The constraint potential in the chiral Gross-Neveu model

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LATTICE 2024, Liverpool, 29.07.2024

in collaboration with
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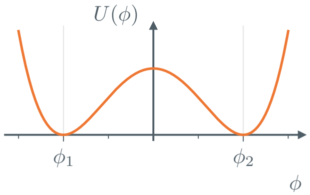
Symmetry breaking in quantum field theories

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + U(\phi^2)$$

Z_2 Symmetry: $\phi \to -\phi$



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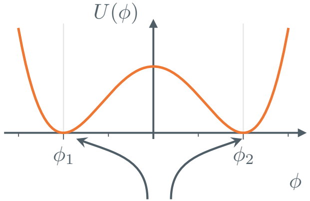

classically spontaneously broken symmetry

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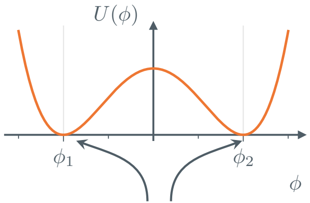

classically spontaneously broken symmetry

- Describe quantum theory via (Euclidean) path integral

$$\mathcal{Z} = \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]} \quad , \; S[\phi] = \int \mathrm{d}\tau\, \mathrm{d}^D x\, \mathcal{L}$$

- Path integral is blind to the broken symmetry, $\langle\phi\rangle = 0$

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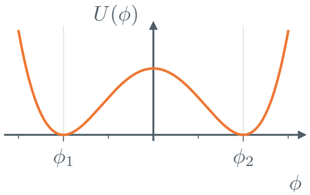

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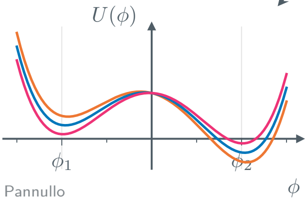

Standard procedure in lattice simulations

- Tilt potential with term $\propto J\phi$
- Calculate $\lim_{J\to 0} \lim_{V\to\infty} \langle\phi\rangle_J$

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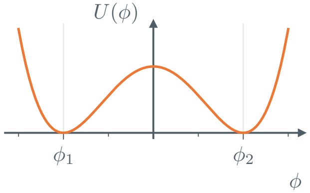

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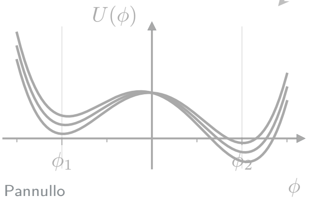
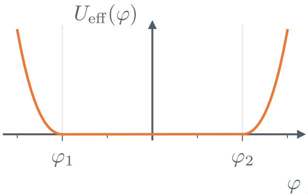


Quantum effective potential of $\varphi = \langle\phi\rangle$

- Convex, flat region when symmetry broken
- Edge of flat region corresponds to realized $\langle\phi\rangle$



Accessing the effective potential via a constraint

- Consider constrained path integral

Constrained potential ↗

$$\mathcal{Z}_{\bar{\phi}} = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]} \, \delta\left(\int \mathrm{d}^D x \, \phi - \bar{\phi} \right) = \mathrm{e}^{-V\Omega(\bar{\phi})}$$

- Only integrates over configurations with spacetime average $\bar{\phi} \Rightarrow \langle \phi \rangle_{\bar{\phi}} = \bar{\phi}$
- Original path integral recovered as

$$\mathcal{Z} = \int \mathrm{d}\bar{\phi} \, \mathcal{Z}_{\bar{\phi}}$$

- U_{eff} is the convex hull of constrained potential Ω, which can be non-convex
- $\Omega(\bar{\phi})$ agrees with $U_{\mathrm{eff}}(\varphi)$ in the infinite volume limit [L. O'Raifeartaigh *et al.*, *Nucl. Phys. B.* **271** (1986)]

Constraining fermionic condensates

- Constraining bosonic degrees of freedom in Monte-Carlo simulations is understood
  [Z. Fodor *et al.*, *PoS.* **LATTICE2007** (2007)] [G. Endrődi *et al.*, *Phys. Rev. Lett.* **127** (2021)]
- What about fermionic condensates? e.g. chiral condensate $\bar{\psi}\psi$ as order parameter of chiral symmetry breaking
- Goal: Develop formalism to constrain fermionic condensates such as $\bar{\psi}\psi$ to calculate

