The perturbative computation of the gradient flow coupling for the twisted Eguchi-Kawai model with the numerical stochastic perturbation theory

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Outline

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   - The gradient flow with NSPT
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1. Motivation

- To compare experimental results written in the $\overline{\mathrm{MS}}$ scheme with the lattice results we need the relation between the $\overline{\mathrm{MS}}$ and a regularization independent schemes.
- The gradient flow coupling independent on regularization.

R.Narayanan, H.Neuberger, JHEP03(2006)064, M. Lüscher, JHEP08(2010)071

→ Important tool to connect lattice and continuum theory

- **The gradient flow in Yang—Mills theory**

t : flow time (= energy scale)

$G_{\mu\nu}(x,t)$: Field strength $B_\mu(x,t)$

  - **The flow equation** in the continuum

$$\frac{\partial}{\partial t}B_\mu(x,t) = -\frac{\delta S}{\delta B_\mu}\ \Big(= D_\nu G_{\nu\mu}(x,t)\Big), \qquad B_\mu(x,0) = A_\mu(x)$$

    - The composite operator does not diverge at positive flow time.

M. Lüscher, P.Weisz, JHEP02(2011)051.

    - The operator composed with the flowed gauge field does not require further renormalization and independent on regularization.

  - We employ **the gradient flow coupling** for the renormalized coupling

$$\lambda_\rho(\mu) = \mathcal{N}^{-1}(t)\left\langle \frac{t^2E(t)}{N}\right\rangle$$

Flow time t and the energy density μ

$\mu^2 t = \rho$

Normalization factor : $\mathcal{N}(t)$

  - To convert the lattice results to the results renormalized with the $\overline{\mathrm{MS}}$ scheme we need the relation $\lambda_\rho(\mu) \leftrightarrow \lambda_S(\mu)$.

1. Motivation

- **The GF coupling**

$$\lambda_\rho(\mu) = \mathcal{N}^{-1}(t)\left\langle \frac{t^2 E(t)}{N} \right\rangle$$

Flow time t and the energy density μ

$\mu^2 t = \rho$

Normalization factor : $\mathcal{N}(t)$

- **The analytic relation between the GF coupling and the $\overline{\mathrm{MS}}$ scheme coupling for the SU(N) Yang—Mills theory in the large-N limit at two-loop level.**

GF coupling : $\lambda_\rho(\mu)$ $\leftrightarrow$ $\overline{\mathrm{MS}}$ coupling : $\lambda_S(\mu)$

R. V. Harlander and T. Neumann, JHEP, vol. 2016, June 2016.
J. Artz, et. al., JHEP, vol. 2019, June 2019.

- $\lambda_\rho(\mu) = \lambda_S(\mu) + e_1\lambda_S(\mu)^2 + e_2\lambda_S(\mu)^3 + \cdots$

▸ **Coefficients** e_1, e_2 $\quad L(z) = \ln(2z) + \gamma_E$

$$e_1 = \frac{1}{2}b_0 L(\rho) + e_{1,0} \quad , e_{1,0} = \frac{1}{16\pi^2}\left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right)$$

$$e_2 = e_{2,0} + \frac{1}{2}\left(2b_0 e_{1,0} + b_1\right)L(\rho) + \left(\frac{1}{2}b_0 L(\rho)\right)^2 \quad , e_{2,0} = \frac{1}{(16\pi)^2}27.978$$

▸ **Beta function** b_0, b_1

$$b_0 = \frac{1}{16\pi^2}\frac{2\cdot 11}{3}$$

$$b_1 = \frac{1}{(16\pi^2)^2}\frac{2\cdot 34}{3}$$

▸ We would like to extract the relation $\lambda_\rho(\mu) \leftrightarrow \lambda_S(\mu)$ at more order for the large-N Yang—Mills theory using the lattice perturbation theory.

2. Strategy

- We employ the numerical stochastic perturbation theory(NSPT).
  - The first study of GF-NSPT is in [**Dalla Brida, M., Lüscher, M. *Eur. Phys. J. C* 77, 308 (2017)**] (for SU(3) YM with SF B.C.).
- The GF-NSPT can evaluate the GF coupling with λ_0 at finite-N

$$\lambda_\rho(\mu) = \lambda_0 + r_1(\hat{t}, N)\lambda_0^2 + r_2(\hat{t}, N)\lambda_0^3 + r_3(\hat{t}, N)\lambda_0^4$$

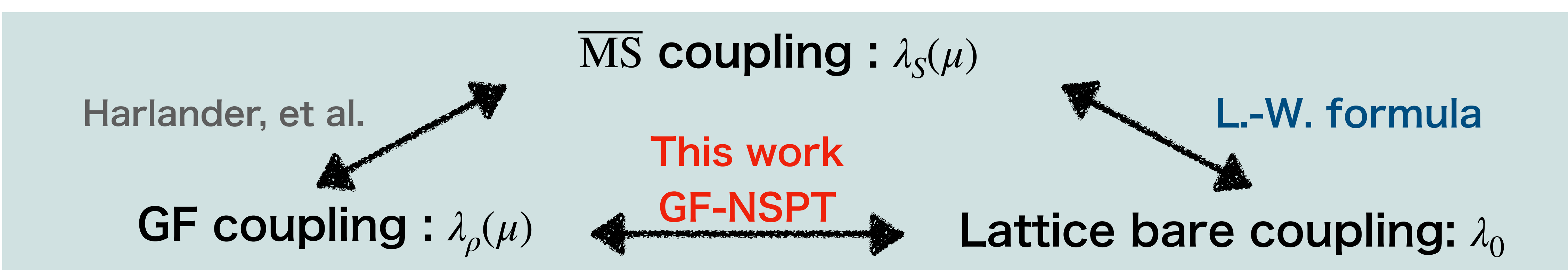
$\hat{t}$: dimensionless flow time

$\rightarrow$ **By taking the large-N limit we obtain** $\lambda_\rho(\mu) = \lambda_0 + r_1(\hat{t})\lambda_0^2 + r_2(\hat{t})\lambda_0^3 + r_3(\hat{t})\lambda_0^4$

- Combining **the Lüscher—Weisz formula ($\lambda_S(\mu) \leftrightarrow \lambda_0$) in the large-$N$ with NSPT,**

$$\lambda_S(\mu) = \lambda_0 + c_1(\mu a)\lambda_0^2 + c_2(\mu a)\lambda_0^3 + \cdots \quad \text{with} \quad \begin{aligned} c_1(\mu a) &= \frac{1}{2}b_0 \ln(\mu a) + k_1 &, k_1 &= 0.1699559992 \\ c_2(\mu a) &= c_1(\mu a)^2 - b_1 \ln(\mu a) + k_3 &, k_3 &= 0.00791012 \end{aligned}$$

we obtain the relation $\lambda_\rho(\mu) \leftrightarrow \lambda_S(\mu)$,



2. Strategy ~ Twisted Eguchi—Kawai model ~

- **Twisted Eguchi—Kawai (TEK) model** : The matrix model on one-site lattice with twisted boundary condition. A. González-Arroyo, M. Okawa, Phys. Lett. B 120 (1983) 174.
- **In the large-N limit** the trace of the closed loop operator $W[U]$

$$\langle W[U]\rangle_{\text{TEK}} \text{ for TEK model } \xrightarrow{N\to\infty} \langle W[U]\rangle \text{ for SU}(N) \text{ Yang—Mills theory}$$

- TEK model is economical model to study **the large-N SU(N) Yang—Mills theory**.

