Liverpool, Lattice2024

Extracting scattering amplitudes from Euclidean correlators

work in collaboration with Nazario Tantalo
based on **arXiv:2407.02069**

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Introduction

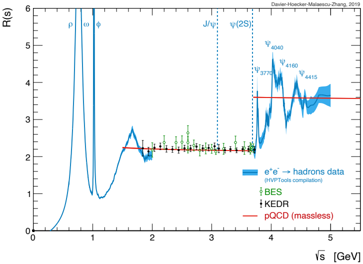
- Scattering amplitudes are inherently Minkowskian observables. Only Euclidean correlators are calculable in lattice QCD. Analytic continuation is needed to real time. Numerically ill-posed problem.

- Scattering amplitudes can be extracted from energy levels in large but finite volume. Energy levels can be calculated from Eucliden correlators. More theory needs to be developed every time a new multi-particle threshold is opened.
  M. Luscher, Commun. Math. Phys. **105** (1986), 153-188
  M. Luscher, Nucl. Phys. B **354** (1991), 531-578
  C. h. Kim, C. T. Sachrajda and S. R. Sharpe, Nucl. Phys. B **727** (2005), 218-243
  M. T. Hansen and S. R. Sharpe, Phys. Rev. D **90** (2014) no.11, 116003
  [...]

- Approximate scattering amplitudes as a linear combination of Euclidean correlators sampled at discrete times.
  J. C. A. Barata and K. Fredenhagen, Commun. Math. Phys. **138** (1991), 507-520
  J. Bulava and M. T. Hansen, Phys. Rev. D **100** (2019) no.3, 034521

An analogy: spectral densities

M. Hansen, A. Lupo and N. Tantalo, Phys. Rev. D **99**, no.9, 094508 (2019)
William Jay, Lattice24 talk, Mon 17:30 (see also references therein)
Matteo Saccardi, Lattice24 talk, Wed 11:55



M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, EurPhysJ. C80, no.3, 241 (2020).

$$\underbrace{C(t)}_{\text{Euclidean correlator}} = \int d^3\boldsymbol{x}\, \langle j_k(t,\boldsymbol{x}) j_k(0) \rangle = \int_0^\infty dE\, e^{-tE}\, \underbrace{\rho(E)}_{\text{Spectral density } (\propto \text{ R-ratio})}$$

Euclidean correlator

Spectral density ($\propto$ R-ratio)

$$C(t) = \int d^3\boldsymbol{x}\, \langle j_k(t,\boldsymbol{x}) j_k(0) \rangle = \int_0^\infty dE\, e^{-tE}\, \rho(E)$$

Euclidean correlator

Spectral density ($\propto$ R-ratio)

1. Target smeared spectral density $$\rho(E) = \lim_{\sigma \to 0^+} \int dE'\, K_\sigma(E'-E) \rho(E') \; .$$

   The smearing kernel must be smooth with $$\lim_{\sigma \to 0} K_\sigma(E) = \delta(E) \; .$$

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2. Approximate of smearing kernel $$K_\sigma(E) \simeq P_{\sigma,\epsilon}(e^{-\tau E}) = \sum_{n=1}^{N} w_n^{\sigma,\epsilon} e^{-n\tau E} \; ,$$

   with given precision $$\| K_\sigma(E) - P_{\sigma,\epsilon}(e^{-\tau E}) \| < \epsilon \; .$$

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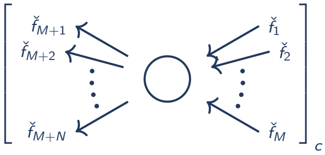
3. Approximation formula $$\rho(E) = \lim_{\sigma \to 0^+} \lim_{\epsilon \to 0^+} \sum_{n=1}^{N} w_n^{\sigma,\epsilon} e^{\tau E} C(n\tau)$$

Can we do anything similar for scattering amplitudes?

Scattering amplitudes

Haag-Ruelle scattering theory



$$= \lim_{\sigma\to 0^+} \int \left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3} \check{f}_A^{(*)}(\boldsymbol{p}_A) \right] \int \left[\prod_A \frac{d\omega_A}{2\pi} \right] \hat{K}_\sigma(\omega,\boldsymbol{p}) \, \rho_c(\omega,\boldsymbol{p})$$

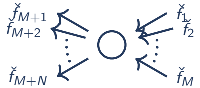
- $\check{f}_A^{(*)}(\boldsymbol{p}_A)$: wave functions of in/out particles
- $\hat{K}_\sigma(\omega,\boldsymbol{p})$: smearing kernel with radius σ
- $\rho_c(\omega,\boldsymbol{p})$: spectral density of $(M+N)$-pt function

Polynomial approx of smearing kernel

$$= \lim_{\sigma\to 0^+} \lim_{\epsilon\to 0^+} \sum_{\substack{n_1,n_2\cdots\geq 1 \\ b\geq 0}} w_{n,b}^{\sigma,\epsilon} \int \left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3} \right] \hat{\Upsilon}_b(n\tau,\boldsymbol{p}) \, C_c(n\tau,\boldsymbol{p})$$

- $w_{n,b}^{\sigma,\epsilon}$: coefficients of polynomial approx
- $\hat{\Upsilon}_b(n\tau,\boldsymbol{p})$: known kinematical functions
- $C_c(n\tau,\boldsymbol{p})$: time-momentum Euclidean $(M+N)$-pt function

Approximation formula



$$\Big[\;\cdots\;\Big]_c = \lim_{\sigma\to 0^+}\lim_{\epsilon\to 0^+}\sum_{\substack{n_1,n_2\cdots\geq 1\\ b\geq 0}} w^{\sigma,\epsilon}_{n,b}\int\left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3}\right]\hat{\Upsilon}_b(n\tau;\boldsymbol{p})\;\hat{C}_c(n\tau;\boldsymbol{p})$$