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{-S}\delta\left(\phi - \int \mathrm{d}^D x\,\bar{\psi}\psi\right) = \mathrm{e}^{-V\Omega(\phi)}$$

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- **But**: in the path integral representation the fermionic fields are Grassmann valued fields
- Investigate:
  - What is $\delta\left(\phi - \int \mathrm{d}^D x\,\bar{\psi}\psi\right)$? How can it be practically realized?
  - Test this realization

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This talk

Practical realization of the fermionic constraint

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- Approximated constraint

$$\mathcal{Z}_\phi = \int \mathcal{D}\Phi \; \mathcal{D}[\bar{\psi}, \psi] \, \mathrm{e}^{-S_b[\Phi]} \mathrm{e}^{\bar{\psi} Q \psi} \prod_{a=0}^{n-1} \delta \left(\phi_a - \tfrac{1}{V} \int \mathrm{d}^D x \, \bar{\psi} \Gamma_a \psi \right)$$

Γ_a: $\mathbb{1}, \mathrm{i}\gamma_5, \mathrm{i}\gamma_5 \tau_i, \tau_i, \ldots$

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$$\to \mathcal{N}\int \mathcal{D}\Phi\, \mathrm{e}^{-S_b[\Phi]} \det Q\, \exp\left[-\frac{V}{2}(\phi_a - \mathcal{M}_a)(\chi^{-1})_{ab}(\phi_b - \mathcal{M}_b) - \frac{1}{2}\ln\det\chi\right]$$

$$\mathcal{M}_a = \frac{\mathrm{Tr}[Q^{-1}\Gamma_a]}{V} \qquad \chi_{ab} = \frac{\mathrm{Tr}[Q^{-1}\Gamma_a Q^{-1}\Gamma_b]}{V}$$

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$$\langle\bar\psi\Gamma_a\psi\rangle_\phi = \phi_a + (\phi_b - \mathcal{M}_b)\chi^{-1}_{bc}\gamma_{cda}\chi^{-1}_{de}(\phi_e - \mathcal{M}_e) - \frac{1}{V}\chi^{-1}_{bc}\gamma_{cba}$$

γ_{cda}: $\dfrac{\mathrm{Tr}\left[Q^{-1}\,\Gamma_c\,Q^{-1}\,\Gamma_d\,Q^{-1}\,\Gamma_a\right] + d \leftrightarrow a}{V}$

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- Test in 2-dimensional chiral Gross-Neveu model

Tests in the chiral Gross-Neveu model

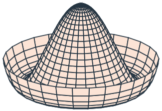
chiral Gross-Neveu model in $2D$

- The chiral Gross-Neveu (chiGN) model in a bosonized form has the desired form

$$Z = \int \mathcal{D}[\rho_0, \rho_1, \bar{\psi}, \psi] \, \mathrm{e}^{-S[\rho_0, \rho_1, \bar{\psi}, \psi]} \,, \; S = \int \mathrm{d}^2 x \left[\bar{\psi}(\partial\!\!\!/ + \rho_0 + \mathrm{i}\gamma_5 \rho_1)\psi + N_f \frac{\rho_0^2 + \rho_1^2}{2g^2} \right]$$

- Continuous $U_A(1)$ symmetry, spontaneously broken

$$\psi \to \mathrm{e}^{\mathrm{i}\gamma_5 \alpha}\psi \,, \; \bar{\psi} \to \bar{\psi}\mathrm{e}^{\mathrm{i}\gamma_5 \alpha} \,, \quad \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} \to \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix}$$



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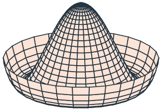

- Large-N_f limit: No quantum fluctuations for ρ_0, ρ_1, relevant configurations are minima of S
- Constrain both channels

$$\delta\left(\phi_0 - \tfrac{1}{V} \int \mathrm{d}^2 x \, \bar{\psi}\psi\right) \delta\left(\phi_1 - \tfrac{1}{V} \int \mathrm{d}^2 x \, \bar{\psi}\mathrm{i}\gamma_5\psi\right) \;\to\; \exp\left[\frac{V}{2}(\phi_a - \mathcal{M}_a)(\chi^{-1})_{ab}(\phi_b - \mathcal{M}_b)\right]$$