Only d link variables : U_μ ($\mu = 1,\cdots,d$)

- **Partition function**

$$Z_{\text{TEK}} = \int \prod_{\mu=1}^{4} dU_\mu e^{-S_{\text{TEK}}[U]}$$

- **Action**

Inverse 't Hooft coupling : $b = \frac{1}{Ng^2} = \frac{1}{\lambda}$

$$S_{\text{TEK}}[U] = Nb \sum_{\mu,\nu=1}^{4} \text{Tr}\left[I - z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right]$$

- Effective volume : $V = (aL)^4 = (a\sqrt{N})^4 = a^4N^2$

- The space-time information is included in the twist eater Γ_μ

$$\Gamma_\mu \Gamma_\nu = z_{\nu\mu} \Gamma_\nu \Gamma_\mu$$

with the twist factor : $z_{\mu\nu} = \exp\left[\frac{2\pi i k}{\sqrt{N}} \epsilon_{\mu\nu}\right]$ k : coprime with $\sqrt{N}$

➡ In the NSPT we perturbatively expand U_μ around the classical vacuum $U_\mu^{(0)} = \Gamma_\mu$

2. Strategy ~ The gradient flow with NSPT ~

- **NSPT : Numerical Stochastic Perturbation Theory**
  - NSPT numerically evaluates the perturbative coefficients for an observable without Feynman diagram => **NSPT allows us to reach higher-order** F. Di Renzo et al. Nucl. Phys. B 426.3(1994)
  - It can be implemented by expanding the field and the action in terms of a coupling constant and integrating **hierarchical stochastic differential equation**. Langevin eq., Molecular dynamics (MD) eq.

    We expanded the link variable as $U_\mu = \sum_{k=0}^{\infty} \lambda_0^{k/2} U_\mu^{(k)}$ (The vacuum : $U_\mu^{(0)}(\hat{t}) = \Gamma_\mu$)
  - We use the HMD-based NSPT for TEK model in [A. González-Arroyo, et al. JHEP 127(2019)].
- **The gradient flow with NSPT**
  - **Hierarchical GF equation**

$$\frac{d}{d\hat{t}} V_\mu^{(k)}(x,\hat{t}) = -\frac{1}{2}\left(F_\mu[V] \star V_\mu(x,\hat{t})\right)^{(k)},$$

$$V_\mu(\hat{t}=0)^{(k)} = U_\mu^{(k)}$$

$V_\mu(\hat{t})^{(k)}$: the coefficient of the flowed link variable

  - **Force for TEK model**

$$F_\mu^{(k)}[U] = \left(S_\mu^{(k)} - S_\mu^{(k)\dagger}\right) - \frac{1}{N}\mathrm{Tr}\left(S_\mu^{(k)} - S_\mu^{(k)\dagger}\right)$$

$$S_\mu^{(k)} = \left[U_\mu \star \sum_{\nu\neq\mu}\left(U_\nu \star U_\mu^\dagger \star U_\nu^\dagger - U_\nu^\dagger \star U_\mu^\dagger \star U_\nu\right)\right]^{(k)}$$

$\star$-symbol : convolutional product

  - We flow the perturbative configuration generated from NSPT.

$$\left\{(U_{\mu,i}^{(0)},\cdots,U_{\mu,i}^{(k)},\cdots)\,|\,i=1,\ldots,N_{\mathrm{sample}}\right\} \xrightarrow{\text{Hierarchical flow}} \left\{(V_{\mu,i}^{(0)}(\hat{t}),\cdots,V_{\mu,i}^{(k)}(\hat{t}),\cdots)\,|\,i=1,\ldots,N_{\mathrm{sample}}\right\}$$

➡We evaluate the coefficients of the GF coupling as the stochastic mean,

$$\langle O[V;\hat{t}]\rangle = \sum_{k=0}^{\infty}\lambda^k \left\langle O^{(k)}[V_\mu^{(0)}(\hat{t}),\cdots,V_\mu^{(k)}(\hat{t})]\right\rangle, \qquad \left\langle O^{(k)}[V_\mu^{(0)},\cdots,V_\mu^{(k)}]\right\rangle \simeq \frac{1}{N_{\mathrm{sample}}}\sum_{i=1}^{N_{\mathrm{sample}}} O^{(k)}[V_{\mu,i}^{(0)},\cdots,V_{\mu,i}^{(k)}]$$

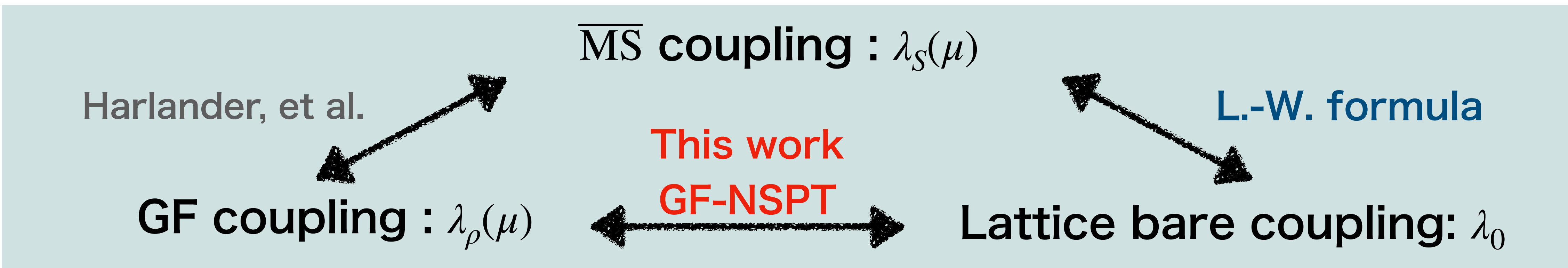
2. Strategy

- The GF-NSPT can evaluate the GF coupling with λ_0 at finite-N

$$\lambda_\rho(\mu) = \lambda_0 + r_1(\hat{t}, N)\lambda_0^2 + r_2(\hat{t}, N)\lambda_0^3 + r_3(\hat{t}, N)\lambda_0^4$$

$\hat{t}$: dimensionless flow time

→ **By taking the large-N limit we obtain** $\lambda_\rho(\mu) = \lambda_0 + r_1(\hat{t})\lambda_0^2 + r_2(\hat{t})\lambda_0^3 + r_3(\hat{t})\lambda_0^4$



- **Combining the two relations (L.-W. formula and Harlander et al. result) we obtain the analytical coefficients for $\lambda_\rho \leftrightarrow \lambda_0$ at two-loop level in the continuum.**

  - $\lambda_\rho(\mu) = \lambda_0 + r_1(\hat{t})\lambda_0^2 + r_2(\hat{t})\lambda_0^3 + \cdots$

    with $r_1(\hat{t}) = b_0\left(\ln\sqrt{2\hat{t}} + \gamma_E\right) + f_1$ $\quad$ $r_2(\hat{t}) = r_1(\hat{t})^2 + b_1\left(\ln\sqrt{2\hat{t}} + \gamma_E\right) + f_2$

    $b_0 = \frac{1}{16\pi^2}\frac{2\cdot 11}{3}$ and $f_1 = 0.21786205$ $\quad$ $b_1 = \frac{1}{(16\pi^2)^2}\frac{2\cdot 34}{3}$ and $f_2 = 0.0067371$

- **From the flow time dependence (running behavior) of $\lambda_\rho(\mu) \leftrightarrow \lambda_0$ we want to extract the beta function $b_{0,1}$ and the constants $f_{1,2}$ (= consistency check the ANA. and NSPT for $\lambda_\rho(\mu) \leftrightarrow \lambda_S(\mu)$).**

3. Result ~ The parameters ~

- **The parameter of the GF-NSPT simulation**
  - **k, SU(N) TEK model**
    - We employ the three matrix sizes $N = 289,\ 441,\ 529$ $\left(L^2 = 17^2,\ 21^2,\ 23^2\right)$.
    - In order to take the smooth large-N limit we have to keep the phase $\theta = 2\pi|\bar{k}|/\sqrt{N}$.

      $$\left(\ k\bar{k} = 1,\ (\mathrm{mod}\ \sqrt{N})\ \right)$$

      Fix the phase parameter $\theta \simeq 0.40$.