- Euclidean correlator:

$$\hat{C}_c(s;\boldsymbol{p}) = \langle\Omega|\hat{\phi}(\boldsymbol{p}_{M+1})e^{-s_{M+N}H}\cdots\hat{\phi}(\boldsymbol{p}_{M+N})e^{-s_M H}\hat{\phi}(\boldsymbol{p}_M)^\dagger\cdots e^{-s_1 H}\hat{\phi}(\boldsymbol{p}_1)^\dagger|\Omega\rangle_c$$

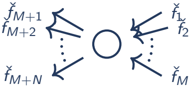
- Kinematic function:

$$\hat{\Upsilon}_b(s;\boldsymbol{p}) = [\Delta(\boldsymbol{p})]^b\,\tilde{h}(\Delta(\boldsymbol{p}))\exp\left\{\sum_{A=1}^{M} s_A\sum_{B=1}^{A}E(\boldsymbol{p}_B) + \sum_{A=M+1}^{M+N-1} s_A\sum_{B=M+1}^{A}E(\boldsymbol{p}_B)\right\}$$

Violation of asympt. energy conservation: $\Delta(\boldsymbol{p}) = \left\{\sum_{A=M+1}^{M+N} - \sum_{A=1}^{M}\right\}E(\boldsymbol{p}_A)$.

$\tilde{h}(\Delta)$ auxiliary function: smooth, compact support, $\tilde{h}(0)=1$.

Approximation formula

$\Big[$  $\Big]_c$

$$= \lim_{\sigma\to 0^+} \lim_{\epsilon\to 0^+} \sum_{\substack{n_1, n_2 \cdots \geq 1 \\ b \geq 0}} w^{\sigma,\epsilon}_{n,b} \int \left[\prod_A \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3} \right] \hat{\Upsilon}_b(n\tau; \boldsymbol{p}) \; \hat{C}_c(n\tau; \boldsymbol{p})$$

- Coefficients of polynomial approximation of smearing kernels:

$$K_\sigma(\omega, \Delta) \quad \simeq \quad P_{\sigma,\epsilon}(e^{-\tau\omega}, \Delta) \quad = \quad \sum_{\substack{n_1, n_2 \cdots \geq 1 \\ b \geq 0}} w^{\sigma,\epsilon}_{n,b} \left[\prod_A \left(e^{-\tau \omega_A} \right)^{n_A} \right] \Delta^b$$

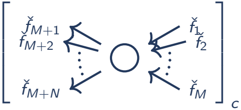
$$\left\| K_\sigma(\omega, \Delta) - P_{\sigma,\epsilon}(e^{-\tau\omega}, \Delta) \right\| < \epsilon$$

Theorem. For every $r > 0$, two constants A, B_r (independent of ϵ and σ) exist such that

$\Big|$  $- \mathrm{approx}(\sigma, \epsilon) \Big| \; < \; A\epsilon + B_r \sigma^r$

assuming that the wave functions have non-overlapping velocities [not essential].

Approximation formula



$$= \lim_{\sigma\to 0^+} \lim_{\epsilon \to 0^+} \sum_{\substack{n_1, n_2 \cdots \geq 1 \\ b \geq 0}} w_{n,b}^{\sigma,\epsilon} \int \left[\prod_A \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3} \right] \hat{\Upsilon}_b(n\tau; \boldsymbol{p}) \; \hat{C}_c(n\tau; \boldsymbol{p})$$

- Coefficients of polynomial approximation of smearing kernels:

$$K_\sigma(\omega, \Delta) \simeq P_{\sigma,\epsilon}(e^{-\tau\omega}, \Delta) = \sum_{\substack{n_1, n_2 \cdots \geq 1 \\ b \geq 0}} w_{n,b}^{\sigma,\epsilon} \left[\prod_A \left(e^{-\tau \omega_A}\right)^{n_A} \right] \Delta^b$$

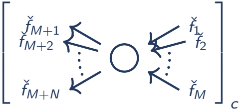
$$\left\| K_\sigma(\omega, \Delta) - P_{\sigma,\epsilon}(e^{-\tau\omega}, \Delta) \right\| < \epsilon$$

What I am not telling you:

- What does the smearing kernel look like?
- What norm do we need to choose?

See paper or backup slides.

Approximation formula

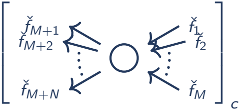


$$= \lim_{\sigma\to 0^+} \lim_{\epsilon\to 0^+} \sum_{\substack{n_1,n_2\cdots\geq 1 \\ b\geq 0}} w^{\sigma,\epsilon}_{n,b} \int \left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3}\right] \hat{\Upsilon}_b(n\tau;\boldsymbol{p}) \; \hat{C}_c(n\tau;\boldsymbol{p})$$

- Smaller ϵ $\Rightarrow$ better approximation of Haag-Ruelle kernel $\Rightarrow$ larger values of n $\Rightarrow$ larger statistical noise.
- Smaller σ $\Rightarrow$ Haag-Ruelle kernel more peaked $\Rightarrow$ harder to approximate $\Rightarrow$ larger values of n $\Rightarrow$ larger statistical noise.
- Also recall: $\Upsilon_h(n\tau;\boldsymbol{p})$ increases exponentially with n.
- Optimization problem: smaller ϵ and σ means larger statistical errors, larger ϵ and σ means larger systematic error. One could design a strategy based on HLT to minimize total error:

$$A[w] = \left\| K_\sigma(\omega,\Delta) - P_{\sigma,\epsilon}(e^{-\tau\omega},\Delta) \right\|^2 \qquad B[w] = \sum_{n,b,n',b'} w^{\sigma,\epsilon}_{n,b} \langle\langle \mathcal{C}_{n,b}\mathcal{C}_{n',b'} \rangle\rangle_c \, w^{\sigma,\epsilon}_{n',b'}$$

Approximation formula



$$= \lim_{\sigma \to 0^+} \lim_{\epsilon \to 0^+} \sum_{\substack{n_1, n_2 \cdots \geq 1 \\ b \geq 0}} w_{n,b}^{\sigma,\epsilon} \int \left[\prod_A \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3} \right] \hat{\Upsilon}_b(n\tau; \boldsymbol{p}) \; \hat{C}_c(n\tau; \boldsymbol{p})$$

- A finite-volume estimator is obtained trivially by replacing $\int \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3}$ with $\frac{1}{L^3} \sum_{\boldsymbol{p}_A}$.