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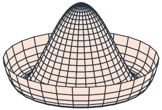

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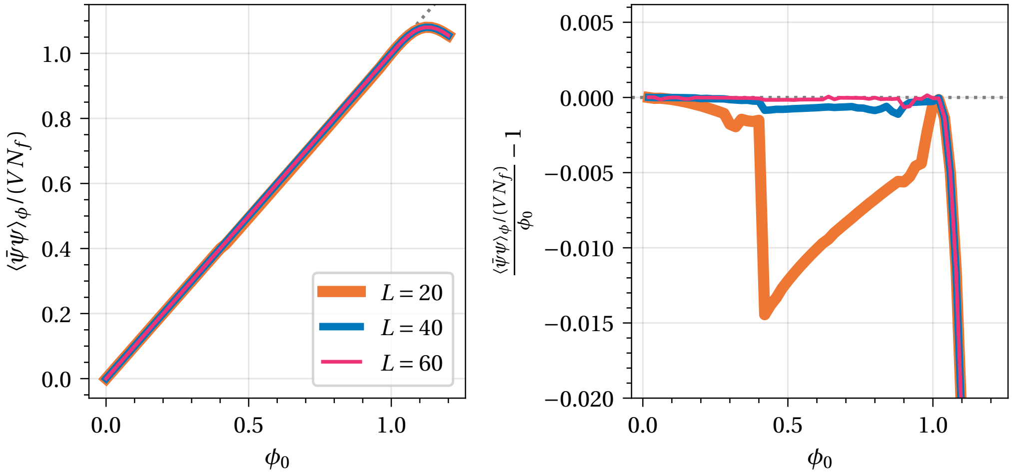
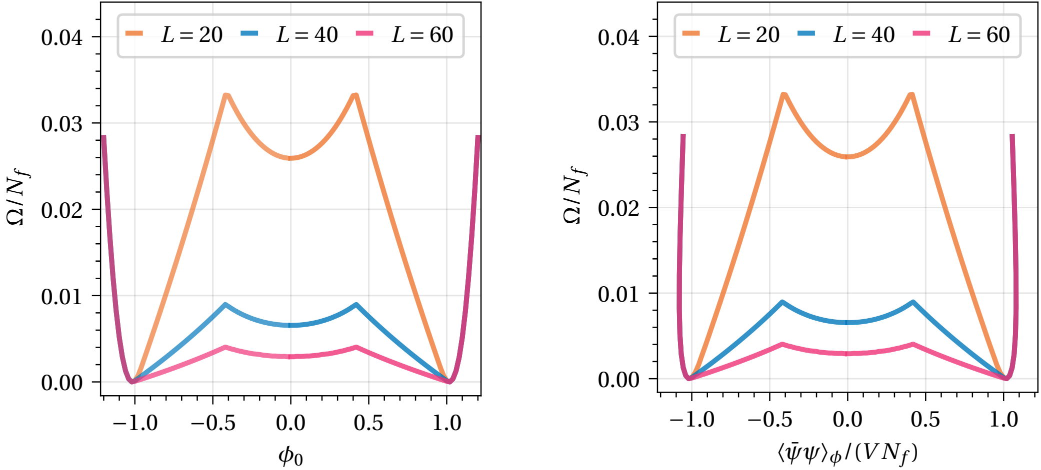
- Naive fermion discretization, single coupling $g^2 = 0.4920$, various lattice sizes $V = L^2$
- In the following results $\phi_1 = 0$ $\to$ slice through the rotationally symmetric potential

chiral GN model: Chiral condensate

$$\langle\bar{\psi}\psi\rangle_\phi = \phi_0 + (\phi_b - \mathcal{M}_b)\chi^{-1}_{bc}\gamma_{cd0}\chi^{-1}_{de}(\phi_e - \mathcal{M}_e)$$