A. González-Arroyo, M. Okawa, JHEP 07(2010) 043.

L	N	k	$\lvert\bar{k}\rvert$	$\theta = \lvert\bar{k}\rvert/L$	τ	N_{MD}	Statics
17	289	5	7	0.41176	1.0	32	5931
21	441	13	8	0.38099	1.0	32	3790
23	529	7	10	0.43479	1.0	32	2600

The parameters for the NSPT and TEK model

  - **HMD-based NSPT**
    - We use the HMD-based NSPT for TEK model in [A. González-Arroyo, et al. JHEP 127(2019)].
  - **Integration for the GF eq.**
    - We use Lüscher's scheme with $\epsilon = 0.01$ as the numerical integration method.

      The integration error is $\mathcal{O}(\epsilon^3)$.

      $\epsilon = 0.01$ is sufficiently small compared to the statistical error.

3. Result ~ The flow time dependence ~

- **The flow time dependence of the GF coupling**
  - We extrapolate the finite-N (289,441,529) results to large-N (Red crosses).

$$\lambda_\rho(\mu) = \lambda_0 + r_1(\hat{t})\lambda_0^2 + r_2(\hat{t})\lambda_0^3$$

  - The analytical result (black dashed line) in the continuum.

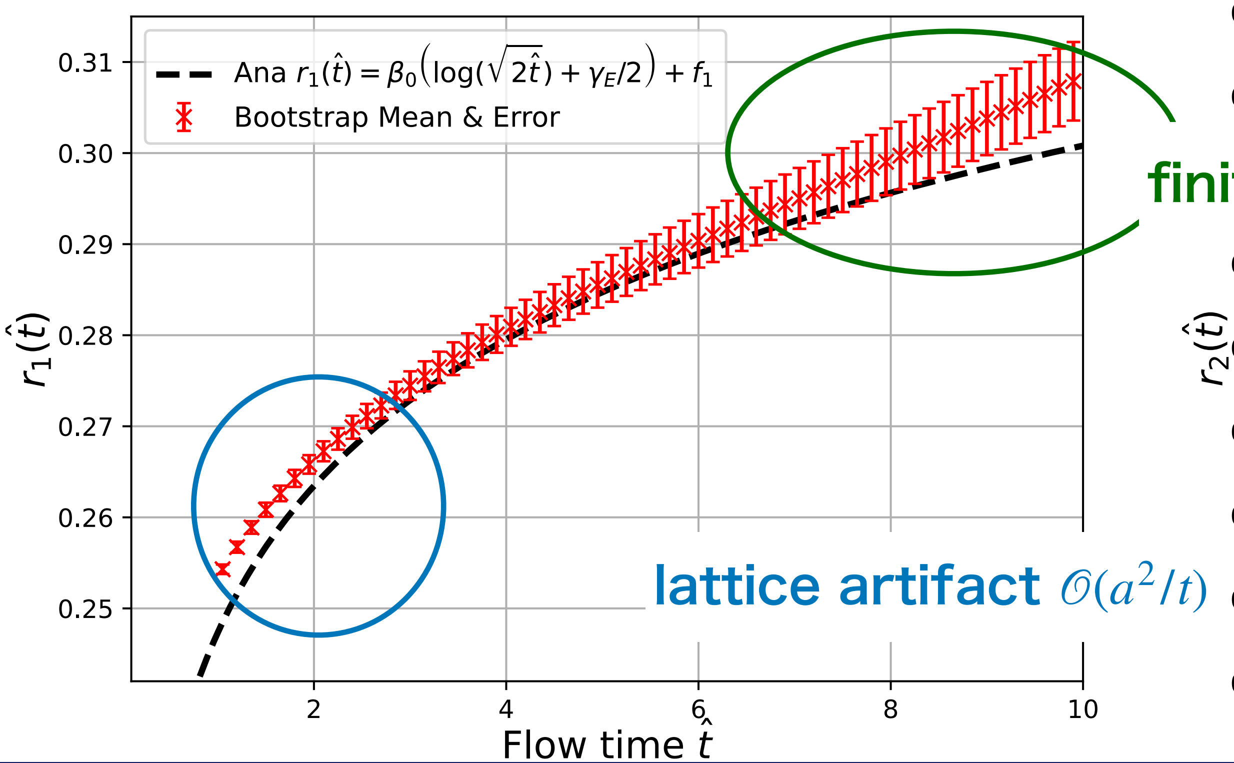
- **The deviation between the GF-NSPT and the continuum result**
  - At large $\hat{t}$ **the effect of a finite volume** becomes large : $\mathcal{O}(t/V)$

    ➡ We need $N > 600$ to suppress the correction under 10% at $\hat{t} = 7.0$! (From the tree-level analysis)

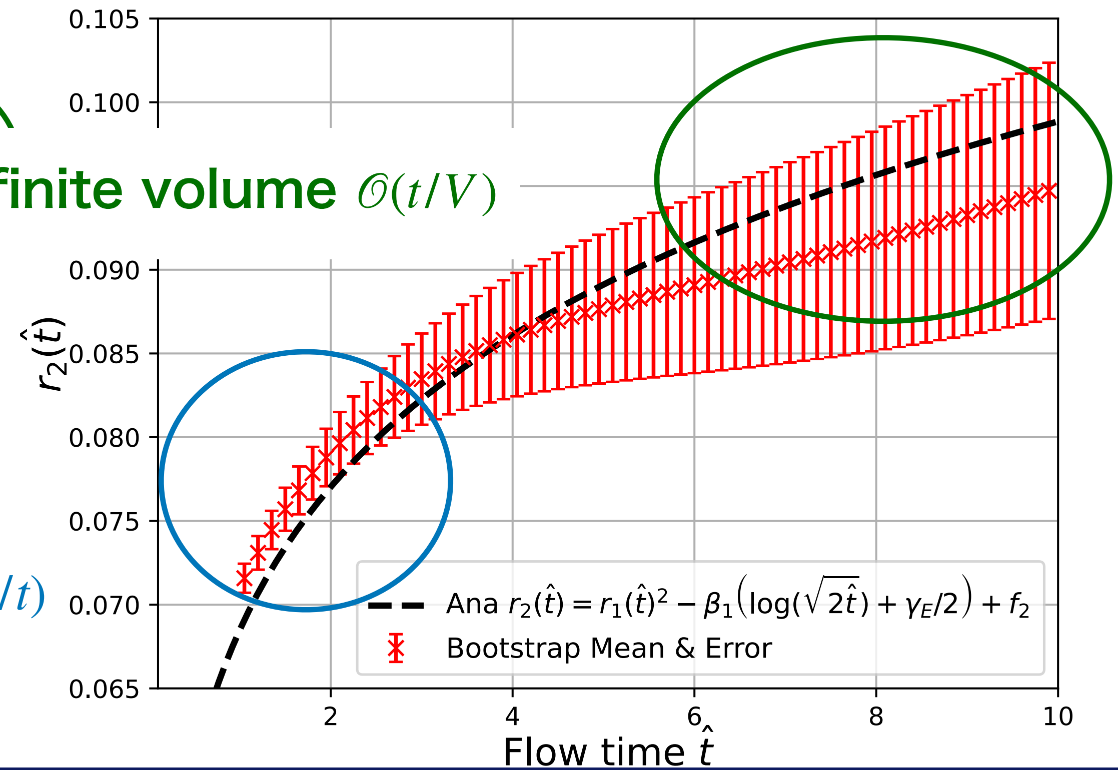
  - At small $\hat{t}$ **the effect of a lattice artifact** becomes large : $\mathcal{O}(a^2/t)$

    ➡ We can control the lattice artifact at small $\hat{t}$.

