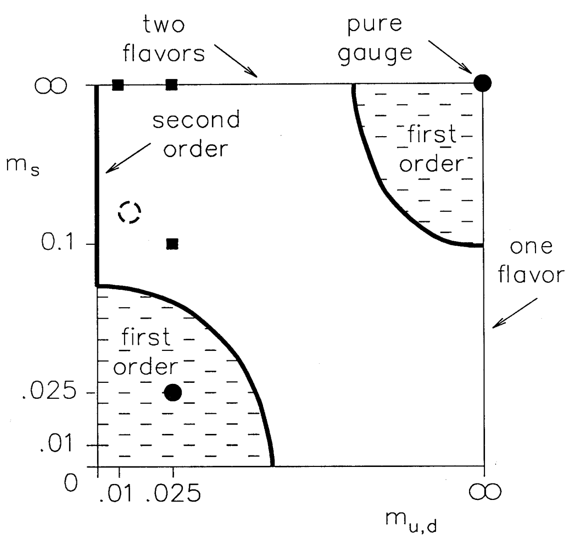


Brown et al., PRL **65** (1990)

Motivation

- QCD in the chiral limit exhibits genuine phase transition.
- The order parameter is the quark condensate $\langle \bar{\psi}\psi \rangle$.
- The study of this phase transition has a long history marked by the evolution of the Columbia-plot.
- Current Monte Carlo methods need $m \neq 0$, numerical extrapolation needed.
- Current successes (T_c, first order region discussion) rely on critical scaling around the transition.



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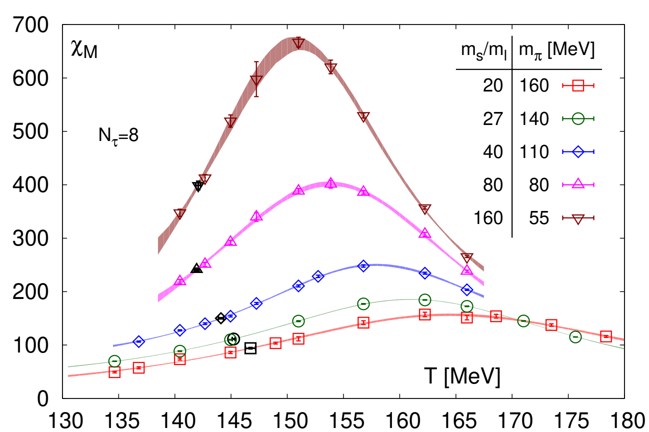
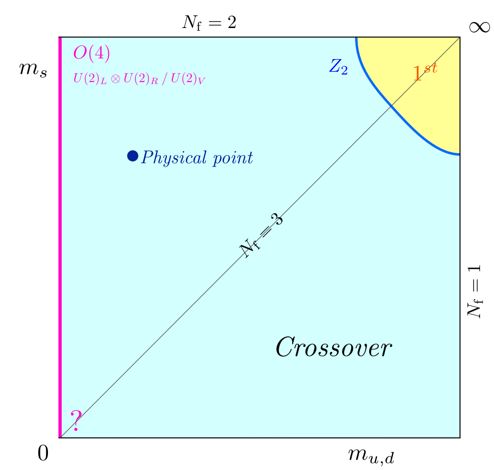


Ding et al., PRL **123** (2019)



Cuteri et al., JHEP **11** (2021)

Spontaneous symmetry breaking

- Spontaneous breaking is defined as a **double-limit**: 1) volume, 2) explicit breaking

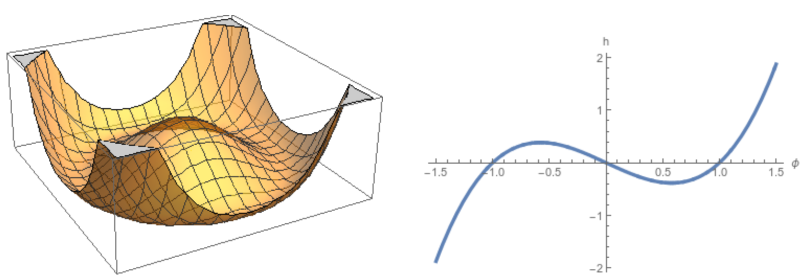
$$\langle \bar{\psi}\psi \rangle_{\mathrm{min}} = \lim_{m \to 0} \lim_{V \to \infty} \langle \bar{\psi}\psi \rangle_{V,m} \, .$$

- The **effective potential** between the different vacua is **flat** (Legendre-transformation), but cannot be accessed by usual simulations.

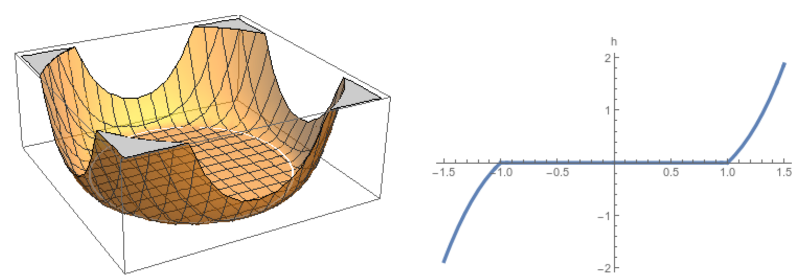
- Is there a way to evaluate the order parameter directly in the $m \to 0$ limit?