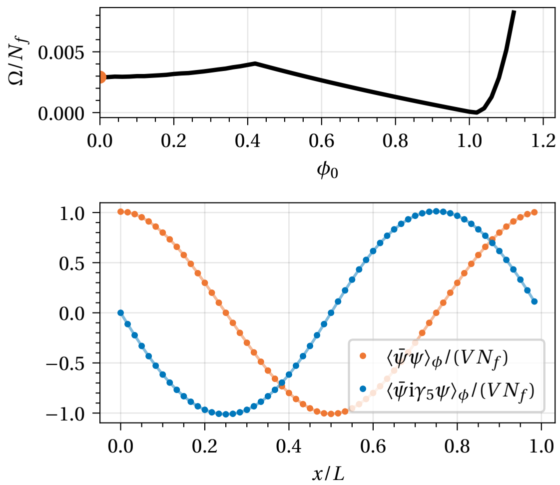


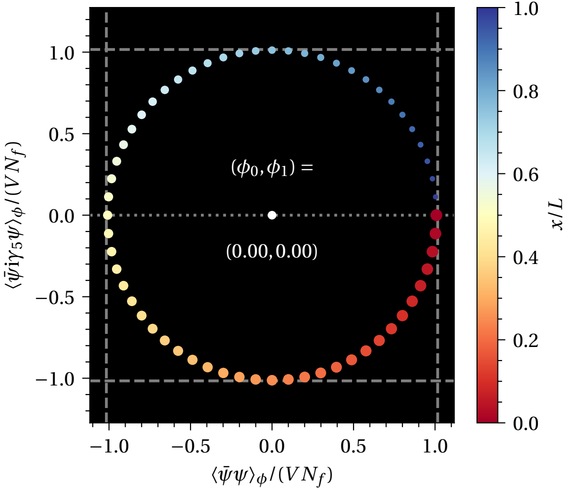
chiral GN model: Constrained potential (I)



chiral GN model: Field configurations

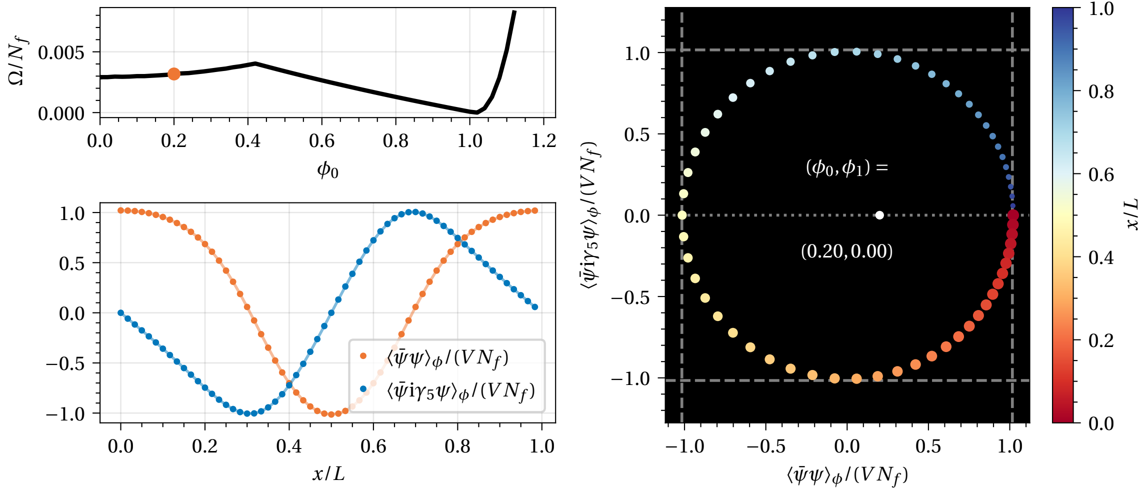
$\bar{\phi}_0 = 0.00, \bar{\phi}_1 = 0.00, L = 60$