One-loop coefficient $r_1(\hat{t})$



Two-loop coefficient $r_2(\hat{t})$



3. Result ~ The flow time dependence ~

- **The flow time dependence of the one-loop coefficient** $r_1(\hat{t})$
  - Highly correlated data (Chi-square worse diverge)→ Use the Cut-off chi-square [arXiv:1101.2248]
  - We correlated fit the NSPT results with $f(\hat{t})$, $g(\hat{t})$ in two regions $\hat{t} \in [2.1,6.3]$ and $[0.9,6.3]$.

$$f(\hat{t}) = B_0\left(\log(\sqrt{2\hat{t}}) + \frac{\gamma_E}{2}\right) + F_1 \quad , g(\hat{t}) = B_0\left(\log(\sqrt{2\hat{t}}) + \frac{\gamma_E}{2}\right) + F_1 + \frac{A_0}{\hat{t}}$$

lattice artifact term $\mathcal{O}(a^2/t)$

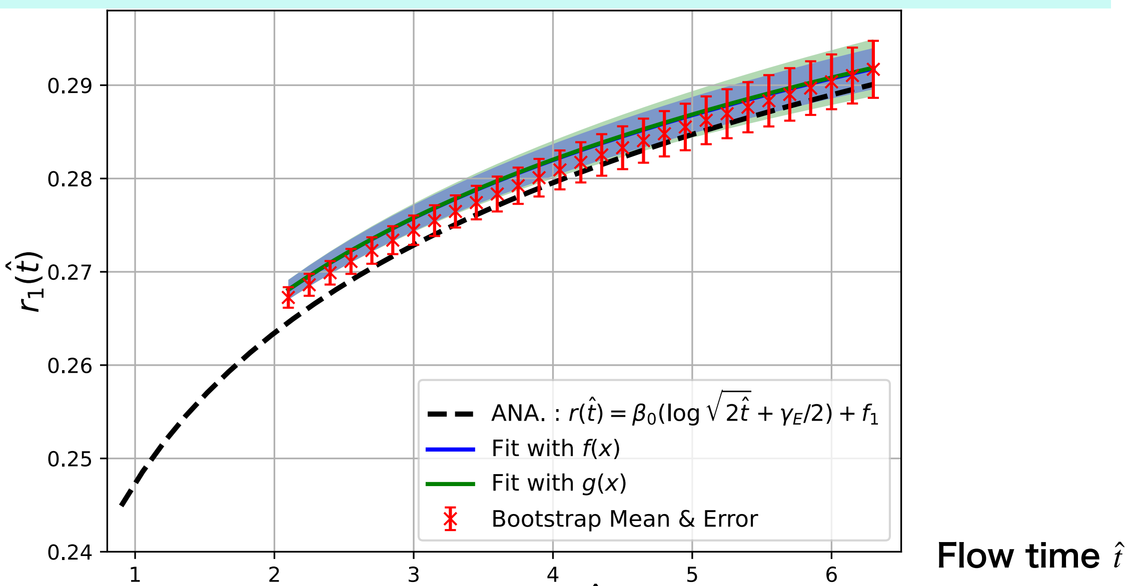
  - The one-loop result well reproduces the beta function b_0 and the coefficient f_1.

① Fit in $\hat{t} \in [2.1,6.3]$

- The small lattice artifact

Fit function	κ	B_0	F_1	A_0	$\chi^2/N_{\rm dof}$
Analytical Value		0.046439	0.217862	–	–
$f(x)$	10^{-10}	0.04315(222)	0.22466(139)	–	12.3
$g(x)$	10^{-10}	0.04349(748)	0.22419(1009)	0.00030(634)	12.1

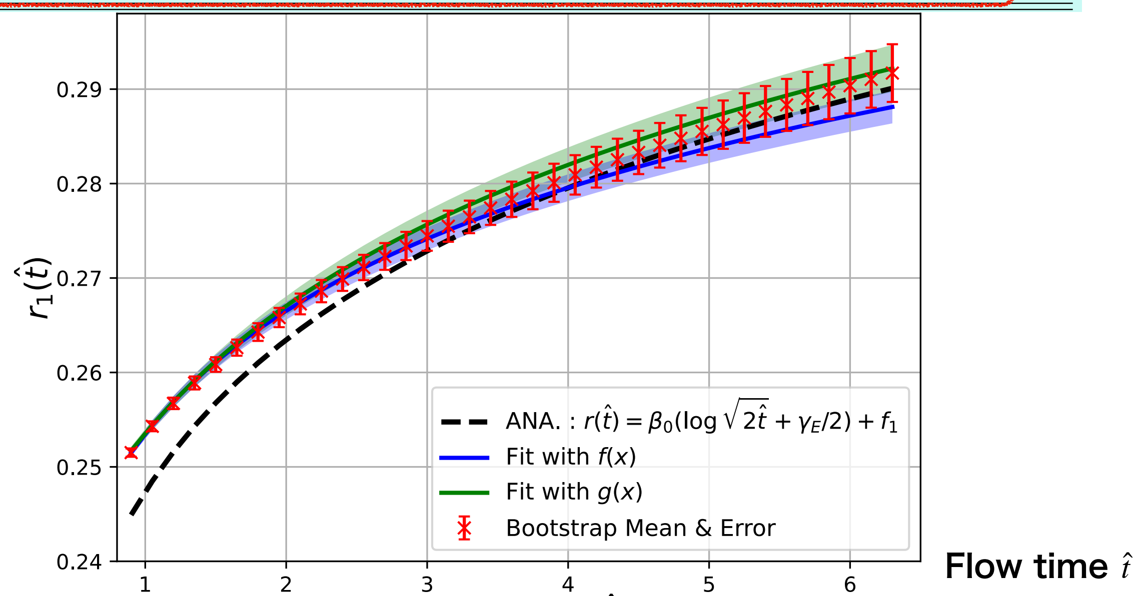
ANA.



② Fit in $\hat{t} \in [0.9,6.3]$ (include small $\hat{t}$)

- The large lattice artifact
- $\mathcal{O}(a^2/\hat{t})$ is sufficiently under control
- Small error (compare with ①)

Fit function	κ	B_0	F_1	A_0	$\chi^2/N_{\rm dof}$
Analytical Value		0.046439	0.217862	–	–
$f(x)$	10^{-8}	0.03762(144)	0.22959(60)	–	3.2
$g(x)$	10^{-9}	0.04725(349)	0.21778(337)	0.00577(139)	4.2