  If coefficients $w_{n,b}$ are kept fixed as the volume is varied, then the $L \to +\infty$ limit is approached exponentially fast. Having Schwartz wave functions is essential for this step.

- The continuum limit of the estimator can be understood in terms of Symanzik effective theory.

- In this approach, the $L \to \infty$ and $a \to 0$ limits must be taken before the $\epsilon \to 0$ and $\sigma \to 0$ limits. In particular τ cannot be identified with the lattice spacing. For the opposit approach, see Barata and Fredenhagen.

Conclusions and outlook

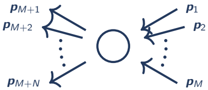
- We have derived an approximation for scattering amplitudes as a linear combination of Euclidean correlators sampled at discrete times.
- This formula provides the blueprints for a potentially viable numerical strategy.
- Our approximation can be calculated from finite-volume correlators and the infinite-volume limit is approached exponentially fast.
- Whether statistical and systematic errors are under control in typical QCD simulations remains to be seen.
- Recent algorithmic methods (e.g. Hansen-Lupo-Tantalo), which have been successful in approximations of spectral densities, can be adapted to this problem.
- The class of operators used to approximate asymptotic states can be generalized by relaxing the constraint that $\tilde{f}^t(p)$ must have compact support. This may make the numerics easier.

Backup slides

Talk given at CERN workshop, July 2024

Introduction

- How do we calculate hadron scattering amplitude from Quantum Chromodynamics? In principle...



$$\propto \lim_{p_0 \to \pm E(\boldsymbol{p})} \left[\prod_A (p_A^2 - m^2) \right] \langle \Omega | \mathrm{T} \tilde{\phi}(p_{M+1}) \cdots \tilde{\phi}(p_{M+N}) \tilde{\phi}(p_M)^\dagger \cdots \tilde{\phi}(p_1)^\dagger | \Omega \rangle$$

- Numerical lattice QCD is the only known tool which allows the calculation of observables in QCD at the nonperturbative level.

- Only Euclidean correlators are calculable in lattice QCD. Analytic continuation is needed to real time. Numerically ill-posed problem.

- Find another way...

Introduction

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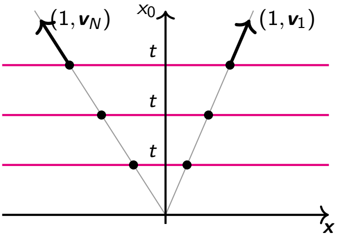
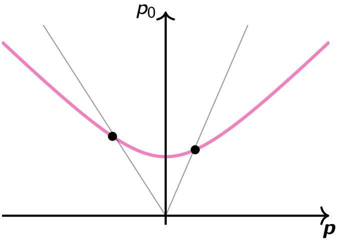
Outlook

Theoretical background

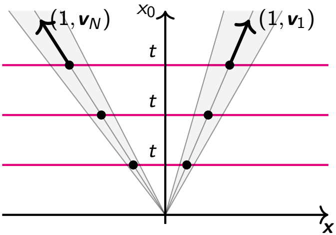
Haag-Ruelle scattering theory





- Black lines = classical trajectories.

- Pink line: $p^2 = m^2$
- Black dots: energy-momentum of particle.



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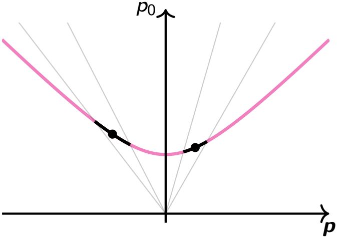
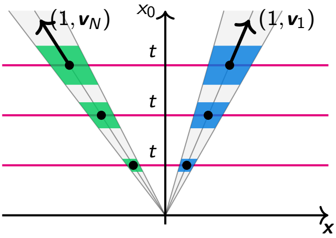
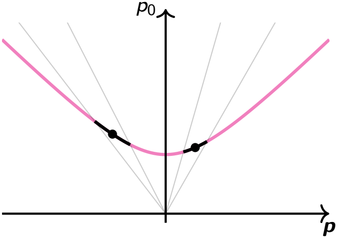
- Allow velocity indetermination.
- Gray regions = cones of classical trajectories.



- Pink line: $p^2 = m^2$
- Black dots: energy-momentum of particle.
- Allow momentum indetermination.
- Black lines: allowed values for energy-momentum.



- Black lines = classical trajectories.
- Allow velocity indetermination.
- Gray regions = cones of classical trajectories.
- Green/blue regions = position of particle at time t.



