Energy-momentum tensor in the 2D Ising CFT in full modular space

Nobuyuki Matsumoto

Boston University

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Based on work in collaboration with
Richard C. Brower (BU), George T. Fleming (Fermilab), J. Y. Lin (Carnegie Melon)

- Lattice field theories on curved spacetimes open up new branches of theoretical study
  - Nonperturbative calculation on curved space
    → *vacuum structure under curvature, BH background, …?*
  - Infinite volume calculations for CFT w/ Riemann projection or radial quantization
    → *infinite volume scattering from lattice, …?* **→ George's poster**
- Difficulty: **Brower-Cheng-Weinberg-Fleming-Gasbarro-Raben-Tan 2018**
  We need to give up rectangular lattice and its symmetries;
  discretization of curved manifolds often done w/ simplicial decomposition
  e.g., Regge 1961, Friedberg-Lee 1984
- Half step forward:
  Flat space but with stressed metrics "affine transformation" **e.g., Owen-Brower 2023**
  → *may be possible to reconstruct theory on curved space from tangential info*
  → Rich's next talk
- An essential quantity in any of these directions: energy-momentum tensor
  - measures the linear response to metric perturbation by definition.
  - In 2D CFT, it is related to the background geometry transparently
    L_0 *changes* τ *on* T^2, $\langle T^\mu_\mu \rangle = -\frac{c}{12} R$ *(trace anomaly)*
  - Even on regular lattices, its definition requires care on discretized spacetime; more for simplicial lattices as translation is even more screwed up

- Thoroughly study EM tensor of the 2D Ising CFT on T^2:
  - w/ arbitrary modulus τ, on hexagonal lattice (dual to simplicial, triangular lattice)
  - Both in spin and Majorana variables
  - Including overall normalization and one-point function

- Previous work on lattice: **Kadanoff-Ceva 1970**
  - On rectangular lattice, before the developments of CFT (cf. **BPZ 1984**)
  - Require antisymmetrization from the original expression to remove contribution from the descendants of ε

- Nontrivial points on non-regular lattice

$\tau \equiv \tau_1 + i\tau_2$: modulus

  - Naive $\tau_{1,2}$ derivatives do not give a suitable EM tensor operator
  - Free Majorana fermion but nontrivial mixing of operators occurs; can be fully described geometrically by the relative shift between the e/o lattices
  - Not all lattice operator works consistently as the EM tensor (under different BC)

This talk mainly focuses on these technicalities

- Ising CFT partition function as free fermion theory:

Onsager 1944, Schultz-Mattis-Lieb 1964, Itzykson 1982, BPZ 1984, Francesco-Saleur-Zuber 1987

$$\begin{cases} L_0 = \sum_{k \in \mathbb{Z}_{>0} - 1/2} k\, a_{-k} a_k & \text{(ABC=NS)} \\ L_0 = \sum_{k \in \mathbb{Z}_{>0}} k\, a_{-k} a_k + \frac{1}{16} & \text{(PBC=R)} \end{cases}$$

(a_k: fermion operator for the Fourier mode k; $\{a_p, a_q\} = \delta_{p+q}$)

$$Z_{\text{cont}} \equiv \mathrm{Tr}_{\mathrm{NS+R}} \left[P_{\mathrm{GSO}}\, q^{L_0 - \frac{1}{48}}\, \bar{q}^{\bar{L}_0 - \frac{1}{48}} \right]$$

$$= \frac{1}{2} \left\{ \begin{array}{l} \mathrm{Tr}_{\mathrm{R}} \left[(-1)^F q^{L_0 - \frac{1}{48}} \bar{q}^{\bar{L}_0 - \frac{1}{48}} \right] + \mathrm{Tr}_{\mathrm{R}} \left[q^{L_0 - \frac{1}{48}} \bar{q}^{\bar{L}_0 - \frac{1}{48}} \right] + \\ + \mathrm{Tr}_{\mathrm{NS}} \left[q^{L_0 - \frac{1}{48}} \bar{q}^{\bar{L}_0 - \frac{1}{48}} \right] + \mathrm{Tr}_{\mathrm{NS}} \left[(-1)^F q^{L_0 - \frac{1}{48}} \bar{q}^{\bar{L}_0 - \frac{1}{48}} \right] \end{array} \right\}$$

$$\equiv \frac{1}{2} \{ Z^{\text{cont}}_{\nu=1} + Z^{\text{cont}}_{\nu=2} + Z^{\text{cont}}_{\nu=3} + Z^{\text{cont}}_{\nu=4} \}$$

$Z^{\text{cont}}_{\nu=1} = 0$ $\because$ zero modes

$\nu = 1, 2, 3, 4 \Leftrightarrow$ PP, PA, AA, AP in (space, time)

($P_{\mathrm{GSO}} \equiv \frac{1 + (-1)^F}{2}$; F: fermion number; $q \equiv \exp(2\pi i \tau)$; $T \equiv T_{zz} = \frac{1}{2}(T_{xx} - i T_{xy})$)

- $T_{\alpha\beta}$ changes τ by the effect of L_0

Eguchi-Ooguri 1986

$$\langle T \rangle = 2\pi i\, \partial_\tau \ln Z_{\text{cont}}(\tau, \bar{\tau})$$

- Spin partition function

$$Z_I \equiv \sum_{\{\sigma\}} \exp \sum_{x\in e,M} \beta_M\, s_x s_{x+\hat{M}} \quad (s_x = \pm 1)$$

Periodic-Periodic in (σ_1, σ_2)

Exact mapping via loop expansion on T^2

$$Z_I = \frac{1}{2}\sum_{\nu} \frac{(-1)^{\delta_{\nu,1}}}{2^V \prod \cosh \beta_{xy}} Z_W^{\nu}$$

$$\tanh \beta_A = \kappa_A \sqrt{\frac{\cos\theta_B \cos\theta_C}{\cos\theta_A}}$$

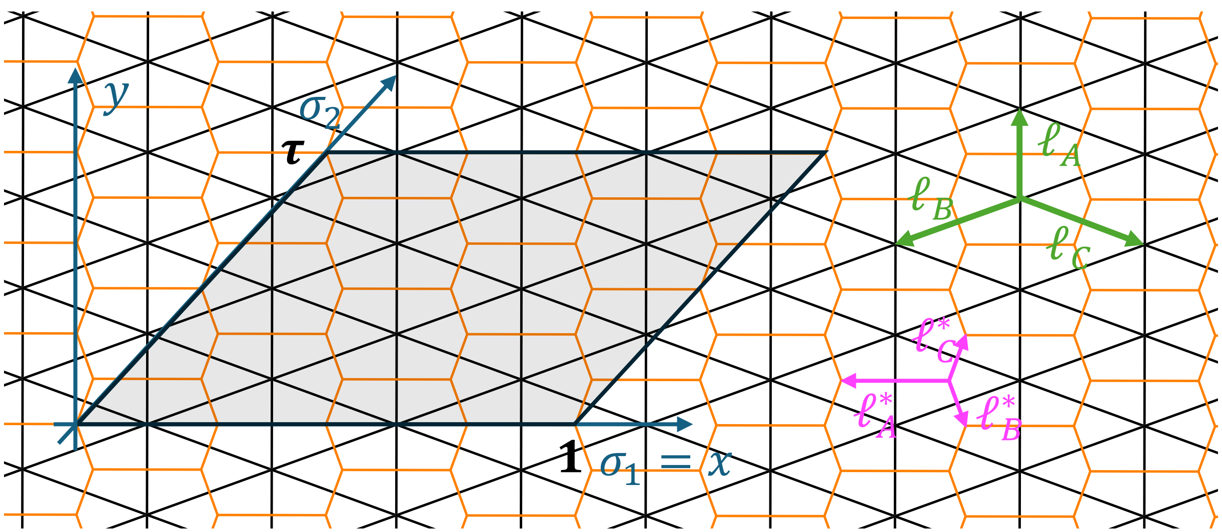
Samuel 1980, Itzykson 1982, Wolff 2020, Brower-Owen 2023

- Wilson-Majorana partition function

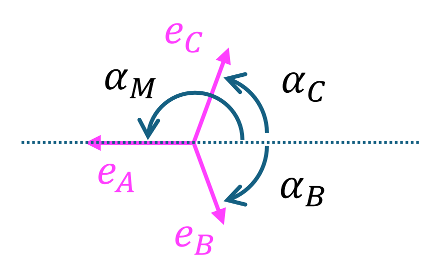
$$Z_W^{\nu} \equiv \int [d\xi]_\nu \exp\left(-\frac{1}{2}\sum_x \bar{\xi}_x \xi_x + \frac{1}{2}\sum_{x\in e,M} \kappa_A\, \bar{\xi}_x (1-\gamma_\mu e_M^\mu) \xi_{x+\hat{M}} \right)$$

$\nu = 1,2,3,4 \Leftrightarrow$ PP, PA, AA, AP in (σ_1, σ_2)