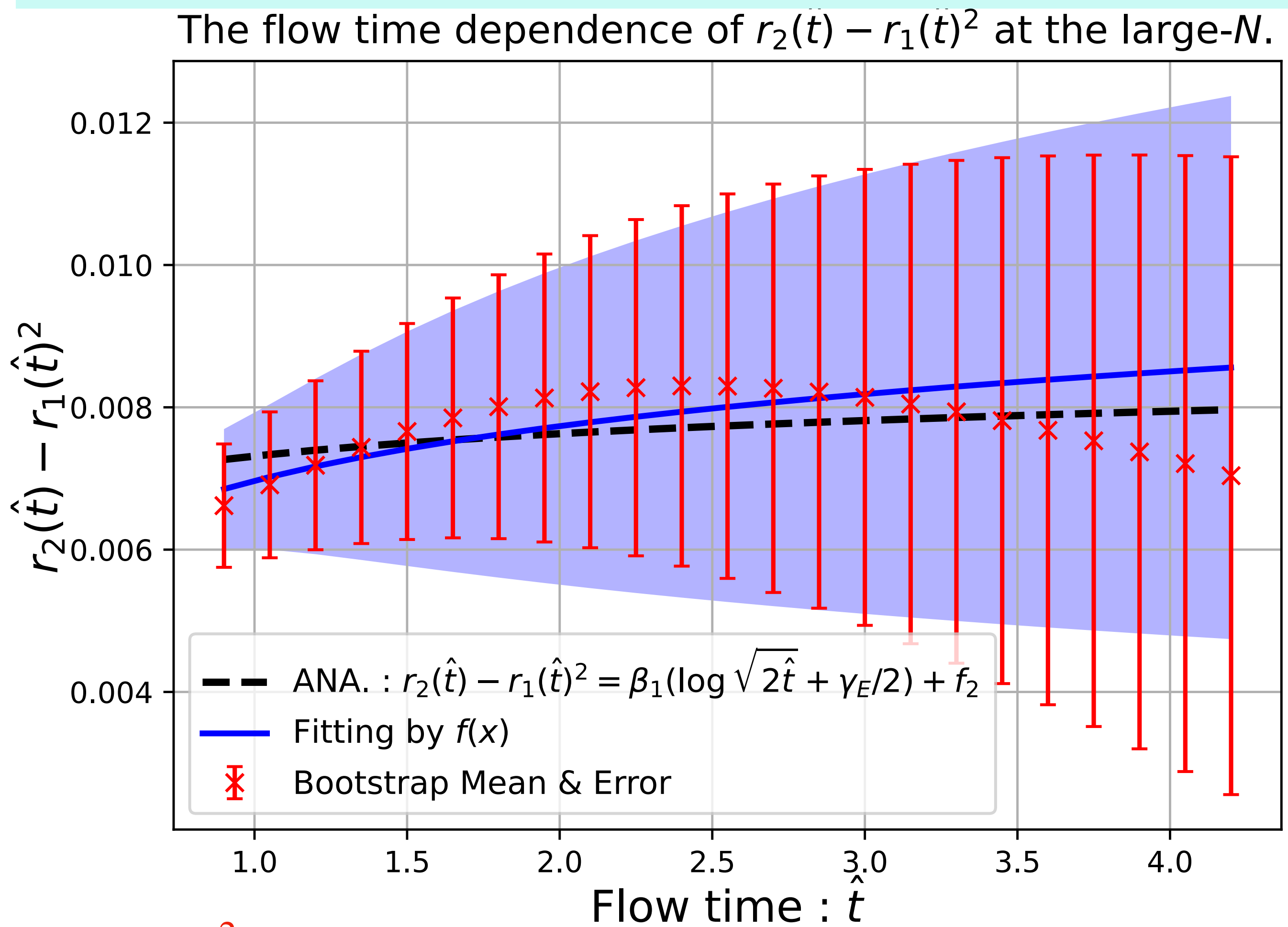
3. Result ~ The flow time dependence ~

- **The flow time dependence of the two-loop coefficient $r_2(\hat{t})$**
  - The analytical form of $r_2(\hat{t})$ is
    $$r_2(\hat{t}) = r_1(\hat{t})^2 + b_1\left(\ln\sqrt{2\hat{t}} + \frac{\gamma_E}{2}\right) + f_2$$
  - We fit $r_2(\hat{t}) - r_1(\hat{t})^2$ with
    $$f(\hat{t}) = B_1\left(\log(\sqrt{2\hat{t}}) + \frac{\gamma_E}{2}\right) + F_2$$
  - The order is consistent with the continuum one.
  - The large error & poor fit result.

    ➡ Need more large N result and stochastic samples

    The large-N factorization reduces the sample size at large-N.

Fit function	Method	κ	B_1	F_2	$\chi^2/N_{\rm dof}$
Analytical Value			0.00090897	0.00673711	–
$f(x)$	Diag.	–	0.00222(444)	0.00556(240)	0.04



Fit in $\hat{t} \in [0.9, 4.2]$

  - The large-N factorization is $\langle O_1 O_2\rangle = \langle O_1\rangle\langle O_2\rangle + \mathcal{O}(N^{-2})$.
  - We confirm the factorization with the variance of the coefficient,
    $$\mathrm{Var}(O) \equiv \langle O^2\rangle - \langle O\rangle^2 \xrightarrow{N\to\infty} 0$$

- **Is there large-N factorization of the coefficient at positive flow time $\hat{t}$?**

We check the variance of $r_i(\hat{t})$ →

3. Result ~ The large-N factorization ~

- **The variance of coefficients** $\mathrm{Var}\left(r_i(\hat{t})\right)$
  - The tree-level is consistent with ANA..

$$\mathrm{Var}(E_W(\hat{t}))|_{\rm tree} = \frac{3}{2N^4}\sum_q{}' e^{-4\hat{t}\hat{q}^2} \leq \frac{3(N^2-1)}{2N^4} \xrightarrow{N\to\infty} 0$$

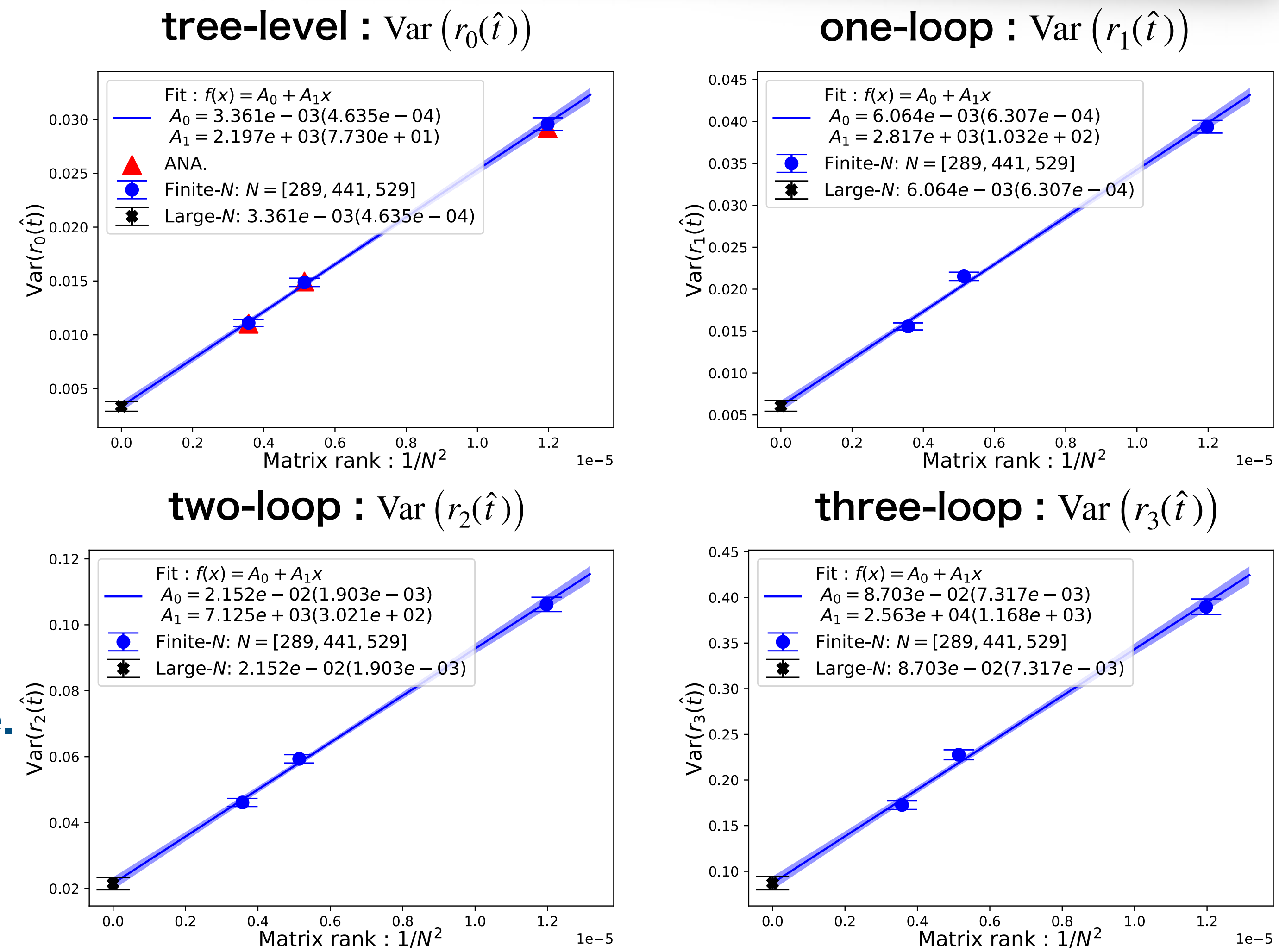
  - We confirm the large-N factorization at positive flow time from tree-level analysis.
  - The variance is reduced with N.