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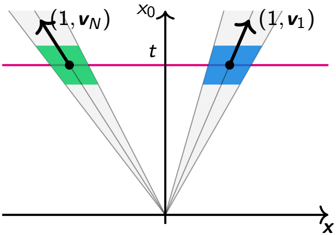
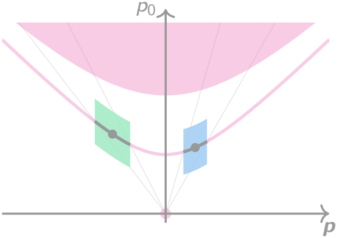
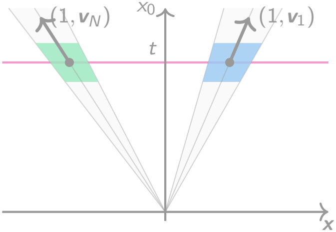
$$|\Psi_{\text{out}}(t)\rangle = \int d^4x_N\, f_N^t(x_N)\phi(x_N)^\dagger \cdots \int d^4x_1\, f_1^t(x_1)\phi(x_1)^\dagger\, |\Omega\rangle$$

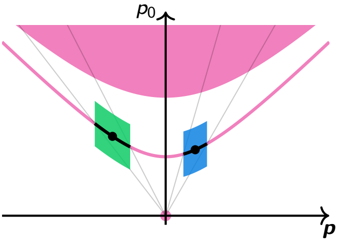




- Gray regions = cones of classical trajectories.
- Green/blue regions scale with t.
- $f_A^t(x)$ is localized in green/blue regions.

$$|\Psi_{\text{out}}(t)\rangle = \int \frac{d^4 p_N}{(2\pi)^4} \tilde{f}_N^t(p_N)\tilde{\phi}(p_N)^\dagger \cdots \int \frac{d^4 p_1}{(2\pi)^4} \tilde{f}_1^t(p_1)\tilde{\phi}(p_1)^\dagger \, |\Omega\rangle$$





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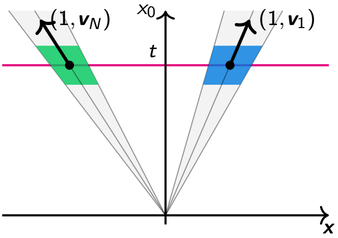
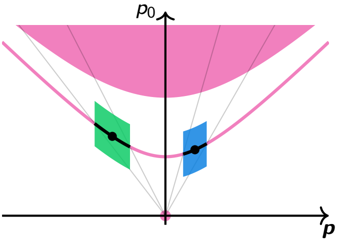
- $f_A^t(x)$ is localized in green/blue regions.

- Pink regions = spectrum of P.
- Green/blue regions intersect spectrum of P on 1-particle mass shell.
- $\tilde{f}_A^t(p)$ has support in green/blue regions.

$$|\Psi_{\text{out}}(t)\rangle = \int \frac{d^4 p_N}{(2\pi)^4}\, \tilde{f}_N^t(p_N)\tilde{\phi}(p_N)^\dagger \cdots \int \frac{d^4 p_1}{(2\pi)^4}\, \tilde{f}_1^t(p_1)\tilde{\phi}(p_1)^\dagger \;|\Omega\rangle$$

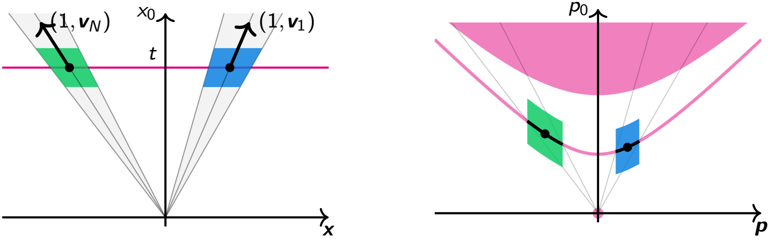




- Gray regions = cones of classical trajectories.
- Green/blue regions scale with t.
- $f_A^t(x)$ is localized in green/blue regions.
- Interaction between particles decreases with t.

- Pink regions = spectrum of P.
- Green/blue regions intersect spectrum of P on 1-particle mass shell.
- $\tilde{f}_A^t(p)$ has support in green/blue regions.

$$|\Psi_{\text{out}}(t)\rangle = \boxed{\int \frac{d^4p_N}{(2\pi)^4}\,\tilde{f}_N^t(p_N)\tilde{\phi}(p_N)^\dagger} \cdots \boxed{\int \frac{d^4p_1}{(2\pi)^4}\,\tilde{f}_1^t(p_1)\tilde{\phi}(p_1)^\dagger}\,|\Omega\rangle$$



$$\tilde{f}_A^t(p) = e^{it[p_0 - E(\boldsymbol{p})]}\zeta_A(p_0 - E(\boldsymbol{p}))\,\check{f}_A(\boldsymbol{p})$$

- $\check{f}_A(\boldsymbol{p}) =$ asymptotic particle wave function and $E(\boldsymbol{p}) = \sqrt{m^2 + \boldsymbol{p}^2}$.
- $\zeta_A(\omega)$ cuts off multi-particle states. $\zeta_A(\omega)$ smooth and compact support, $\zeta_A(0) = 1$.
- Support of $\tilde{f}_A^t(p)$ intersects spectrum of P only on 1-particle mass shell.

Haag-Ruelle scattering theory

$$|\Psi_{\text{out}}(t)\rangle = \prod_A \int \frac{d^4 p_A}{(2\pi)^4} \tilde{f}_A^t(p_A)\tilde{\phi}(p_A)^\dagger \ |\Omega\rangle$$

$$\stackrel{t\to\pm\infty}{=} \prod_A \int \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3} \check{f}_A(\boldsymbol{p}_A) a^\dagger_{\text{out}}(\boldsymbol{p}_A) \ |\Omega\rangle + O(|t|^{-\infty})$$

- $\tilde{f}_A^t(p) = e^{it[p_0 - E(\boldsymbol{p})]}\zeta_A(p_0 - E(\boldsymbol{p}))\check{f}_A(\boldsymbol{p})$
- Error is $O(|t|^{-\infty})$ for non-overlapping velocities, otherwise $O(|t|^{-1/2})$.
- $a^\dagger_{\text{out}}(\boldsymbol{p})$ are standard creation operators:

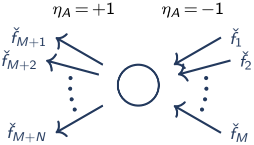
$$[a_{\text{out}}(\boldsymbol{p}), a^\dagger_{\text{out}}(\boldsymbol{p}')] = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{p}') \qquad [a_{\text{out}}(\boldsymbol{p}), a_{\text{out}}(\boldsymbol{p}')] = 0$$

$$[\boldsymbol{P}, a^\dagger_{\text{out}}(\boldsymbol{p})] = \boldsymbol{p}\, a^\dagger_{\text{out}}(\boldsymbol{p}) \qquad [H, a^\dagger_{\text{out}}(\boldsymbol{p})] = E(\boldsymbol{p})\, a^\dagger_{\text{out}}(\boldsymbol{p})$$