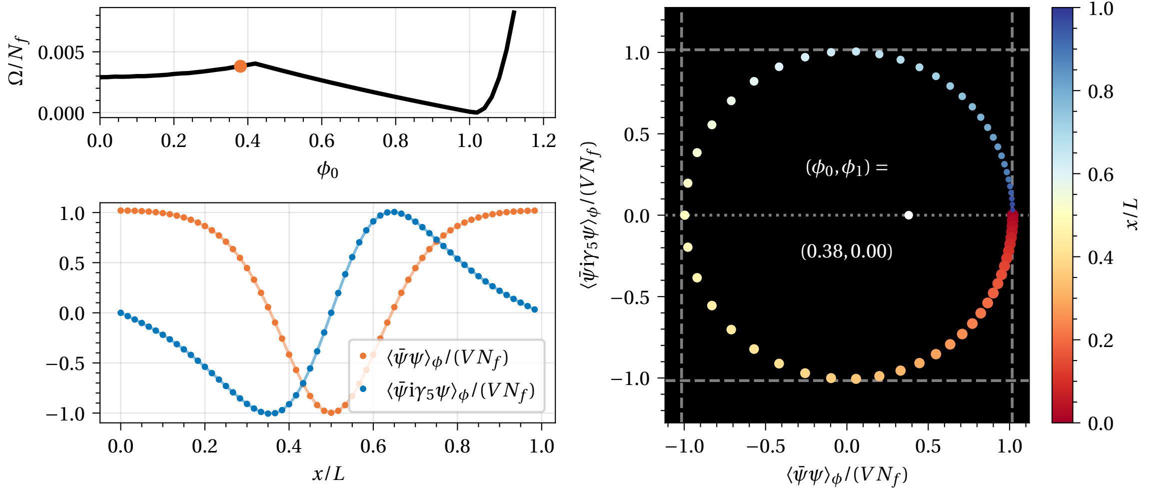
chiral GN model: Field configurations

$$\bar{\phi}_0 = 0.20,\ \bar{\phi}_1 = 0.00,\ L = 60$$



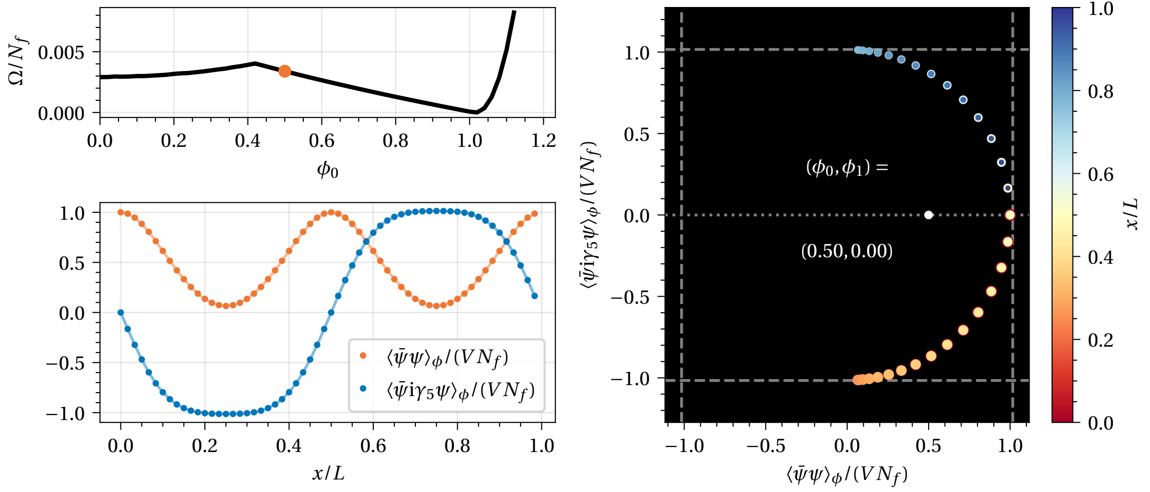
chiral GN model: Field configurations

$\bar{\phi}_0 = 0.38,\ \bar{\phi}_1 = 0.00,\ L = 60$



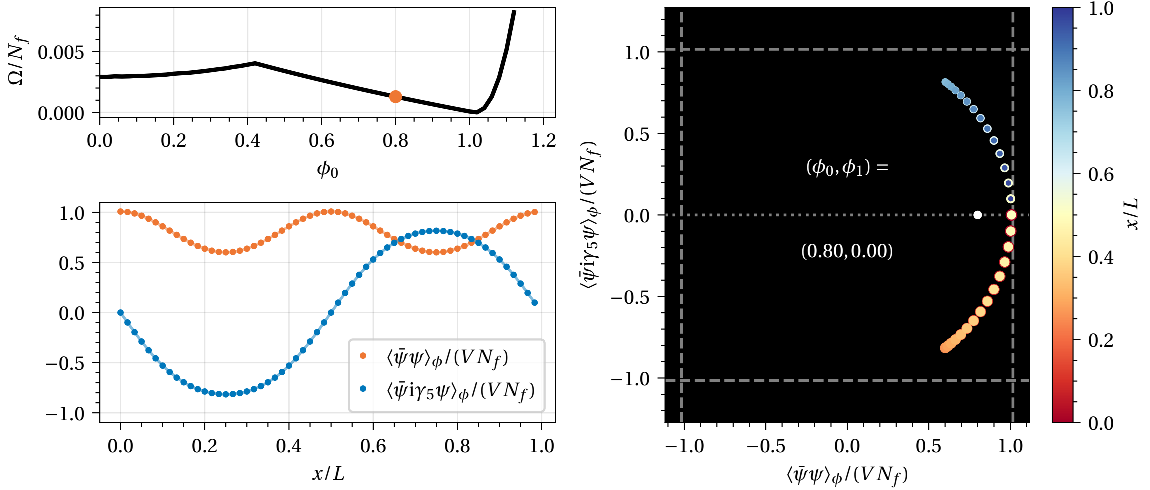
chiral GN model: Field configurations

$\bar{\phi}_0 = 0.50$, $\bar{\phi}_1 = 0.00$, $L = 60$



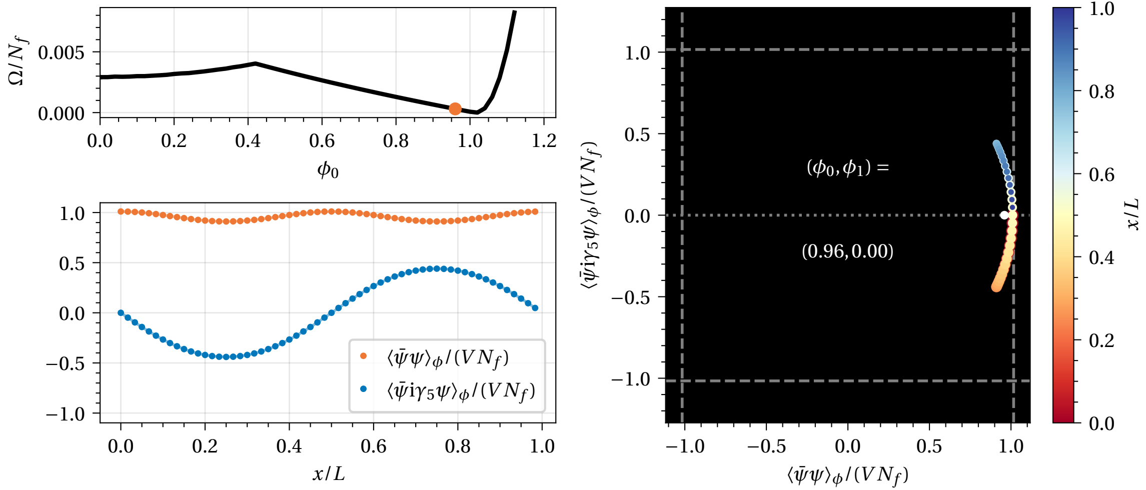
chiral GN model: Field configurations

$\bar{\phi}_0 = 0.80, \bar{\phi}_1 = 0.00, L = 60$