- Parametrization of the hexagonal lattice



$$e_M^\alpha \equiv \frac{(\ell_M^*)^\alpha}{|\ell_M^*|} \quad (M = A, B, C)$$



$\tau_{1,2}$ derivatives on the lattice $\quad (\tau = \tau_1 + i\tau_2)$

Utilize $\tau_{1,2}$ derivatives? $\quad \langle T_{xx}\rangle_\nu = 2\pi\partial_{\tau_2}\ln Z_\nu^{\text{cont}}(\tau_1,\tau_2)$

$$\approx 2\pi\partial_{\tau_2}\ln\{\mathcal{N}^{-1}(\tau_1,\tau_2;L)\, Z_\nu^{\text{lat}}(\tau_1,\tau_2;L)\}$$

- Fermion bilinear part:

$$\partial_{\tau_2}\ln Z_\nu^{\text{lat}} = -\sum_M \left\langle\left(\frac{\partial\kappa_M}{\partial\tau_2}\frac{\partial S_\nu^{\text{lat}}}{\partial\kappa_M} + \frac{\partial e_M^\alpha}{\partial\tau_2}\frac{\partial S_\nu^{\text{lat}}}{\partial e_M^\alpha}\right)\right\rangle_\nu^{\text{lat}}$$

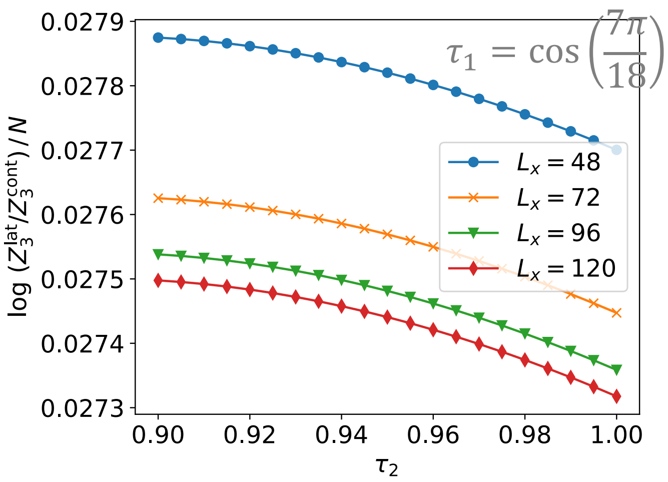
$$\frac{\partial\kappa_M}{\partial\tau_2}\frac{\partial S_\nu^{\text{lat}}}{\partial\kappa_M} + \frac{\partial e_M^\alpha}{\partial\tau_2}\frac{\partial S_\nu^{\text{lat}}}{\partial e_M^\alpha} = \frac{\partial\kappa_M}{\partial\tau_2}\sum_{x\in e}\frac{1}{2}\bar{\xi}_x(1-\gamma\cdot\, e_M)\,\xi_{x+\hat{M}} + \underbrace{\sum_M \frac{\partial e_M^\alpha}{\partial\tau_2}\sum_{x\in e}\bar{\xi}_x\gamma_\alpha\xi_{x+\hat{M}}}$$

Not easy to map to the spin system

- Constant part

  $\tau_{1,2}$ dependence on $\mathcal{N}(\tau_1,\tau_2;L)$
  remains in the $L\to\infty$ limit,
  that would be only canceled by
  a divergent part of the fermion bilinear operator

  (Usually, the continuum path integral is regularized with zeta function regularization, which does so cleanly w/o such $\tau_{1,2}$ dependence.)

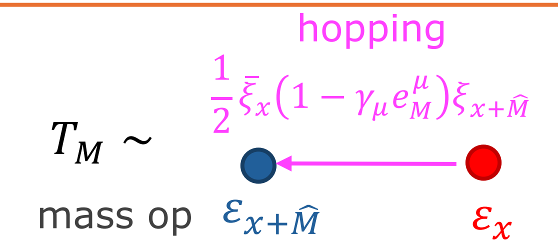


- Defining a local operator from a global discussion is ambiguous

(cf. need of antisymmetrization for **Kadanoff-Ceva 1970**)

We rather take a conventional lattice strategy

Coming back to Symanzik-type construction



- We consider the lattice operator:

$$\hat{T}^{\text{lat}}_{x,M} \equiv \frac{1}{2}\bar{\xi}_x(1-\gamma_\mu e_M^\mu)\xi_{x+\hat{M}} - \frac{1}{4}(\bar{\xi}_x\xi_x + \bar{\xi}_{x+\hat{M}}\xi_{x+\hat{M}})$$

$$T^{\text{lat}}_{x,M} \equiv \frac{2\pi}{s}\frac{1}{|\ell^*_M|}\hat{T}^{\text{lat}}_{x,M}$$

easily mappable to the spin system via loop expansion

($s \equiv \frac{\sum_M |\ell^*_M|}{2}$: semiperimeter; supplies dimension)

- Mixing of $T, \bar{T}$ (and 1) can be resolved by the three projected components T_M

- To calculate the mixing matrix, naively, one may use:

$$\xi_{x+\hat{M}} \stackrel{?}{=} \xi_x + |\ell^*_M| e_M^\nu \partial_\nu \xi_x + O(a^2),$$

which implies:

$$\hat{T}^{\text{lat}}_{x,M} \stackrel{?}{=} -|\ell^*_M| \cdot e_M^\mu e_M^\nu \frac{1}{2}\bar{\xi}_x\gamma_\mu\partial_\nu\xi_x \cdot (1+O(a))$$

projected EM tensor: $e_M^\alpha e_M^\beta T_{\alpha\beta}$

However, "?" turns out to be negative for nonregular lattices

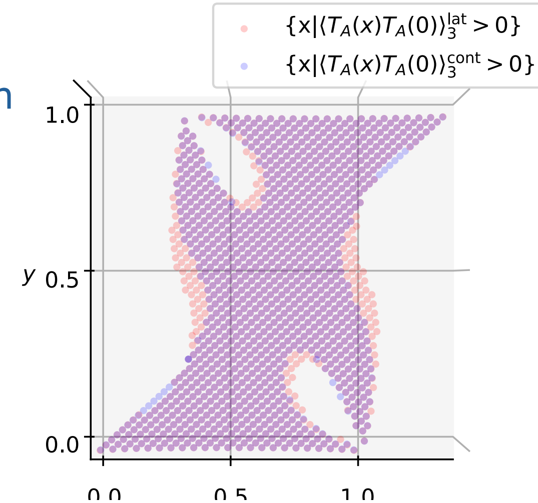
Deviation from the prediction of classical expansion

- Contribution from 1 dropped by taking connected part
- Mixing of T and $\bar{T}$ differs from the prediction from the classical expansion:

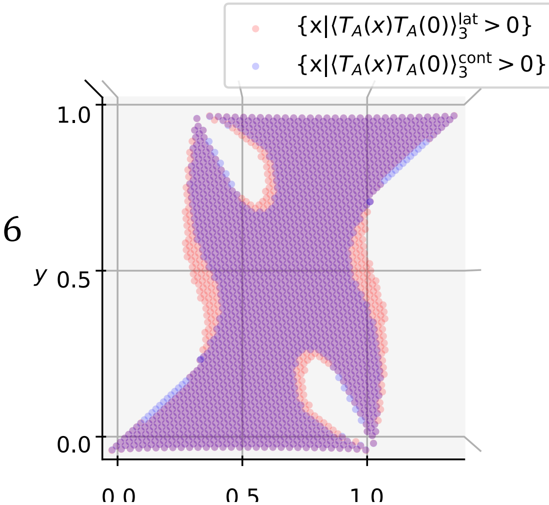
$\langle T_A(x) T_A(0) \rangle_{3,c}$

red: from lattice op T_A
blue: predicted exact expression
positive region is shown