The large-N variance is small but does not becomes ZERO!

The finite volume effect grow with the flow time.

$\mathcal{O}(t^2/N^4)$ **become large**

➡ We can not see the zero variance at large-N from simple linear extrapolation with $1/N^2$.



The variance $\mathrm{Var}\left(r_i(\hat{t})\right), (i = 0,1,2,3)$ **at** $\hat{t} = 6.0$ **v.s. volume** $1/N^2$

- **The estimation of sample size**
  - We can estimate $N_{\rm sample}$ at large N from

$$N_{\rm sample} \simeq \frac{\mathrm{Var}(O(\hat{t}))}{\langle O(\hat{t})\rangle^2}\left(\frac{\langle O(\hat{t})\rangle}{\delta O(\hat{t})}\right)^2$$

Relative error

A. González-Arroyo, et. al, Modern Physics A Vol. 37, No. 36 (2022).

  - $N_{\rm sample}$ with relative error $1\,\%$ at $\hat{t} = 7.0$, $N = 729$.

one-loop : $N_{\rm sample} \simeq 600$

two-loop : $N_{\rm sample} \simeq 15200$

We can compute within a year.

Coeff.	$\mathrm{Var}(r_i)\vert_{N\to\infty}$
$r_0(\hat{t} = 6.0)$	$3.36(0.46) \times 10^{-3}$
$r_1(\hat{t} = 6.0)$	$6.06(0.63) \times 10^{-3}$
$r_2(\hat{t} = 6.0)$	$2.15(0.19) \times 10^{-2}$
$r_3(\hat{t} = 6.0)$	$8.70(0.73) \times 10^{-2}$

Non-zero value !

The simple linear extrapolation of $\mathrm{Var}\left(r_i(\hat{t})\right)$ **at** $\hat{t} = 6.0$**, assuming** $1/N^2$ **dependence.**

4. Conclusion

- **We compute the coefficients of the GF coupling by using the NSPT for TEK model and analyze the running behavior (flow time dependence).**
  - The one-loop result is consistent with continuum result.
  - The two-loop result has the consistency with ANA. but has the large statistical error.

We need more large matrix size N and statistical samples.

➡ Reduce the finite volume effect

- **We confirm the large-N factorization at finite flow time at tree-level result.**
- **The large-N factorization exist at positive flow time.**
- **Our finite N results does not reproduce zero variance at large-N from simple linear extrapolation $1/N^2$.**
  - The reason is that the finite volume effect grows with $\hat{t}$. (Large $\mathcal{O}(t^2/N^4)$ term)

- **Future work**
  - We will compute the GF coupling with more large N.
  - We want to relate GF coupling to an other renormalized coupling by using only NSPT calculation.

Thank you !

Backup

Backup : HMD based NSPT for TEK model

- **NSPT for TEK model**
  - We generate the perturbative configuration with the HMD based NSPT
    Dalla Brida, M., Lüscher, M. *Eur. Phys. J. C* 77, 308 (2017)
  - We use the HMD-based NSPT for TEK model in [A. González-Arroyo, et al. JHEP 127(2019)].

The expanded link variable : $U_\mu = \sum_{k=0}^{\infty} \lambda^{k/2} U_\mu^{(k)}$ (The vacuum : $U_\mu^{(0)} = \Gamma_\mu$)

Hierarchy molecular dynamics equation for TEK model

$$\frac{dU_\mu^{(k)}}{d\tau} = i\left(P_\mu \star U_\mu\right)^{(k)} \qquad F_\mu^{(k)}[U] = \left(S_\mu^{(k)} - S_\mu^{(k)\dagger}\right) - \frac{1}{N}\mathrm{Tr}\left(S_\mu^{(k)} - S_\mu^{(k)\dagger}\right)$$

$$\frac{dP_\mu^{(k)}}{d\tau} = F_\mu^{(k)}[U] \qquad S_\mu^{(k)} = \left(U_\mu \star \sum_{\nu\neq\mu}\left(U_\nu \star U_\mu^\dagger \star U_\nu^\dagger - U_\nu^\dagger \star U_\mu^\dagger \star U_\nu\right)\right)^{(k)}$$

Perturbation order : k

$\star$-symbol : convolutional product

  - We accumulate the perturbative configuration by integrating MD eq.

$$\left\{(U_{\mu,i}^{(0)}, \cdots, U_{\mu,i}^{(k)}, \cdots) \,|\, i = 1, \ldots, N_{\mathrm{sample}}\right\}$$

  - The coefficient of the expectation value are evaluate as the stochastic mean

$$\langle O[U]\rangle \simeq \sum_{k=0}^{\infty} \lambda^k \left\langle O^{(k)}[U_\mu^{(0)}, \cdots, U_\mu^{(k)}]\right\rangle, \qquad \left\langle O^{(k)}[U_\mu^{(0)}, \cdots, U_\mu^{(k)}]\right\rangle = \frac{1}{N_{\mathrm{sample}}} \sum_{i=1}^{N_{\mathrm{sample}}} O^{(k)}[U_{\mu,i}^{(0)}, \cdots, U_{\mu,i}^{(k)}]$$

Backup : Numerical integration for GF eq.

- **Integration for the GF eq.**
  - We test the three integration method on single configuration for $k = 7$, SU(529) TEK model.