- And to access the flat region of the potential?

Classically:



QFT:



Constraint potential

Define the **constraint** effective potential

$$\exp\left(-V\Omega(\bar{\phi})\right) = \int \mathcal{D}\varphi \exp\left(-S[\varphi]\right)\, \delta\left(\int \varphi - V\phi\right) \equiv \mathcal{Z}_\phi \,.$$

- Full partition function recovered as

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- For scalar fields Markov chain Monte Carlo techniques can be constructed which **satisfy** the constraint.

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- Analogous to changing from **canonical** to **microcanonical** ensemble.

Bosonic order parameter

- Toy model example: $D = 3$, $O(2)$ symmetric φ^4 model.
- Constrain: $\delta\left(\frac{1}{V}\int d^3x\varphi(x) - \bar{\phi}\right)$.

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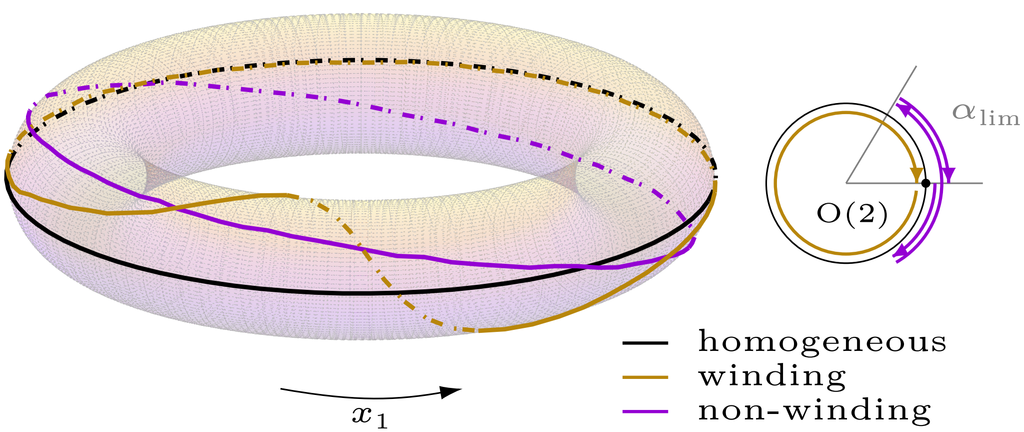
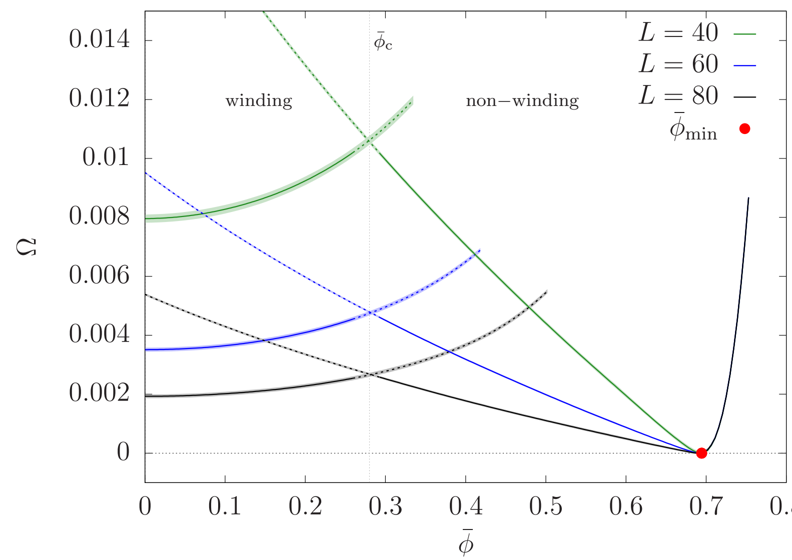
- Toy model example: $D = 3$, $O(2)$ symmetric φ^4 model.
- Constrain: $\delta\left(\frac{1}{V}\int d^3x\varphi(x) - \bar{\phi}\right)$.
- Inhomogenenous configurations dominate the path integral in the flat region.
- Two distinct topology of configurations.
- Constraint potential flattens towards infinite volume limit.