$L = 72$

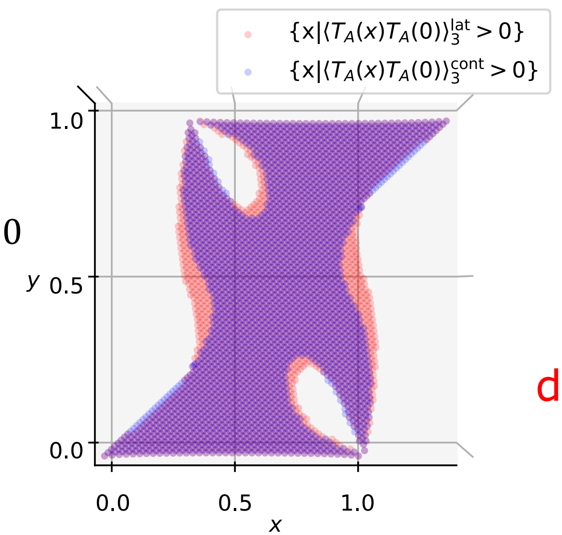


$L = 96$

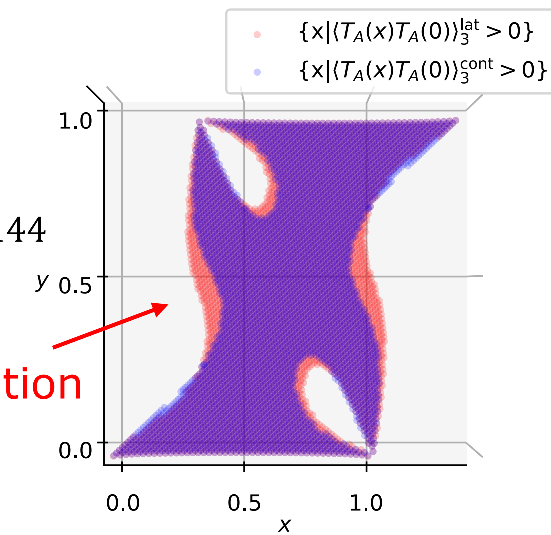


$\tau = e^{7\pi i/18}$

$L = 120$



$L = 144$

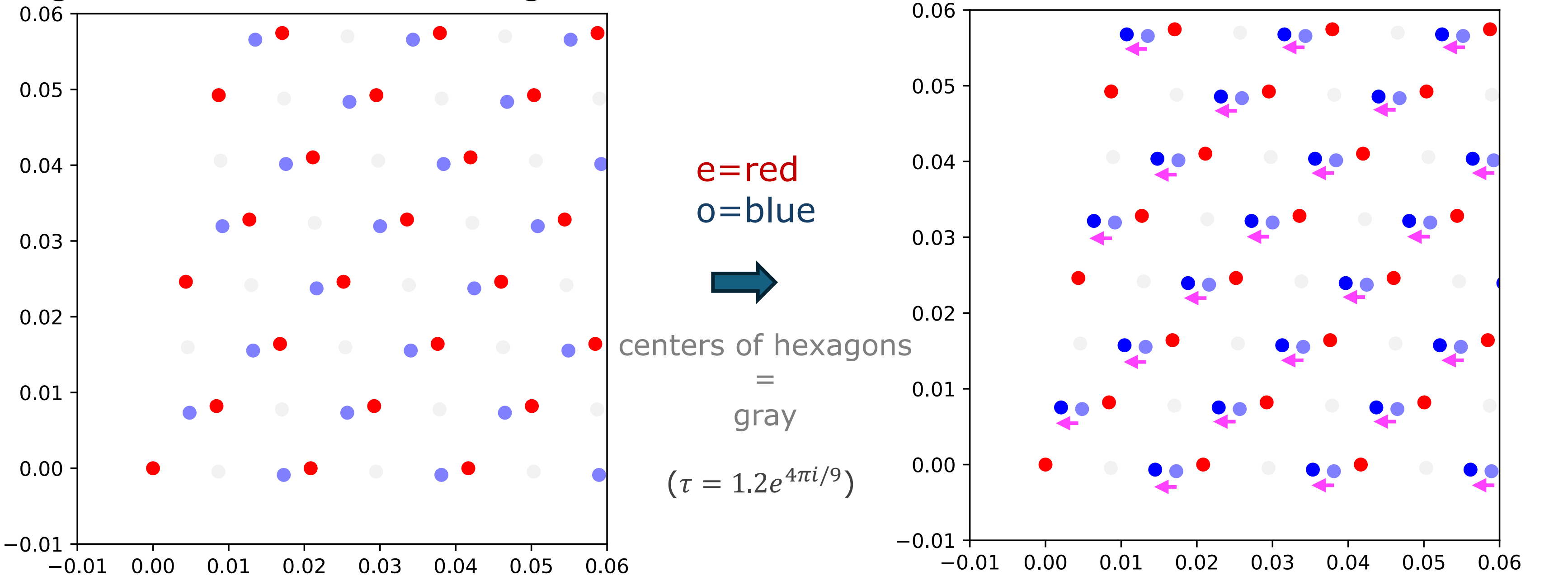


deviation

Deviation remains in the continuum limit.

- Lattice translation holds only for e/o sublattices, which cannot constrain their relative position to the classical prediction:

original sites when constructing the action — shifted sites in constructing the observables



This allows $\xi_{x+\hat{M}}$ to float and redeclare its location in the observables:

$$\xi_{x+\hat{M}} = \xi_x + \tilde{\ell}_M^{*\nu}\partial_\nu \xi_x + O(a^2)$$
$$= \xi_x + \left|\tilde{\ell}_M^*\right| \tilde{e}_M^\nu \partial_\nu \xi_x + O(a^2)$$

different from original ℓ_M^*

$$\left(\begin{array}{l} (\tilde{e}_M^\nu) \equiv [\cos(\alpha_M + \delta\alpha_M), \sin(\alpha_M + \delta\alpha_M)]^T \\ x \in e \end{array} \right)$$

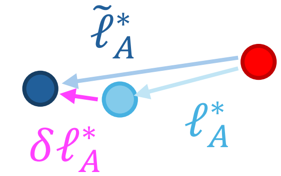
- With such possibility:

$$\hat{T}_{x,M}^{\text{lat}} = \frac{1}{2}\bar{\xi}_x (1 - \gamma_\mu e_M^\mu)\xi_{x+\hat{M}} - \frac{1}{4}(\bar{\xi}_x \xi_x + \bar{\xi}_{x+\hat{M}}\xi_{x+\hat{M}})$$
$$\propto e_M^\alpha \tilde{e}_M^\beta T_{\alpha\beta}$$

two different vectors; one known, one unknown

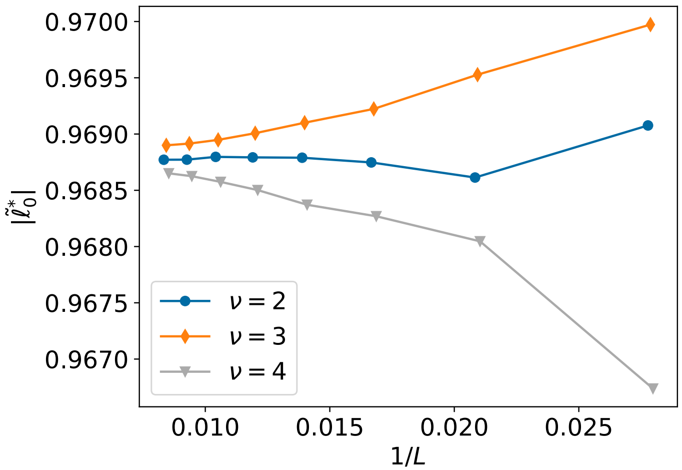
- Fit an IR part of the correlators $\langle T_M^{\mathrm{lat}}(x) T_N^{\mathrm{lat}}(0) \rangle_{\nu,\mathrm{conn}}$

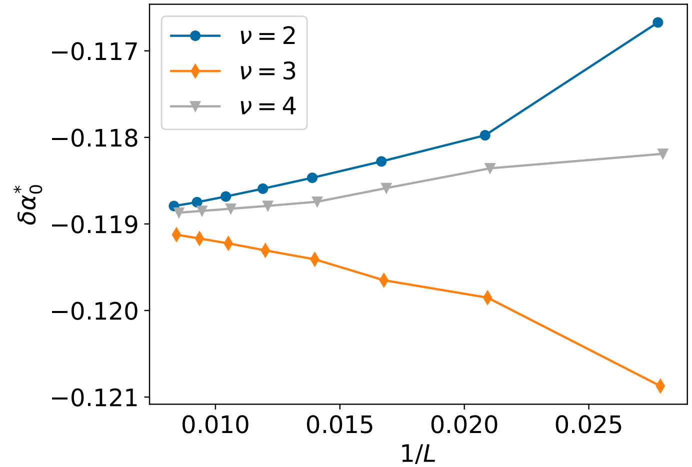
➡ Shift params converges to a universal value as $L \to \infty$ irresp of ν



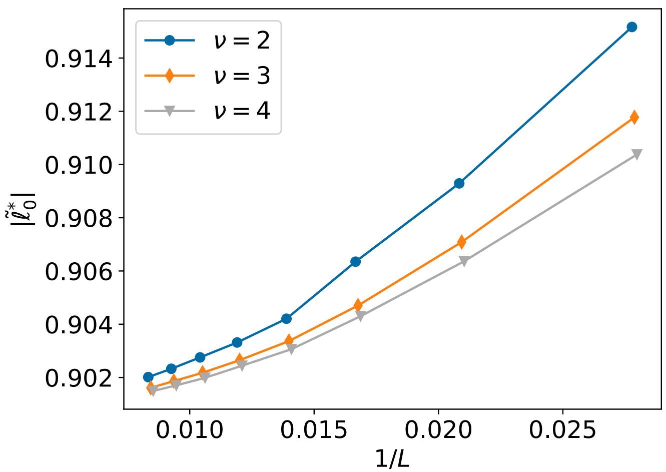
modified length $|\tilde{\ell}_A^*|$ — angle shift $\delta\alpha_A$