Approximation formula for scattering amplitudes

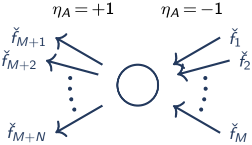
Rough sketch of derivation

Scattering amplitude



$$= \lim_{t \to +\infty} \langle \Psi_{\mathrm{out}}(t) | \Psi_{\mathrm{in}}(-t) \rangle$$

Scattering amplitude

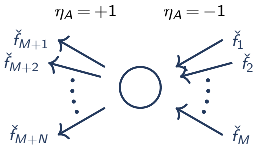


$$= \lim_{t\to+\infty} \langle \Psi_{\text{out}}(t) | \Psi_{\text{in}}(-t) \rangle$$

$$= \lim_{t\to+\infty} \int \left[\prod_A \frac{d^4 p_A}{(2\pi)^4} \check{f}_A^{(*)}(\boldsymbol{p}_A) \zeta_A^{(*)}(p_A^0 - E(\boldsymbol{p}_A)) \right] e^{it\sum_A \eta_A [p_A^0 - E(\boldsymbol{p}_A)]}$$

$$\times \langle \Omega | \tilde{\phi}(p_{M+1}) \cdots \tilde{\phi}(p_{M+N}) \tilde{\phi}(p_M)^\dagger \cdots \tilde{\phi}(p_1)^\dagger | \Omega \rangle$$

Scattering amplitude



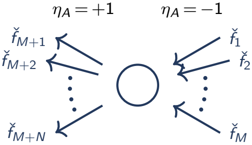
$$= \lim_{t\to+\infty} \langle \Psi_{\text{out}}(t) | \Psi_{\text{in}}(-t) \rangle$$

$$= \lim_{t\to+\infty} \int \left[\prod_A \frac{d^4 p_A}{(2\pi)^4} \check{f}_A^{(*)}(\boldsymbol{p}_A) \zeta_A^{(*)}(p_A^0 - E(\boldsymbol{p}_A)) \right] e^{it \sum_A \eta_A [p_A^0 - E(\boldsymbol{p}_A)]}$$

$$\times \langle \Omega | \tilde{\phi}(p_{M+1}) \cdots \tilde{\phi}(p_{M+N}) \tilde{\phi}(p_M)^\dagger \cdots \tilde{\phi}(p_1)^\dagger | \Omega \rangle$$

- Wildly oscillating phase for $t \to +\infty$.
- Not good for numerics.
- Cancallation of regions with $\sum_A \eta_A [p_A^0 - E(\boldsymbol{p}_A)] \neq 0$.
- Can we achieve the same effect in a different way? Some mathematical trickery...

Scattering amplitude



$$= \lim_{t\to+\infty} \langle \Psi_{\mathrm{out}}(t) | \Psi_{\mathrm{in}}(-t) \rangle$$

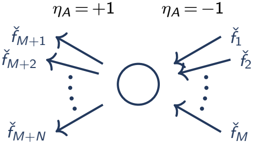
Introduce two auxiliary functions:

- $\Phi(t)$ Schwartz with unit integral and closed support in $(0,+\infty)$;
- $h(t)$ Schwartz with unit integral.

$$\lim_{\sigma\to0^+} \int dt\, ds\, \Phi(t)\, h(s) \left\langle \Psi_{\mathrm{out}}\left(\tfrac{t}{2\sigma}-s\right) \middle| \Psi_{\mathrm{in}}\left(-\tfrac{t}{2\sigma}-s\right) \right\rangle$$

$$= \int ds\, h(s) \int_0^{+\infty} dt\, \Phi(t) \lim_{\sigma\to0^+} \left\langle \Psi_{\mathrm{out}}\left(\tfrac{t}{2\sigma}-s\right) \middle| \Psi_{\mathrm{in}}\left(-\tfrac{t}{2\sigma}-s\right) \right\rangle = \langle \Psi_{\mathrm{out}}(+\infty) | \Psi_{\mathrm{in}}(-\infty) \rangle$$

Scattering amplitude (2)

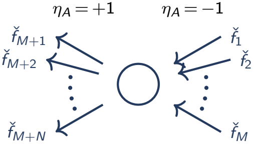


$$= \lim_{\sigma \to 0^+} \int dt\, ds\, \Phi(t)\, h(s) \left\langle \Psi_{\mathrm{out}}\left(\tfrac{t}{2\sigma} - s\right) \middle| \Psi_{\mathrm{in}}\left(-\tfrac{t}{2\sigma} - s\right) \right\rangle$$

$$= \lim_{\sigma \to 0^+} \int \left[\prod_A \frac{d^4 p_A}{(2\pi)^4} \check{f}_A^{(*)}(\boldsymbol{p}_A) \zeta_A^{(*)}(p_A^0 - E(\boldsymbol{p}_A)) \right] \tilde{h}\left(\sum_A \eta_A E(\boldsymbol{p}_A) \right) \tilde{\Phi}\left(\frac{1}{\sigma} \sum_A \eta_A [p_A^0 - E(\boldsymbol{p}_A)] \right)$$

$$\times \langle \Omega | \tilde{\phi}(p_{M+1}) \cdots \tilde{\phi}(p_{M+N}) \tilde{\phi}(p_M)^\dagger \cdots \tilde{\phi}(p_1)^\dagger | \Omega \rangle$$