Endrődi, Kovács and GM, PRL 127 (2021)

Fermionic order parameter

- We turn to a general fermionic model with the action

$$\mathcal{Z} = \int \mathcal{D}\Phi \mathcal{D}\bar{\psi}\mathcal{D}\psi \; \exp[\bar{\psi} Q \psi - S_b[\Phi]] = \int \mathcal{D}\Phi \, \mathrm{e}^{\,-S_b[\Phi]} \det Q[\Phi] \, .$$

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$$\rightarrow \quad \text{e.g.} \quad \delta\big(\phi - \bar{\psi}\psi\big) = \sum_k^{N_{\rm lat}} \frac{(-1)^k}{k!}(\bar{\psi}\psi)^k \frac{\partial}{\partial\phi^k}\delta(\phi)\,.$$

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- Makes sense as a distribution

$$\int d\phi\, \mathcal{Z}_\phi \phi^m = \big\langle (\bar{\psi}\psi)^m \big\rangle \;,$$

- but impractical: each term in the $\sum_k$ needs simulations with (strangely) modified fermion determinant.

Fermionic order parameter

- Alternatively, use Fourier-representation: $\delta(x) = \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \, \mathrm{e}^{\,i\eta x}$

$$\mathcal{Z}_\phi = \int \frac{d\eta}{2\pi} \tilde{\mathcal{Z}}_\eta \, \mathrm{e}^{\,i\eta\phi} \, .$$

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- The characteristic function, $\tilde{\mathcal{Z}}_\eta$ looks more familiar

$$\tilde{\mathcal{Z}}_\eta = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[\bar{\psi}\left(Q - \frac{i\eta}{V}\right)\psi\right] = \det\left[Q - \frac{i\eta}{V}\right] \, .$$

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- If the spectrum of Q is directly known, physical information can be extracted from the η-dependence.

- This approach was used in the compact Schwinger model in Azcoiti et al., PLB **354** (1995).

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- If the spectrum of Q is directly known, physical information can be extracted from the η-dependence.
- This approach was used in the compact Schwinger model in Azcoiti et al., PLB **354** (1995).
- Larger lattices: approximate or partial spectrum lead to numerical instabilities (similar to Lee-Yang zeros).

Fermionic order parameter

- We follow a different route, use $\det X = \mathrm{e}^{\,\mathrm{Tr}\,\log X} \to$ expand $\log$ in V^{-1}.

$$\tilde{\mathcal{Z}}_\eta = \det Q \times \exp\left[-\sum_k \left(\frac{i}{V}\right)^k \mathrm{Tr}\,\left(Q^{-1}\eta\right)^k\right] .$$

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- In the thermodynamic limit we can treat the η-integrals in a saddle-point approximation, keeping up to NLO in $1/V$

$$\mathcal{Z}_\phi = \int \frac{d\eta}{2\pi}\,\mathrm{e}^{\,i\eta\phi}\det Q \exp\Big[-\,i\eta\underbrace{\frac{\mathrm{Tr}\,Q^{-1}}{V}}_{\mathcal{M}} + \frac{\eta^2}{V}\underbrace{\frac{\mathrm{Tr}\,Q^{-2}}{V}}_{-\chi}\Big]$$

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$$= \det Q\,\exp\left[-\frac{V}{2}(\phi-\mathcal{M})\chi^{-1}(\phi-\mathcal{M}) - \frac{1}{2}\log\det\chi\right] .$$

- Moments are correct up to V^{-2}

$$\int d\phi\mathcal{Z}_\phi\phi = \mathcal{M} \equiv \left\langle\bar{\psi}\psi\right\rangle , \quad \int d\phi\mathcal{Z}_\phi\phi^2 = \mathcal{M}^2 + \frac{\chi}{V} \equiv \left\langle(\bar{\psi}\psi)^2\right\rangle$$

$$\int d\phi\mathcal{Z}_\phi\phi^3 = \left\langle(\bar{\psi}\psi)^3\right\rangle + \mathcal{O}(V^{-2}) .$$

Fermionic order parameter

Putting the bosonic fields back

$$\mathcal{Z}_\phi = \int \mathcal{D}\Phi \, \mathrm{e}^{-S_b[\Phi]} \frac{\det Q[\Phi]}{\sqrt{\det \chi[\Phi]}} \exp\left[-\frac{V}{2}(\phi - \mathcal{M}[\Phi]) \cdot \chi^{-1}[\Phi] \cdot (\phi - \mathcal{M}[\Phi])\right]$$

- Simulations with a **modified action**.

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- Relies on **large volume expansion**, the constraint gets stronger towards the thermodynamic limit.
- Also relies on $\det \chi > 0$, which is ensured in the continuum limit.

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- Simulations with a **modified action**.
- Relies on **large volume expansion**, the constraint gets stronger towards the thermodynamic limit.
- Also relies on $\det \chi > 0$, which is ensured in the continuum limit.
- Similar to a **density of states** approach, but has a **natural width**, $\chi \to$ no need for extra extrapolation!
- **Explicit results** in the chiral GN model:

see the next talk by
L. Pannullo

Summary and outlook

- The **constraint potential** is a tool to discuss **spontaneous symmetry breaking**.
- Directly at **vanishing** explicit breaking.
- Monte Carlo simulations for bosonic order parameters has been used.
- A generalization to fermionic order parameters is not straightforward.
- We gave a **generalization** which becomes **exact** in the **thermodynamic limit**.
- It is also feasible to simulate.
- **First results:**

  see the next talk by L. Pannullo

- More to follow!