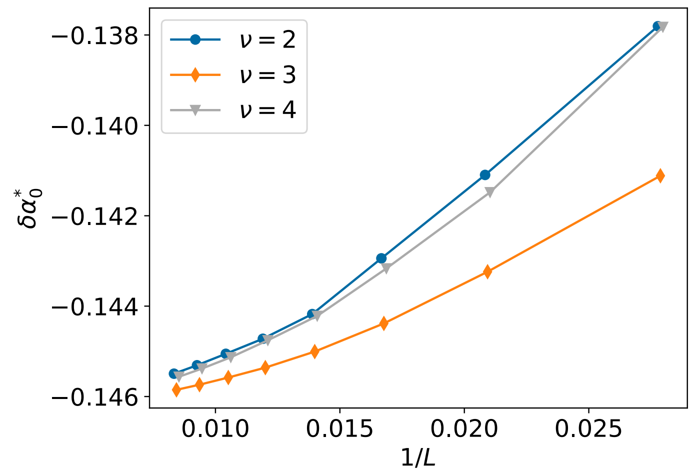
$\tau = e^{7\pi i/18}$





$\tau = 1.2e^{4\pi i/9}$

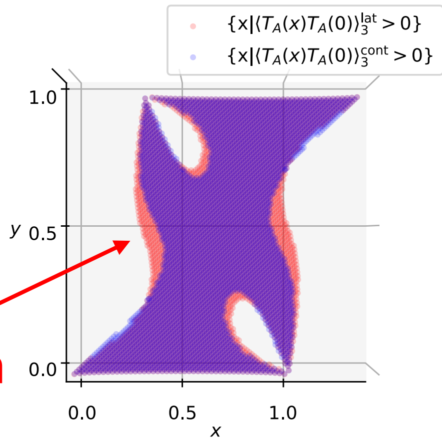




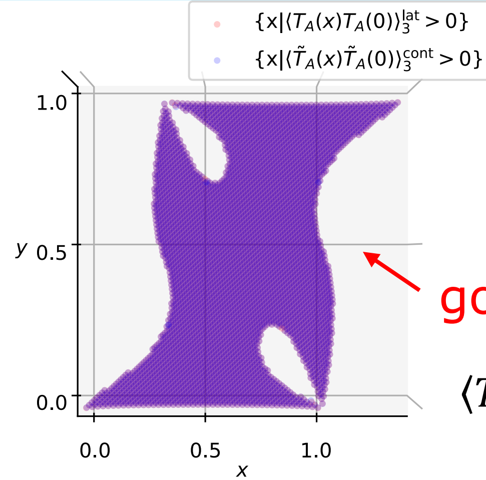
suggesting the existence of a consistent continuum limit

Confirming the correction

$$\begin{pmatrix} L = 144 \\ \tau = e^{7\pi i/18} \\ \nu = 3 \end{pmatrix}$$



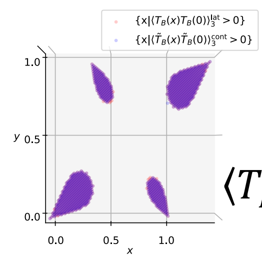
deviation



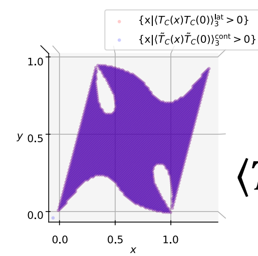
gone!

$\langle T_A(x)T_A(0)\rangle_{3,c}$

Possibly 2 x 3 mixing parameters explained only with 2 param shift



$\langle T_B(x)T_B(0)\rangle_{3,c}$

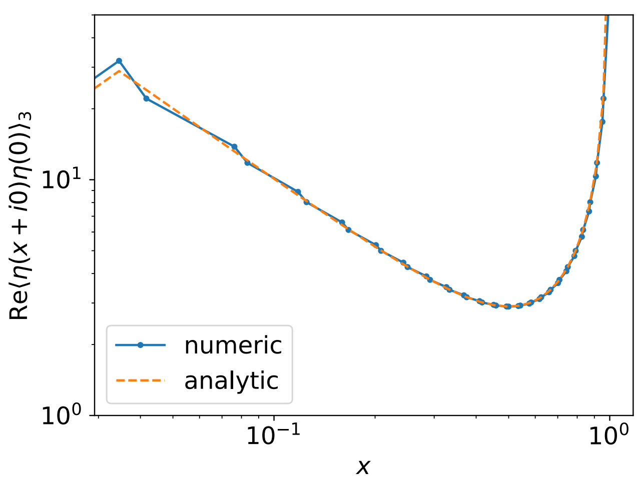


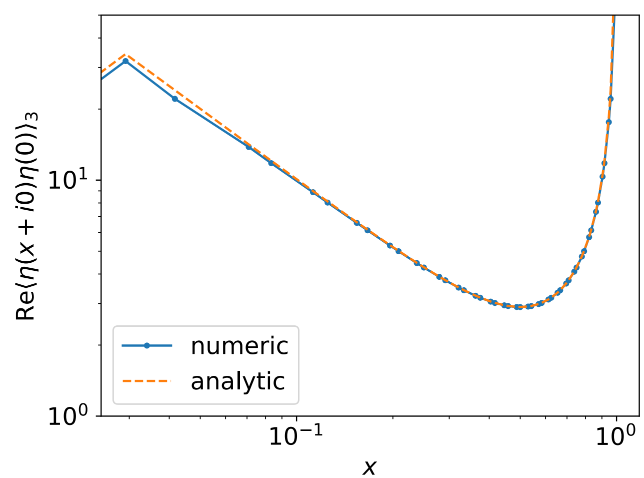
$\langle T_C(x)T_C(0)\rangle_{3,c}$

- In fact, staggered bumps disappears in the fermion correlators

$\langle \eta(z)\eta(0)\rangle$: holomorphic

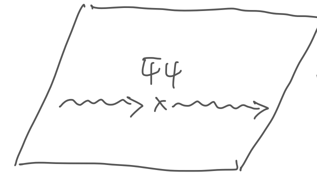
$L = 72$ $\tau = 1.2e^{4\pi i/9}$
$\nu = 3$





Mixing with 1: One point function

- Finite volume (torus) → $\langle T\rangle_\nu \neq 0$ in the continuum



wrap around
propagation

- $\langle T_M^{\mathrm{lat}}\rangle_\nu$ further has a divergent part on the lattice because of the Wilson term:

$$\bar\psi\psi \cdot a\int_x \bar\psi\partial^2\psi = O(1/a)$$

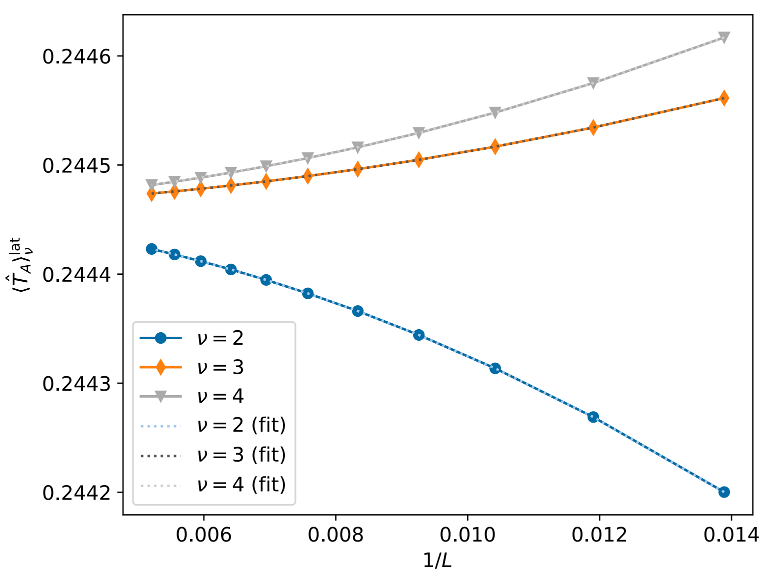
When properly regularized,
$(½)\langle(\bar\psi\psi)_{\mathrm{reg}}\rangle_{\nu\neq1} = \langle\varepsilon\rangle_{\nu\neq1} = 0$

Ferdinand-Fisher 1969,
Francesco-Saleur-Zuber 1987

This contribution dropped here for simplification

Divergent part again converges to a universal value as $L \to \infty$ irresp of ν:

$\tau = e^{7\pi i/18}$



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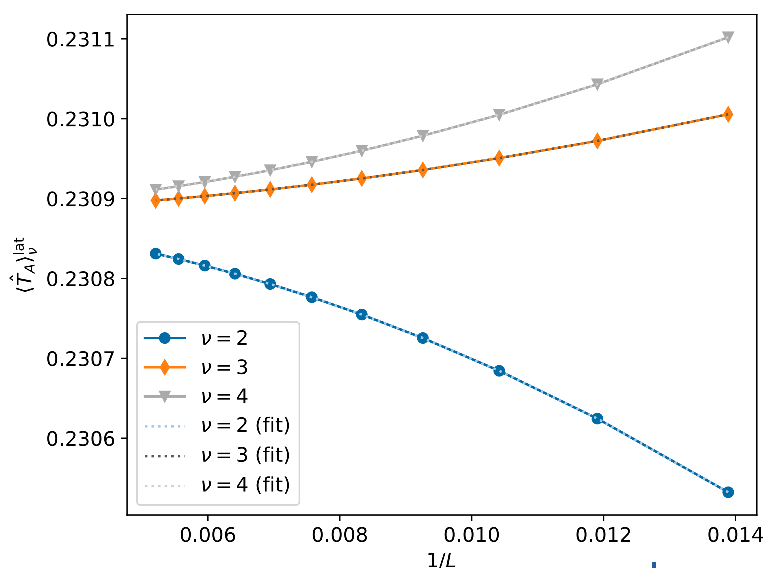

clean a^2 scaling

<u>NB</u> Determined independently of the shift parameters

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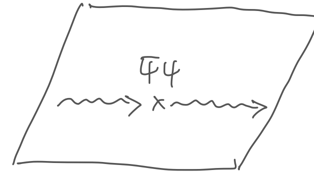

propagation
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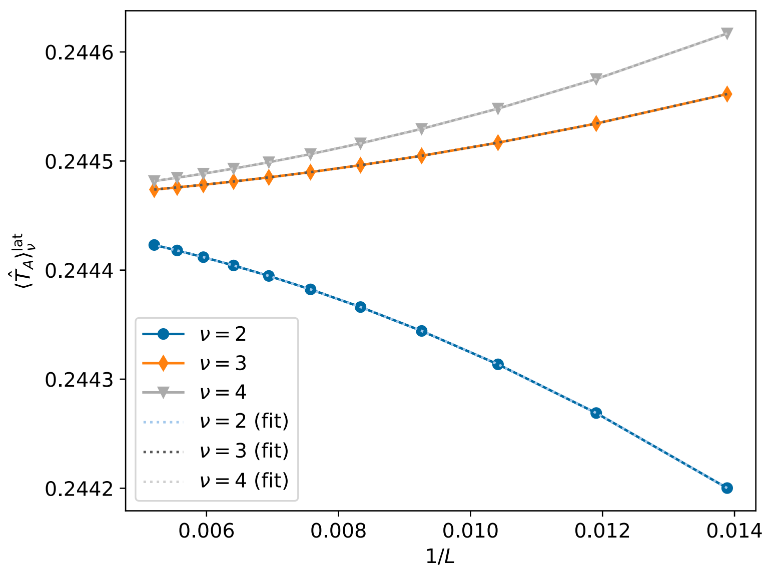

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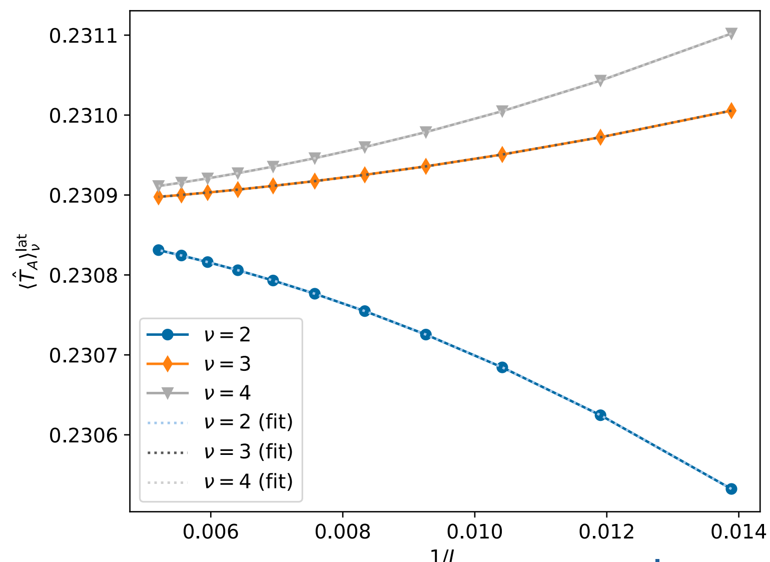


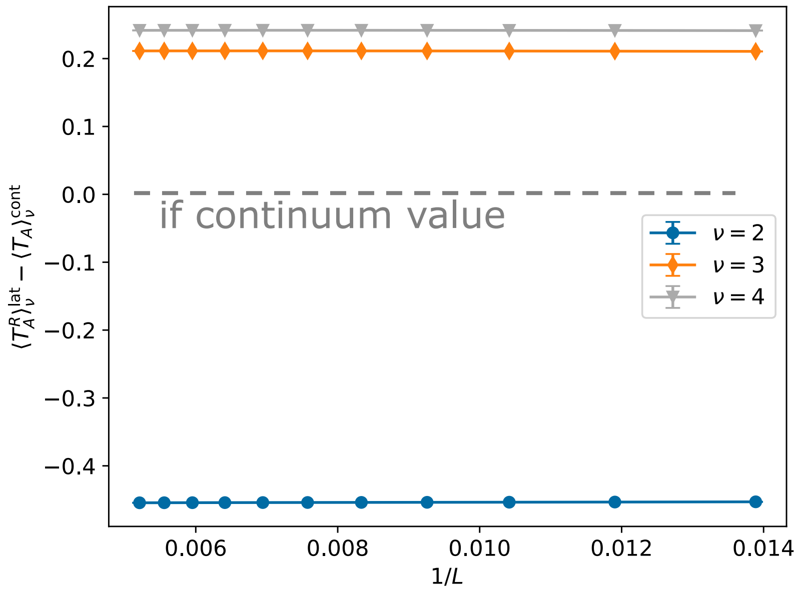
clean a^2 scaling

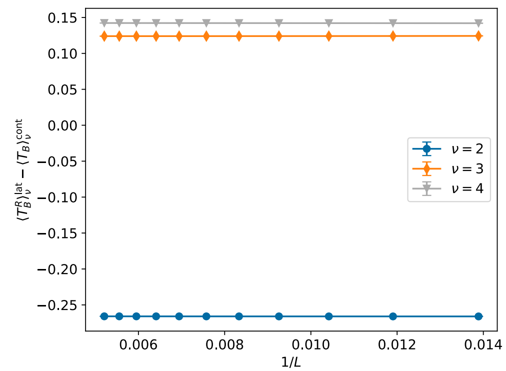
NB Determined independently of the shift parameters

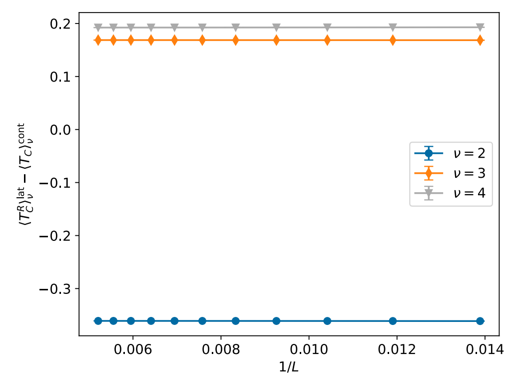
Nonuniversal finite part of $\langle T_M^{\mathrm{lat}} \rangle_\nu$

- One might regularize T_M^{lat} by subtracting the divergent part: $T_M^{\mathrm{lat},R} \equiv T_M^{\mathrm{lat}} - \dfrac{C_M}{a}$
- However, $\langle T_M^{\mathrm{lat},R} \rangle_\nu$ does not approach the continuum value $\left\langle e_M^\alpha \tilde{e}_M^\beta T_{\alpha\beta} \right\rangle_\nu$ and the deviation differs by ν:

$\tau = 1.2e^{4\pi i/9}$







- This finite shift may *not* be regarded as a finite part of the renormalization as it depends on the BC ν.

$T_M^{\mathrm{lat},R}$ themselves do not behave consistently as the projected EM tensor operator when including one-point function.