Scattering amplitude (2)



$$= \lim_{\sigma\to 0^+} \int dt\, ds\, \Phi(t)\, h(s) \left\langle \Psi_{\mathrm{out}}\left(\tfrac{t}{2\sigma} - s\right) \middle| \Psi_{\mathrm{in}}\left(-\tfrac{t}{2\sigma} - s\right) \right\rangle$$

$$= \lim_{\sigma\to 0^+} \int \left[\prod_A \frac{d^4 p_A}{(2\pi)^4} \check{f}_A^{(*)}(\boldsymbol{p}_A) \zeta_A^{(*)}(p_A^0 - E(\boldsymbol{p}_A)) \right] \tilde{h}\left(\sum_A \eta_A E(\boldsymbol{p}_A) \right) \tilde{\Phi}\left(\frac{1}{\sigma} \sum_A \eta_A [p_A^0 - E(\boldsymbol{p}_A)] \right)$$

$$\times \langle \Omega | \tilde{\phi}(p_{M+1}) \cdots \tilde{\phi}(p_{M+N}) \tilde{\phi}(p_M)^\dagger \cdots \tilde{\phi}(p_1)^\dagger | \Omega \rangle$$

- $\tilde{\Phi}$ regularizes the wildly-oscillating phase factor and selects the desired time-ordering. **It must be complex!**
- $\tilde{h}(\Delta)$ can be chosen with compact and arbitrarily narrow support around $\Delta = 0$. It cuts away contributions characterized by non-zero violations of the asymptotic energy conservation.
- Wightman function in momentum space $\simeq$ spectral density.

Wightman function $\simeq$ spectral density

$$\langle\Omega|\tilde{\phi}(p_{M+1}) \quad \tilde{\phi}(p_{M+2}) \quad \cdots \quad \tilde{\phi}(p_{M+N}) \quad \tilde{\phi}(p_M)^\dagger \quad \cdots \quad \tilde{\phi}(p_2)^\dagger \quad \tilde{\phi}(p_1)^\dagger|\Omega\rangle$$

Wightman function $\simeq$ spectral density

$$\mathcal{E}_{M+1} = p^0_{M+1} \qquad \mathcal{E}_{M+2} = p^0_{M+1} + p^0_{M+2} \qquad \mathcal{E}_M = p^0_1 + \cdots + p^0_M \qquad \mathcal{E}_{M-1} = p^0_1 + \cdots + p^0_{M-1} \qquad \mathcal{E}_2 = p^0_1 + p^0_2 \qquad \mathcal{E}_1 = p^0_1$$

$$\langle\Omega|\tilde{\phi}(p_{M+1}) \uparrow \tilde{\phi}(p_{M+2}) \uparrow \cdots \quad \tilde{\phi}(p_{M+N}) \uparrow \tilde{\phi}(p_M)^\dagger \uparrow \cdots \uparrow \tilde{\phi}(p_2)^\dagger \uparrow \tilde{\phi}(p_1)^\dagger|\Omega\rangle$$

Wightman function $\simeq$ spectral density

$$\mathcal{E}_M = p_1^0 + \cdots + p_M^0$$
$$\mathcal{E}_{M-1} = p_1^0 + \cdots + p_{M-1}^0$$
$$\mathcal{E}_{M+2} = p_{M+1}^0 + p_{M+2}^0 \qquad \mathcal{E}_2 = p_1^0 + p_2^0$$
$$\mathcal{E}_{M+1} = p_{M+1}^0 \qquad \mathcal{E}_1 = p_1^0$$

$$\langle\Omega|\tilde{\phi}(p_{M+1}) \uparrow \tilde{\phi}(p_{M+2}) \uparrow \cdots \quad \tilde{\phi}(p_{M+N}) \uparrow \tilde{\phi}(p_M)^\dagger \uparrow \cdots \uparrow \tilde{\phi}(p_2)^\dagger \uparrow \tilde{\phi}(p_1)^\dagger|\Omega\rangle$$

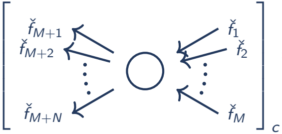
$$= 2\pi\delta(\mathcal{E}_{M+N} - \mathcal{E}_M)$$
$$\times \langle\Omega|\hat{\phi}(\boldsymbol{p}_{M+1})2\pi\delta(H-\mathcal{E}_{M+1})\cdots\hat{\phi}(\boldsymbol{p}_{M+N})2\pi\delta(H-\mathcal{E}_M)\hat{\phi}(\boldsymbol{p}_M)\cdots 2\pi\delta(H-\mathcal{E}_1)\hat{\phi}(\boldsymbol{p}_1)|\Omega\rangle$$

definitions: $\hat{\phi}(\boldsymbol{p}) = \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{p}\boldsymbol{x}}\phi(0,\boldsymbol{x})$

$\omega_A = \mathcal{E}_A - [\mathcal{E}_A]_{\text{on-shell}}$

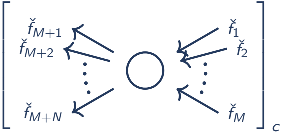
$$= 2\pi\delta(\mathcal{E}_{M+N} - \mathcal{E}_M)\rho(\omega,\boldsymbol{p})$$

Approximation formula



$$= \lim_{\sigma\to 0^+} \int \left[\prod_A \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3}\, \check{f}_A^{(*)}(\boldsymbol{p}_A)\right] \tilde{h}(\Delta(\boldsymbol{p})) \times \int \left[\prod_A \frac{d\omega_A}{2\pi}\right] K_\sigma(\omega,\Delta(\boldsymbol{p}))\, \rho_c(\omega,\boldsymbol{p})$$

Approximation formula



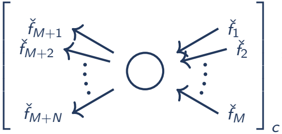
$$= \lim_{\sigma\to 0^+} \int \left[\prod_A \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3} \check{f}_A^{(*)}(\boldsymbol{p}_A) \right] \tilde{h}(\Delta(\boldsymbol{p})) \times \int \left[\prod_A \frac{d\omega_A}{2\pi} \right] K_\sigma(\omega, \Delta(\boldsymbol{p})) \, \rho_c(\omega, \boldsymbol{p})$$