    Euler method : $\mathcal{O}(\epsilon)$

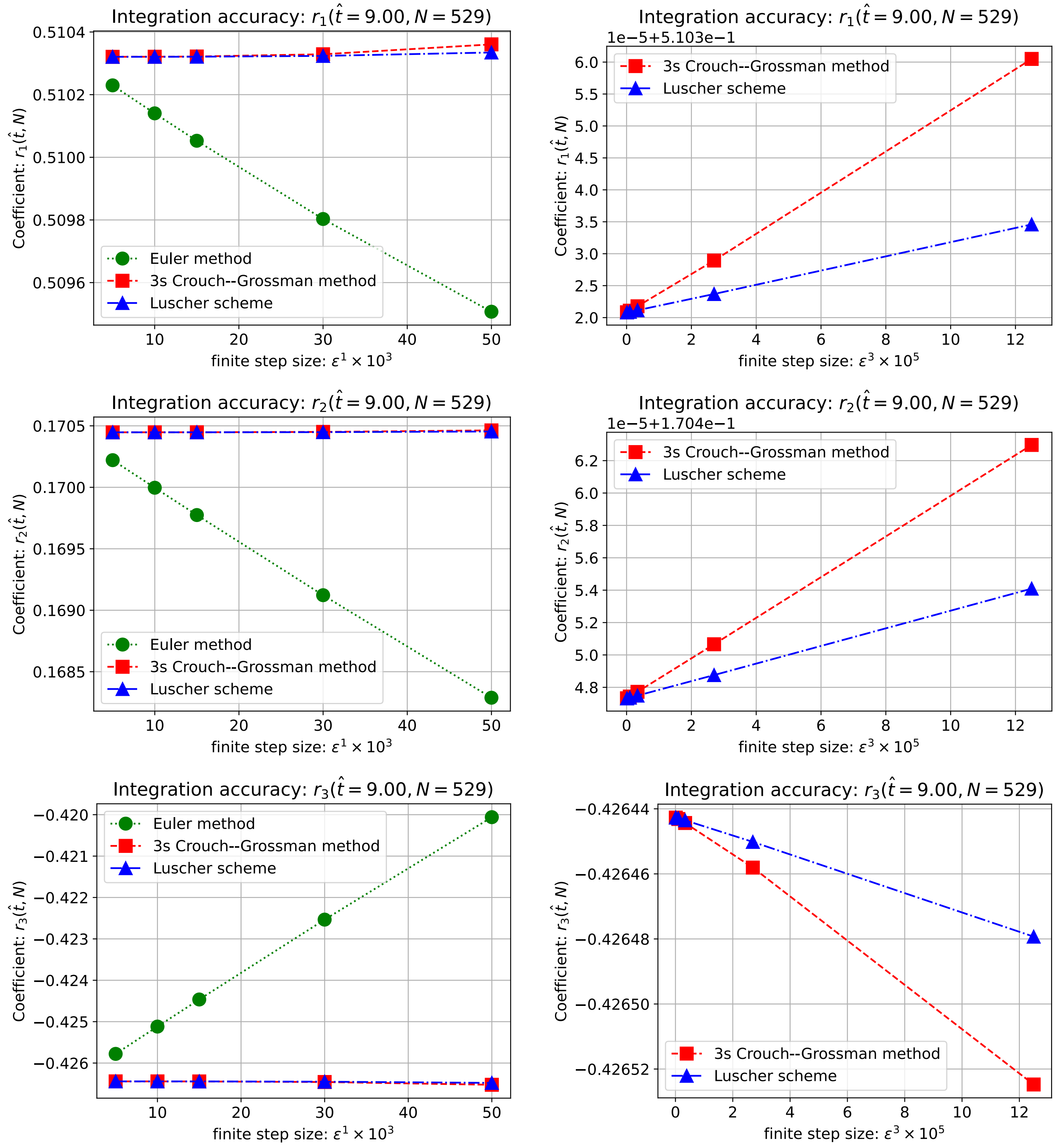
    3-s. Crouch—Grossman method : $\mathcal{O}(\epsilon^3)$

    Lüscher's scheme : $\mathcal{O}(\epsilon^3)$

  ➡All order coefficients have the same scaling for the finite step size ϵ.

  - $\epsilon < 0.5$ sufficiently small.
  - **We use Lüscher's scheme with $\epsilon = 0.01$.**

(Right Figures are without Euler method)



The scaling test for three method on a single configuration at $k = 7$, $SU(529)$ TEK model.

Backup : Asymptotic form of $E(\hat{t})$

- **The energy density** $E_W(\hat{t})$

$$E_W(\hat{t}) = \frac{1}{N}\mathrm{Tr}\left(I - z_{\mu\nu}V_\mu(\hat{t})V_\nu(\hat{t})V_\mu(\hat{t})^\dagger V_\nu(\hat{t})^\dagger\right)$$

- The tree-level solution of $\hat{t}^2 E_W(\hat{t})$

$$\left.\frac{\hat{t}^2 E_W(\hat{t})}{N}\right|_{\text{tree}} = \frac{3\hat{t}^2}{2N^2}\sum_q{}' e^{-2\hat{t}\hat{q}^2} = \frac{3\hat{t}^2}{2}e^{-16\hat{t}}\left[I_0(4\hat{t}) + 2\sum_{k=1}^{\infty} I_{k\sqrt{N}}(4\hat{t})\right]^4 - \frac{3\hat{t}^2}{2N^2}$$

- Asymptotic form in the large flow time

$$\sim \frac{3\hat{t}^2}{2N^2}\sum_q{}' e^{-2\hat{t}\hat{q}^2} = \frac{3\hat{t}^2}{2}e^{-16\hat{t}}\left[I_0(4\hat{t}) + 2\sum_{k=1}^{\infty}\frac{1}{\sqrt{2\pi k\sqrt{N}}}\left(\frac{2e\hat{t}}{k\sqrt{N}}\right)^{k\sqrt{N}}\right]^4 - \frac{3\hat{t}^2}{2N^2}$$

- In the large-N

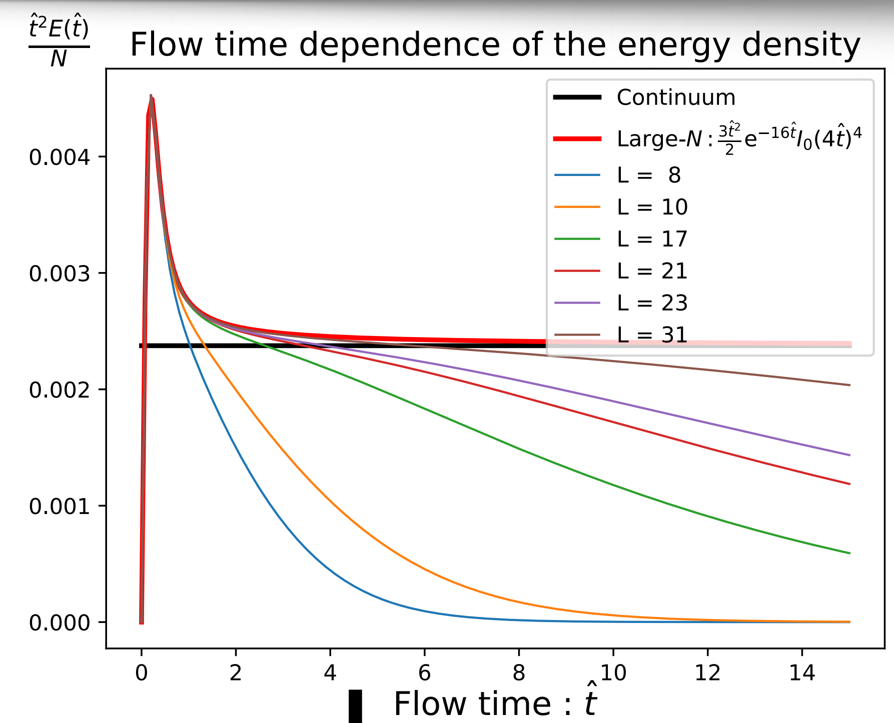
$$\sim \frac{3}{128\pi^2}\left[1 + \frac{1}{8\hat{t}} + \cdots\right] - \frac{3\hat{t}^2}{2N^2}$$

(In continuum $3/128\pi^2$)