chiral GN model: Field configurations

$\bar\phi_0 = 0.96,\ \bar\phi_1 = 0.00,\ L = 60$



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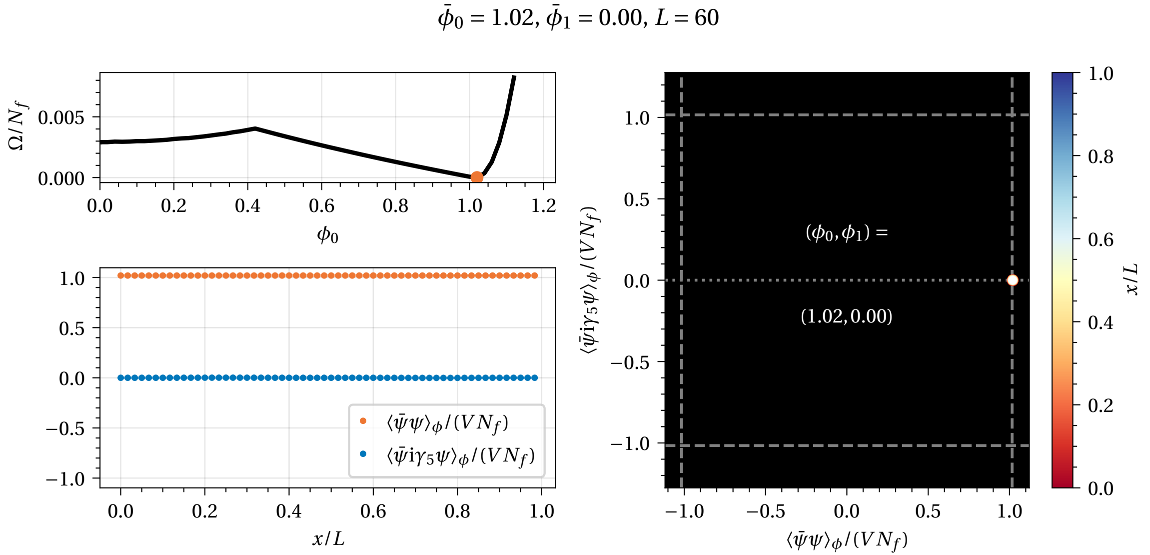


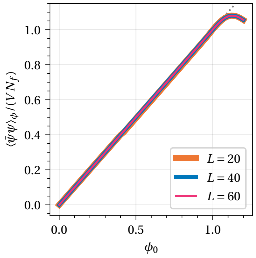
Summary and outlook

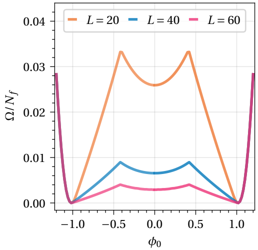
Summary:

- Developed a method to constrain fermionic condensates
- Tested in the chiral Gross-Neveu model
- Observed the flattening of the constrained potential
- Inhomogeneous configurations in flat region

Outlook:

- Perform full constrained Monte-Carlo simulations to test the constraint beyond the large-N limit





Appendix

Bosonic constraint in the Gross-Neveu model

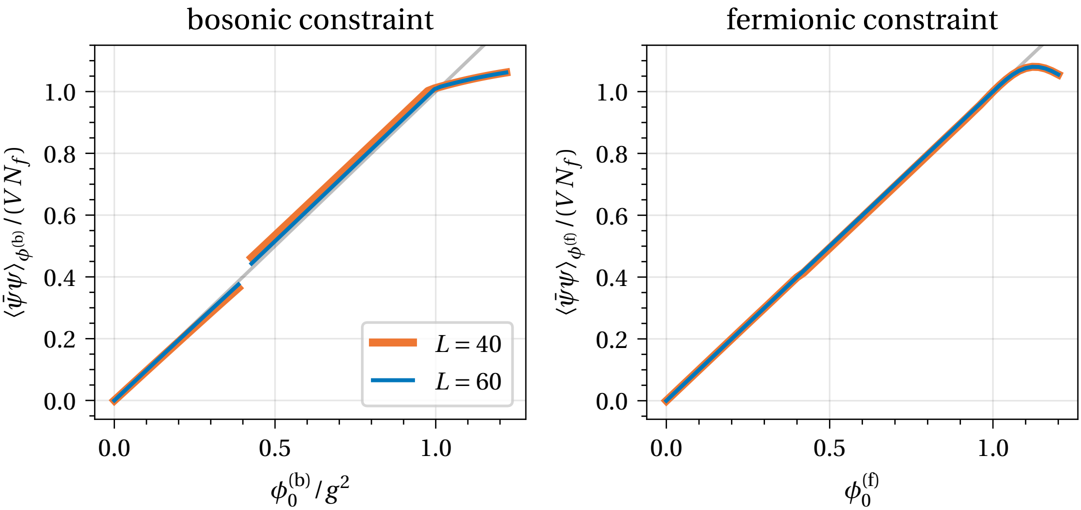
- The (chiral) GN model offers an alternative way to constrain the fermionic condensate
- Recall Ward identities: $\langle(\bar{\psi}\psi)(x)\rangle = \frac{-N_f}{g^2}\langle\rho_0(x)\rangle\,, \quad \langle(\bar{\psi}\mathrm{i}\gamma_5\psi)(x)\rangle = \frac{-N_f}{g^2}\langle\rho_1(x)\rangle$
- Constrain bosonic fields and with it implicitly $\bar{\psi}\psi$ and $\bar{\psi}\mathrm{i}\gamma_5\psi$