Haag-Ruelle kernel $K_\sigma(\omega,\Delta)$ smears the spectral density in the energy variable ω. The parameter σ plays the role of the smearing radius.

$$K_\sigma(\omega,\Delta) = \tilde{\Phi}\left(\frac{2\omega_M - \Delta}{2\sigma}\right) \zeta_1(\omega_1) \left[\prod_{A=2}^{M-1} \zeta_A(\omega_A - \omega_{A-1}) \right] \zeta_M(\omega_M - \omega_{M-1}) \times \zeta^*_{M+1}(\omega_{M+1}) \left[\prod_{A=M+2}^{M+N-1} \zeta^*_A(\omega_A - \omega_{A-1}) \right] \zeta^*_{M+N}(\omega_M - \omega_{M+N-1} - \Delta)$$

Violation of asymptotic energy conservation: $\Delta(\boldsymbol{p}) = \sum_A \eta_A E(\boldsymbol{p}_A)$.

Approximation formula



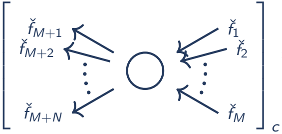
$$= \lim_{\sigma\to 0^+} \int \left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3}\, \check{f}_A^{(*)}(\boldsymbol{p}_A) \right] \tilde{h}(\Delta(\boldsymbol{p})) \times \int \left[\prod_A \frac{d\omega_A}{2\pi} \right] K_\sigma(\omega,\Delta(\boldsymbol{p}))\, \rho_c(\omega,\boldsymbol{p})$$

Approximation is obtained by replacing the Haag-Ruelle kernel with a polynomial in the variables $e^{-\tau\omega}$ and Δ:

$$K_\sigma(\omega,\Delta) \quad \longrightarrow \quad P_{\sigma,\epsilon}(e^{-\tau\omega},\Delta) = \sum_{n_1,n_2\cdots\geq 1} \sum_{b\geq 0} w_{n,b}^{\sigma,\epsilon} \left[\prod_A \left(e^{-\tau\omega_A}\right)^{n_A} \right] \Delta^b$$

$$\left\| K_\sigma(\omega,\Delta) - P_{\sigma,\epsilon}(e^{-\tau\omega},\Delta) \right\|_{???} < \epsilon$$

Approximation formula



$$= \lim_{\sigma\to 0^+} \int \left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3}\, \check{f}_A^{(*)}(\boldsymbol{p}_A)\right] \tilde{h}(\Delta(\boldsymbol{p})) \times \int \left[\prod_A \frac{d\omega_A}{2\pi}\right] K_\sigma(\omega,\Delta(\boldsymbol{p}))\, \rho_c(\omega,\boldsymbol{p})$$

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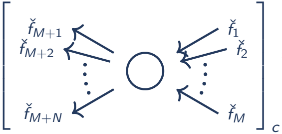
$$\left\|K_\sigma(\omega,\Delta) - P_{\sigma,\epsilon}(e^{-\tau\omega},\Delta)\right\|_{???} < \epsilon$$

Integrating $P_{\sigma,\epsilon}(e^{-\tau\omega},\Delta)$ against the spectral density yields the Euclidean correlator!



$$= \lim_{\sigma\to 0^+} \lim_{\epsilon\to 0^+} \int \left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3} \check{f}_A^{(*)}(\boldsymbol{p}_A) \right] \times \sum_{n_1, n_2 \cdots \geq 1} \sum_{b\geq 0} w_{n,b}^{\sigma,\epsilon} [\Delta(\boldsymbol{p})]^b \, \Upsilon_h(n\tau;\boldsymbol{p}) \, \hat{C}_c(n\tau;\boldsymbol{p})$$

- Euclidean correlator:

$$\hat{C}_c(s;\boldsymbol{p}) = \langle\Omega|\hat{\phi}(\boldsymbol{p}_{M+1})e^{-s_{M+N}H}\cdots\hat{\phi}(\boldsymbol{p}_{M+N})e^{-s_M H}\hat{\phi}(\boldsymbol{p}_M)^\dagger\cdots e^{-s_1 H}\hat{\phi}(\boldsymbol{p}_1)^\dagger|\Omega\rangle_c$$

- Kinematic function:

$$\Upsilon_h(s;\boldsymbol{p}) = \tilde{h}(\Delta(\boldsymbol{p}))\exp\left\{ \sum_{A=1}^{M} s_A \sum_{B=1}^{A} E(\boldsymbol{p}_B) + \sum_{A=M+1}^{M+N-1} s_A \sum_{B=M+1}^{A} E(\boldsymbol{p}_B) \right\}$$

Which norm?

$$\sum_{\substack{\|\alpha\|_1=\mathfrak{N}_\omega \\ 0\le b\le \mathfrak{N}_{\boldsymbol{p}}}} \bar{\Delta}^b \int_{\mathbb{K}} \left[\prod_{A=1}^{M+N-1} \frac{d\omega_A}{2\pi}\right] d\Delta\, e^{\tau \sum_A \omega_A} \left| D_\omega^\alpha \partial_\Delta^b \left[K_\sigma(\omega,\Delta) - P_{\sigma,\epsilon}(e^{-\tau\omega},\Delta)\right]\right|^2 < \epsilon^2$$

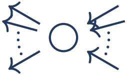
- One can choose some linear combinations of weighted L^2 norm for various derivatives.
- The integration domain $\mathbb{K}$ is completely determined by kinematics.
- The number of derivatives that one needs to control ($\mathfrak{N}_\omega$, $\mathfrak{N}_{\boldsymbol{p}}$) depend on how singular the spectral density is.
- The l.h.s. is a quadratic function of the polynomial coefficients $w_{n,b}^{\sigma,\epsilon}$. Minimizing the l.h.s. can be done by solving a system of linear equations.
- Some speculative argument suggests $\mathfrak{N}_\omega = M+N$ and $\mathfrak{N}_{\boldsymbol{p}} = 0$. We need to understand this better...