- Nontrivial point:

  By constructing $T_{\alpha\beta}^{\mathrm{lat}}$ as a linear combination of T_M^{lat} s.t. the divergent part cancels, the finite part likely also cancels for every ν.

i.e.,

$$T_M^{\mathrm{lat}} \simeq \cos(2\alpha_M + \delta\alpha_M) T_{xx} + \sin(2\alpha_M + \delta\alpha_M)\tilde{T}_{xy} + \frac{A_M}{a} + B_{M,\nu} + O(a)$$

$$T_{xx}^{\mathrm{lat}} \equiv \sum_M C_{xx,M}\, T_M^{\mathrm{lat}} \simeq 1 \cdot T_{xx} + 0 \cdot T_{xy} + \frac{0}{a} + \sum_M C_{xx,M} B_{M,\nu} + O(a)$$

by the choice of $C_{xx,M}, C_{xy,M}$

can exist in principle but cancels

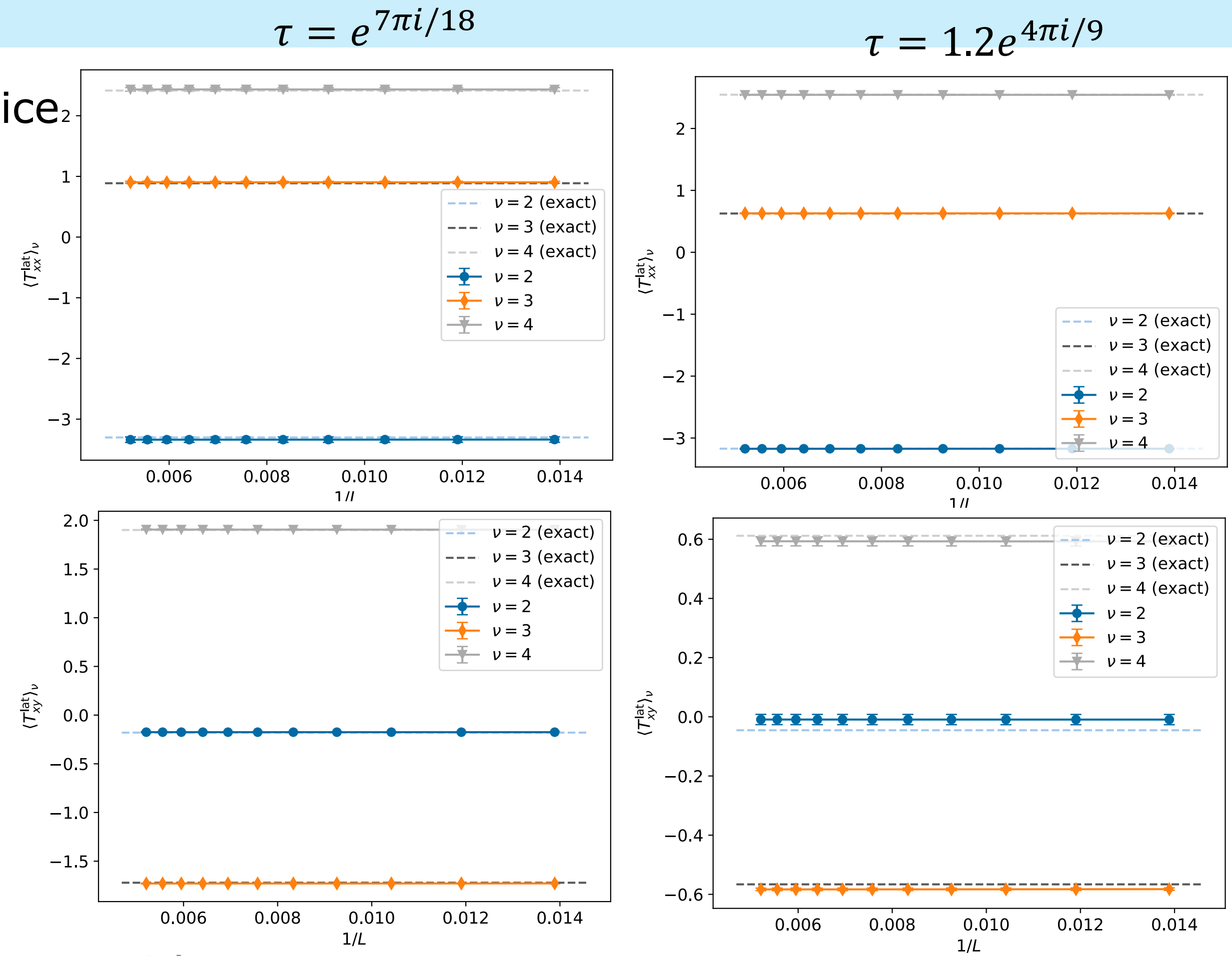
$$T_{xy}^{\mathrm{lat}} \equiv \sum_M C_{xy,M}\, T_M^{\mathrm{lat}} \simeq 0 \cdot T_{xx} + 1 \cdot T_{xy} + \frac{0}{a} + \sum_M C_{xy,M} B_{M,\nu} + O(a)$$

$T_{\alpha\beta}^{\mathrm{lat}}$ uniquely constructed from T_M^{lat}

Numerics

Thus-obtained $\langle T_{\alpha\beta}\rangle_\nu$ from lattice

→ reasonable agreement



<u>$c_0 + c_1 a + c_2 a^2$ fit</u> (errors fully systematic)

$\langle T\rangle_2 \approx -1.5869(43) - 0.0873(33)$
exact: $-1.5860 \quad - 0.0899$

$\langle T\rangle_3 \approx +0.3147(30) - 0.8645(24)$
exact: $+0.3132 \quad - 0.8606$

$\langle T\rangle_4 \approx +1.2718(41) + 0.9530(35)$
exact: $+1.2727 \quad + 0.9506$

$\langle T\rangle_2 \approx -1.669(23) - 0.0038(88)$
exact: $-1.651 \quad - 0.0227$

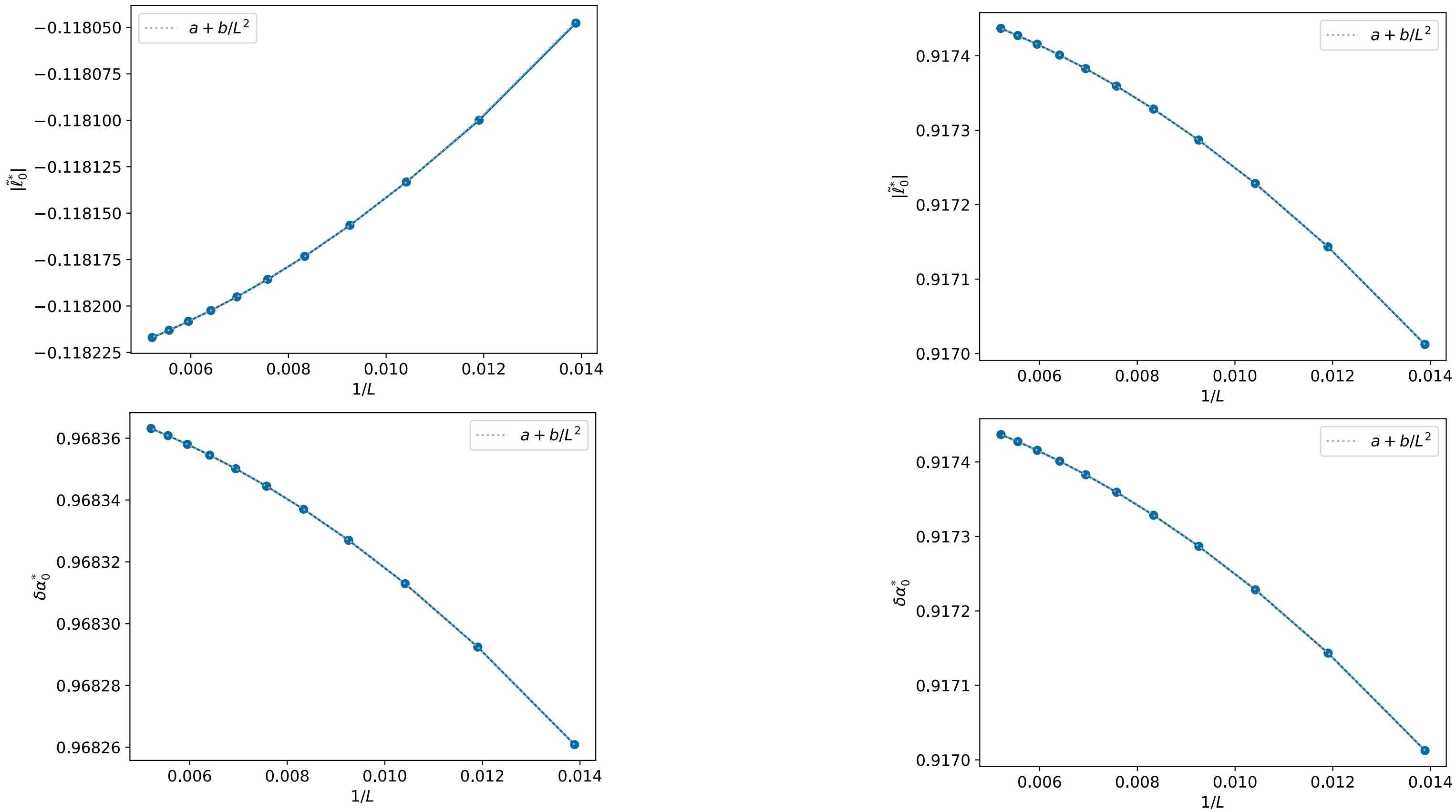
$\langle T\rangle_3 \approx +0.4524(88) - 0.2909(26)$
exact: $+0.4437 \quad - 0.2831$