$$\mathcal{Z}_\phi = \int \mathcal{D}\rho_0\,\mathcal{D}\rho_1\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathrm{e}^{-S_{\mathrm{bos}}}\;\delta\left(\phi_0^{(\mathrm{b})} - \frac{1}{V}\int_V \rho_0\right)\delta\left(\phi_1^{(\mathrm{b})} - \frac{1}{V}\int_V \rho_1\right)$$

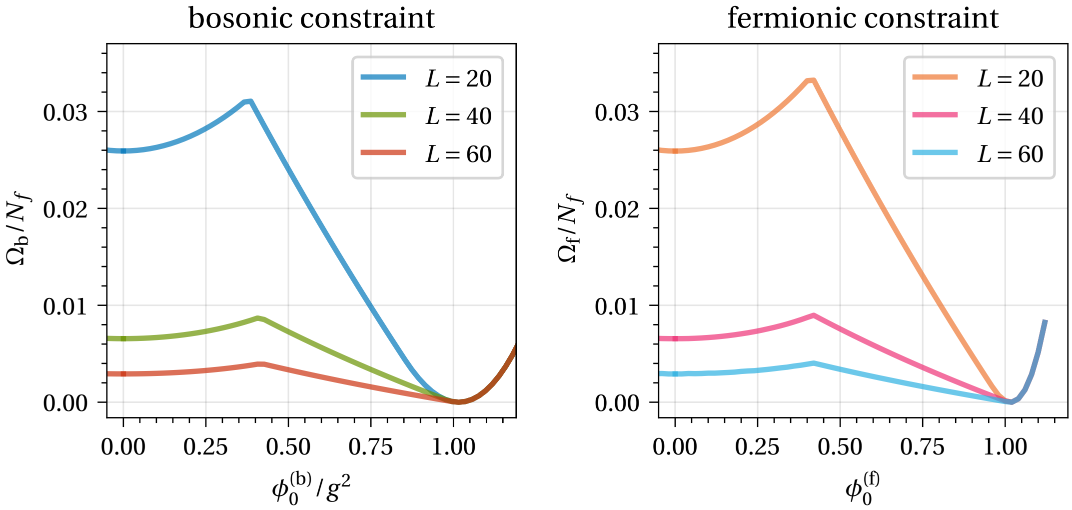
- Modifies Ward identity

$$\langle\bar{\psi}\psi\rangle_\phi = \frac{-N_f}{g^2}\langle\sigma\rangle_\phi + \frac{1}{V}\frac{\partial\ln\mathcal{Z}_\phi}{\partial\phi_0^{(\mathrm{b})}}\,, \quad \langle\bar{\psi}\mathrm{i}\gamma_5\psi\rangle_\phi = \frac{-N_f}{g^2}\langle\pi\rangle_\phi + \frac{1}{V}\frac{\partial\ln\mathcal{Z}_\phi}{\partial\phi_1^{(\mathrm{b})}}$$

chiral GN with Bosonic constraint: chiral condensate



chiral GN with Bosonic constraint: chiral condensate



chiral GN with Bosonic constraint: chiral condensate