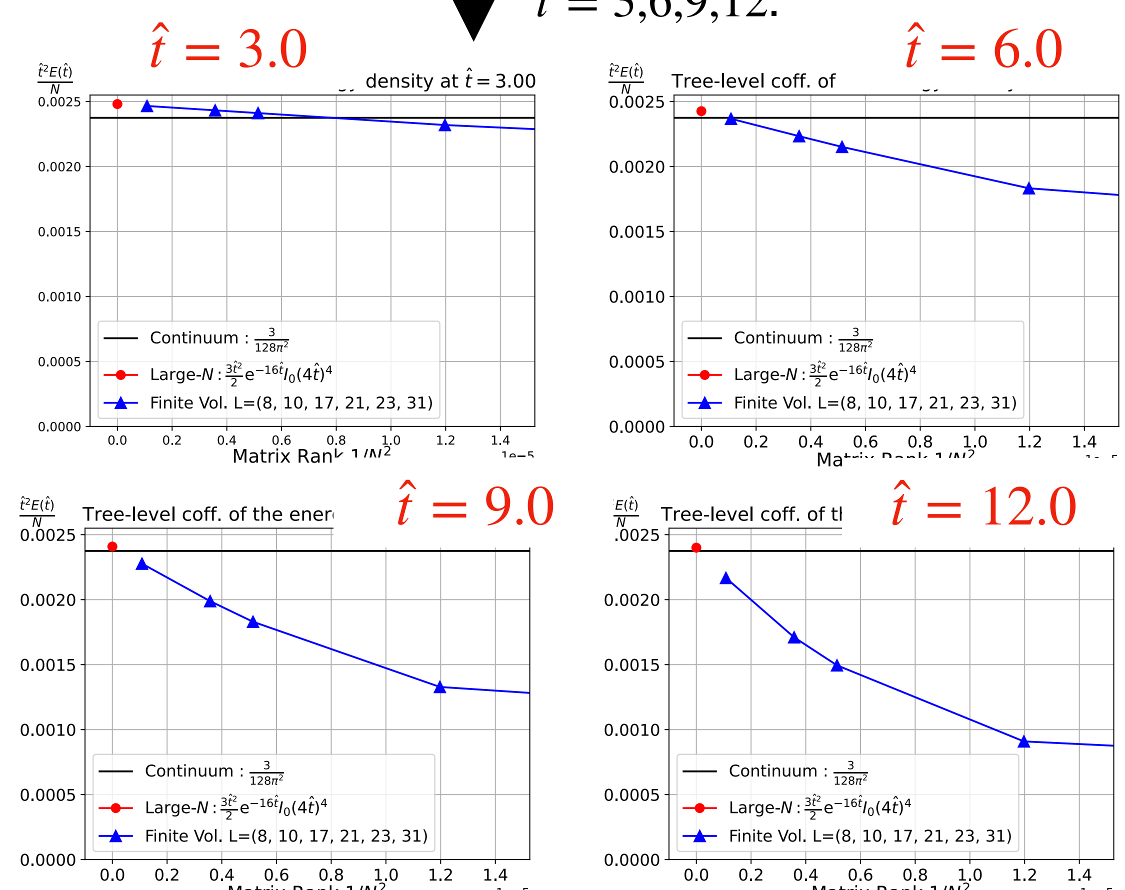
- The finite volume effect grow with flow time.

➡ We can not ignore $\mathcal{O}(t^4/N^4)$

The simple linear extrapolation with $f(N) = A_0 + A_1/N^2$ is difficult→



Time slice at $\hat{t} = 3,6,9,12.$



Backup : Asymptotic form of $\mathrm{Var}(\hat{t}^2 E(\hat{t}))$

- **The variance of $E_W(\hat{t})$ at tree-level**

$$\mathrm{Var}(E_W(\hat{t})) \equiv \left\langle E_W(\hat{t})^2 \right\rangle - \left\langle E_W(\hat{t}) \right\rangle^2$$

- The tree-level solution of the variance

$$\mathrm{Var}(E_W(\hat{t}))\,|_{\mathrm{tree}} = \frac{3}{2N^4} \sum_q{}' \, e^{-4\hat{t}\hat{q}^2}$$

- **There is the large-N factorization at positive flow time**

$$\mathrm{Var}(E_W(\hat{t}))\,|_{\mathrm{tree}} \leq \frac{3(N^2-1)}{2N^4} \xrightarrow[N\to\infty]{} 0$$

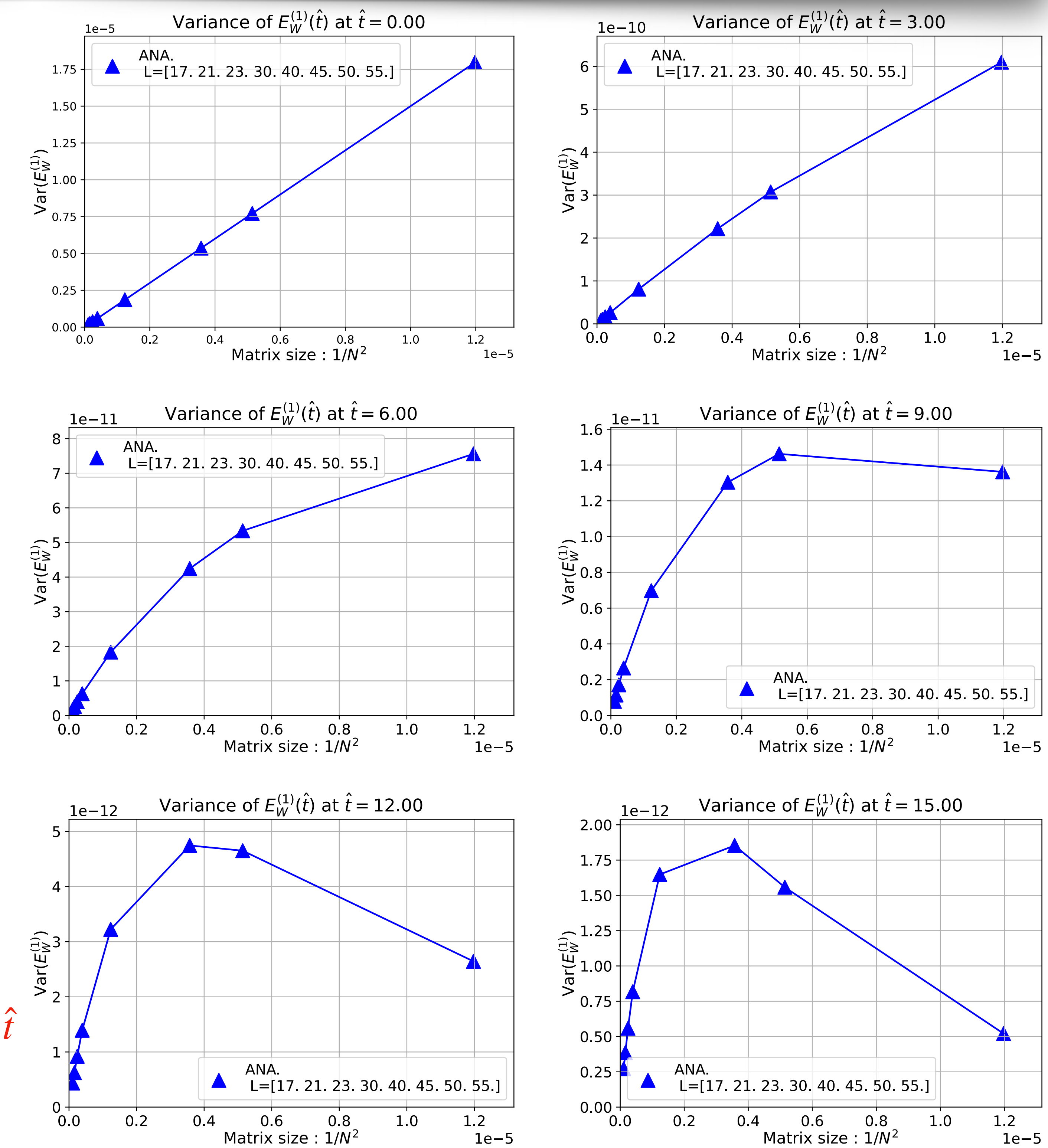
- In the large-N & large flow time
  The asymptotic form is

$$\sim \frac{3}{2N^2}\left(\frac{1}{(16\pi^2\hat{t})^2}\left(1+\frac{1}{16\hat{t}}+\cdots\right)-\frac{1}{N^2}\right)$$

$\mathcal{O}(1/N^4)$ **become large with $\hat{t}$**

- The $\mathrm{Var}\left(r(\hat{t})\right)$ can be obtained from

$$\mathrm{Var}(r(\hat{t})) = \left(\frac{\hat{t}^2}{\mathcal{N}(\hat{t})}\right)^2 \mathrm{Var}(E_W(\hat{t}))$$



The variance v.s. matrix size : $1/N^2$ at $\hat{t} = 0,3,6,9,12,15$.