Summary

$$\sum_{\substack{\|\alpha\|_1=\mathfrak{N}_\omega\\ 0\le b\le \mathfrak{N}_{\boldsymbol{p}}}} \bar{\Delta}^b \int_{\mathbb{K}} \left[\prod_{A=1}^{M+N-1}\frac{d\omega_A}{2\pi}\right] d\Delta\, e^{\tau\sum_A\omega_A}\left|D^\alpha_\omega\partial^b_\Delta\left[K_\sigma(\omega,\Delta)-P_{\sigma,\epsilon}(e^{-\tau\omega},\Delta)\right]\right|^2<\epsilon^2$$

$$\mathrm{approx}(\sigma,\epsilon)=\sum_{\substack{n_1,n_2\cdots\ge 1\\ b\ge 0}} w^{\sigma,\epsilon}_{n,b}\int\left[\prod_A \frac{d^3\boldsymbol{p}_A}{(2\pi)^3}\,\check{f}_A^{(*)}(\boldsymbol{p}_A)\right][\Delta(\boldsymbol{p})]^b\,\Upsilon_h(n\tau;\boldsymbol{p})\,\hat{C}_c(n\tau;\boldsymbol{p})$$

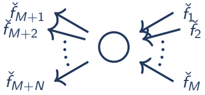
Theorem. For every $r>0$, two constants A, B_r (independent of ϵ and σ) exist such that

$\Big|$  $-\,\mathrm{approx}(\sigma,\epsilon)\Big| \;<\; A\epsilon+B_r\sigma^r$

assuming that the wave functions have non-overlapping velocities [not essential].

Approximation formula for scattering amplitudes

How can we use it?

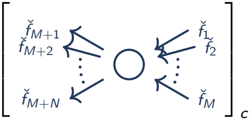


$$\left[\quad \cdots \quad \right]_c = \lim_{\sigma \to 0^+} \lim_{\epsilon \to 0^+} \sum_{n_1, n_2 \cdots \geq 1} \sum_{b \geq 0} w^{\sigma,\epsilon}_{n,b} \mathcal{C}_{n,b}$$

$$\mathcal{C}_{n,b} = \int \left[\prod_A \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3} \check{f}_A^{(*)}(\boldsymbol{p}_A) \right] [\Delta(\boldsymbol{p})]^b \Upsilon_h(n\tau; \boldsymbol{p}) \hat{C}_c(n\tau; \boldsymbol{p})$$

- Smaller ϵ $\Rightarrow$ better approximation of Haag-Ruelle kernel $\Rightarrow$ larger values of n $\Rightarrow$ larger statistical noise.
- Smaller σ $\Rightarrow$ Haag-Ruelle kernel more peaked $\Rightarrow$ harder to approximate $\Rightarrow$ larger values of n $\Rightarrow$ larger statistical noise.
- Also recall: $\Upsilon_h(n\tau; \boldsymbol{p})$ increases exponentially with n.
- Optimization problem: smaller ϵ and σ means larger statistical errors, larger ϵ and σ means larger systematic error. One could design a strategy based on HLT to minimize total error:

$$A[w] = \left\| K_\sigma(\omega, \Delta) - P_{\sigma,\epsilon}(e^{-\tau\omega}, \Delta) \right\|^2_{???} \qquad B[w] = \sum_{n,b,n',b'} w^{\sigma,\epsilon}_{n,b} \langle\langle \mathcal{C}_{n,b} \mathcal{C}_{n',b'} \rangle\rangle_c w^{\sigma,\epsilon}_{n',b'}$$



$$= \lim_{\sigma\to 0^+} \lim_{\epsilon\to 0^+} \sum_{n_1, n_2 \dots \geq 1} \sum_{b\geq 0} w^{\sigma,\epsilon}_{n,b} \mathcal{C}_{n,b}$$

$$\mathcal{C}_{n,b} = \int \left[\prod_A \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3} \check{f}^{(*)}_A(\boldsymbol{p}_A) \right] [\Delta(\boldsymbol{p})]^b \Upsilon_h(n\tau;\boldsymbol{p}) \hat{C}_c(n\tau;\boldsymbol{p})$$

- A finite-volume estimator is obtained trivially by replacing $\int \frac{d^3 \boldsymbol{p}_A}{(2\pi)^3}$ with $\frac{1}{L^3} \sum_{\boldsymbol{p}_A}$.

  If coefficients $w_{n,b}$ are kept fixed as the volume is varied, then the $L \to +\infty$ limit is approached exponentially fast. Having Schwartz wave functions is essential for this step.

- The continuum limit of the estimator can be understood in terms of Symanzik effective theory.

- In this approach, the $L \to \infty$ and $a \to 0$ limits must be taken before the $\epsilon \to 0$ and $\sigma \to 0$ limits. In particular τ cannot be identified with the lattice spacing. For the opposit approach, see Barata and Fredenhagen.

Conclusions and outlook

- We have derived an approximation for scattering amplitudes as a linear combination of Euclidean correlators sampled at discrete times.
- This formula provides the blueprints for a potentially viable numerical strategy.
- Our approximation can be calculated from finite-volume correlators and the infinite-volume limit is approached exponentially fast.
- Whether statistical and systematic errors are under control in typical QCD simulations remains to be seen.
- Recent algorithmic methods (e.g. Hansen-Lupo-Tantalo), which have been successful in approximations of spectral densities, can be adapted to this problem.
- The class of operators used to approximate asymptotic states can be generalized by relaxing the constraint that $\tilde{f}^t(p)$ must have compact support. This may make the numerics easier.