$\langle T\rangle_4 \approx +1.217(15) + 0.2974(78)$
exact: $+1.207 \quad + 0.3058$

Alternative scheme for shift parameters: fix the 1pt functions

- Fitting all ν simultaneously → clean a^2 scaling

length $|\ell_A^*|$ angle shift $\delta\alpha_A$



- Gives more precise values, two schemes in a tolerable agreement:

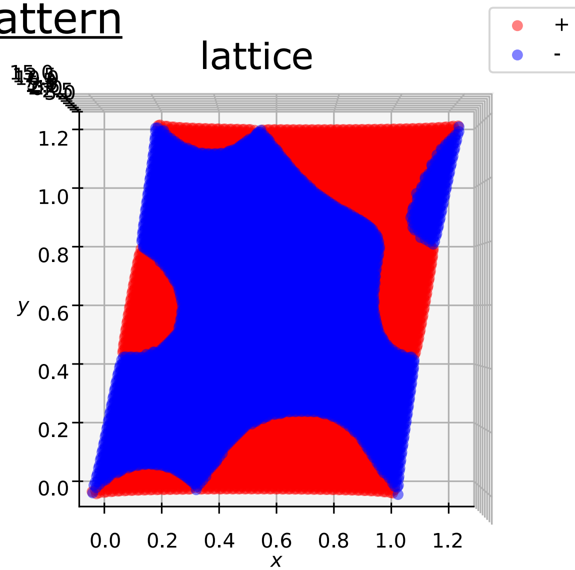
	$\lvert\ell_A^*\rvert$	$\delta\alpha_A$	$\lvert\ell_A^*\rvert$	$\delta\alpha_A$
<u>1pt</u>	0.96837989(13)	-0.11824455(48)	0.917506302(89)	-0.136754468(41)
<u>IR of 2pt</u>	0.9688(11)	-0.11897(44)	0.9015(63)	-0.1458(34)

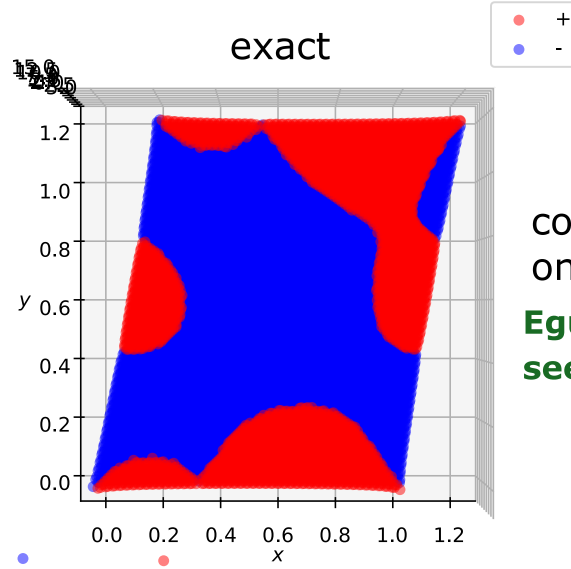
We use the values from the 1pt scheme below (errors fully systematic)

Conformal Ward identities – I. fermion variable

- $\langle T_{xx}(z,\bar{z})\varepsilon(z_1,\bar{z}_1)\varepsilon(z_2,\bar{z}_2)\rangle_{3,c}$ with fermionic variables $z_1 = 0, z_2 = 1/3,\ L = 144$

<u>+ pattern</u>



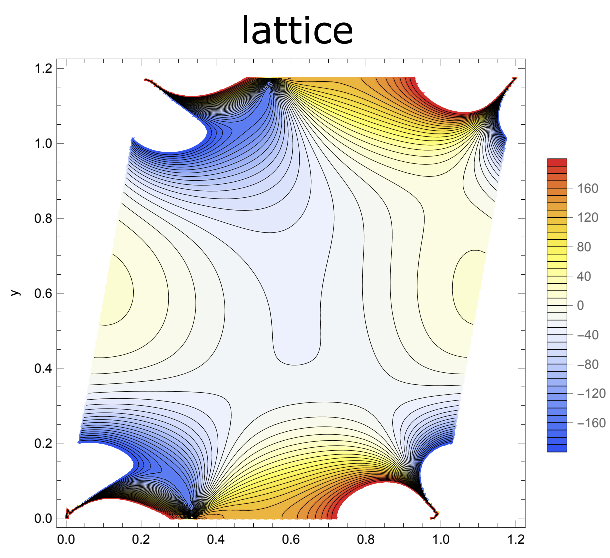


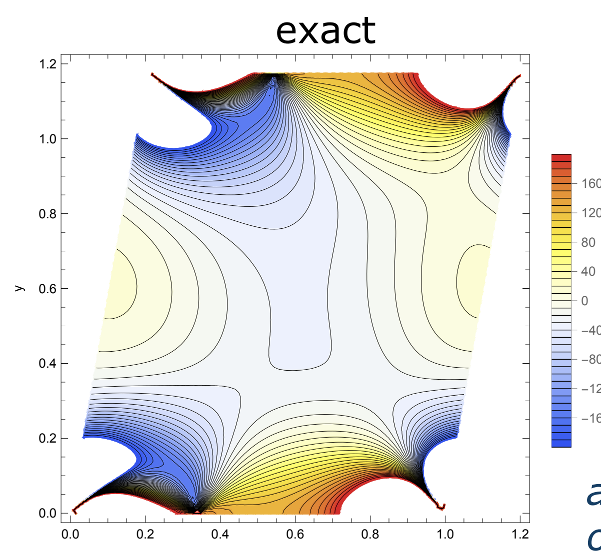
conformal Ward identity on torus:

Eguchi-Ooguri 1986

see also Felder-Silvotti 1989

<u>Full contour plot</u>

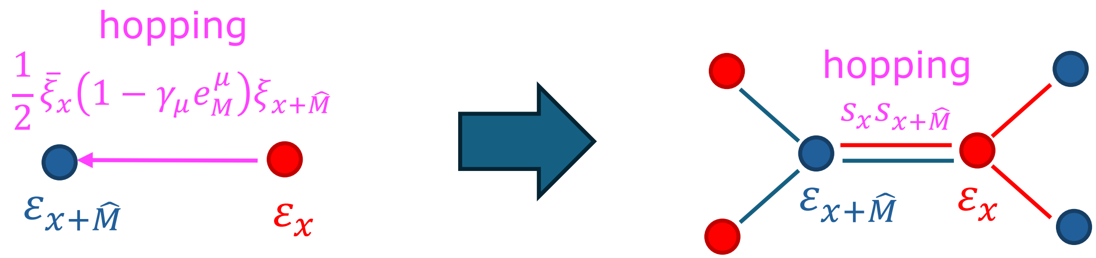




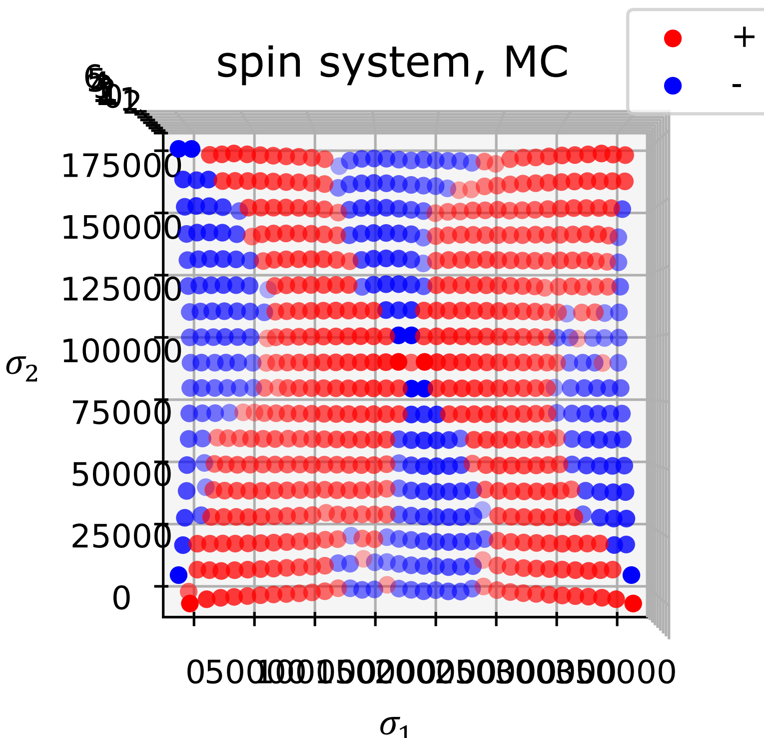
agreement also in the overall scaling

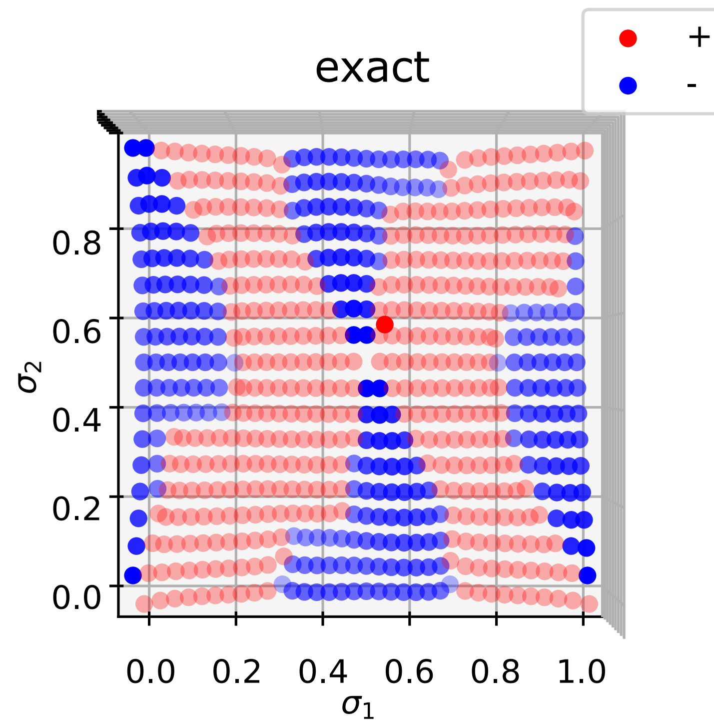
- Constructed $T_{\alpha\beta}$ can be readily mapped to spin operator via loop expansion

$T_M \sim$



$\underline{\langle T_{xx}(z,\bar{z})\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle}$ $\left(z_1 = 0, z_2 = \frac{(\tau+1)}{2}\right)$ $\tau = e^{\pi i/3}$ (regular hex lattice)





Good agreement (including the overall scaling)

Summary

- We constructed a lattice EM tensor in the Ising CFT
  - for arbitrary affine parameter, on hexagonal lattice
  - both in spin and fermion variables
  - including overall normalization and one-point function
- For nonregular lattices:
  - Extra mixing of T and $\bar{T}$ can be understood as the geometrical staggered shift
  - Not all lattice operator works consistently as the EM tensor:
    By canceling the diverging part of T_M^{lat}, the nonuniversal finite part disappears.

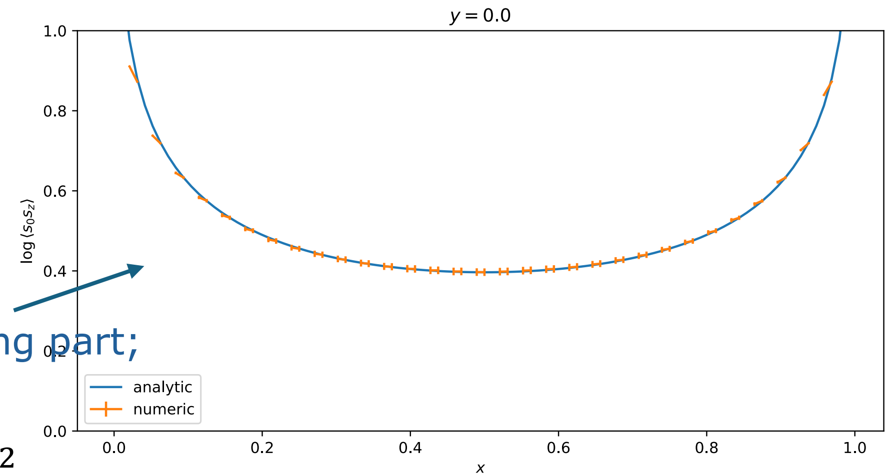
Outlook

- Map it further to the triangular lattice by taking the dual.
- Get a solid theoretical understanding of the shift parameter and diverging part.
  The two seem related; both are relevant to the normalization of σ_x operator.

$\tau = 1.2e^{4\pi i/9}$

$L = 96$

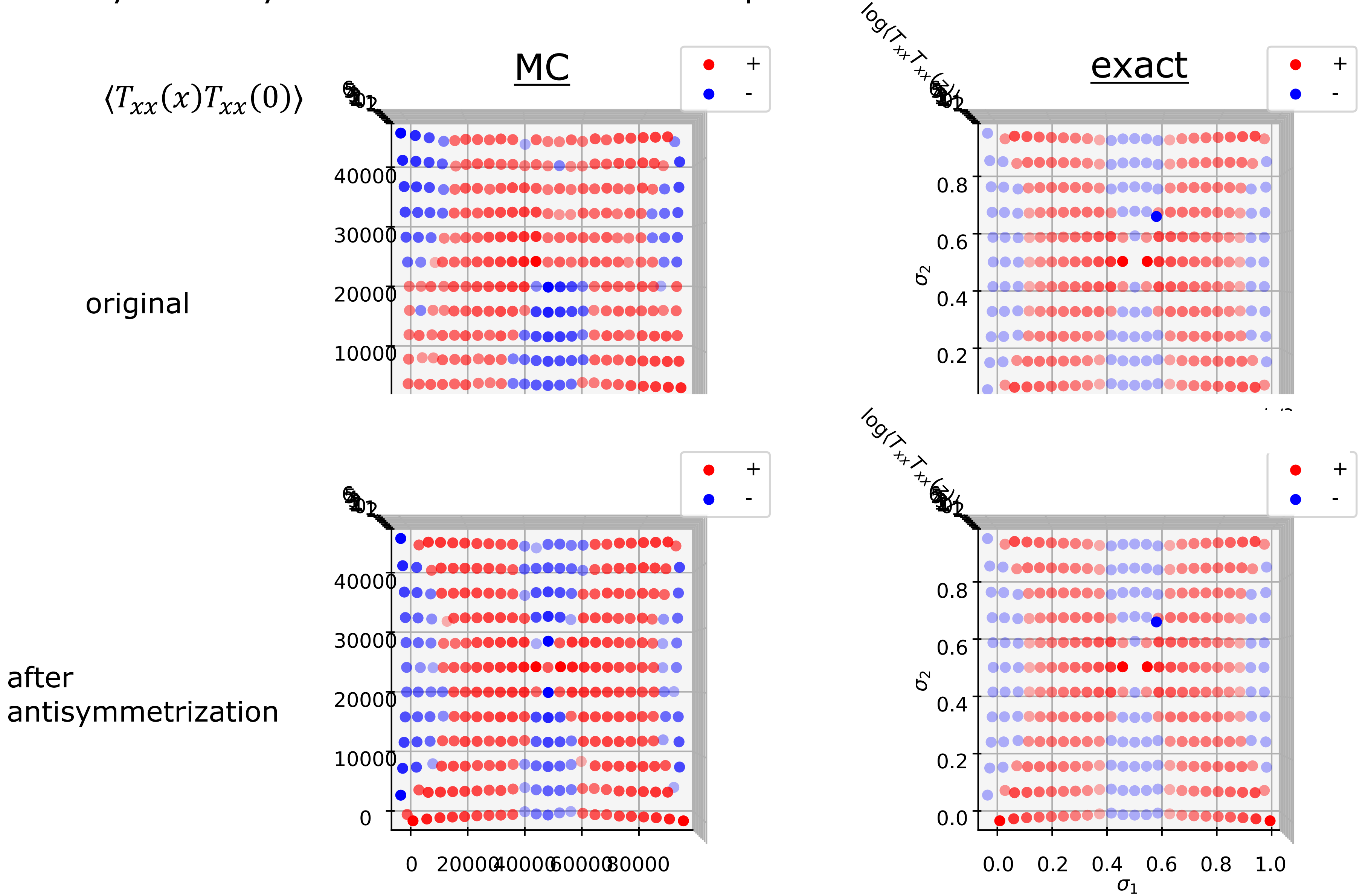
Normalization between
s_x and σ_x is determined from
geometrical quantities and diverging part;
working quite nicely



- EM tensor of the Ising CFT on S^2.

Thank you!

Necessity of antisymmetrization in Kadanoff's operator



Antisymmetrization removes $\partial\varepsilon, \bar{\partial}\varepsilon$, leaving $T, \bar{T}$

- Wilson-Majorana fermion **Wolff 2020 Brower-Owen 2023**

$$Z_\nu^{\text{lat}} \equiv \int (d\xi)\ e^{-S_\nu^{\text{lat}}}$$

$$S_\nu^{\text{lat}} \equiv \frac{1}{2}\sum_x \bar{\xi}_x \xi_x - \sum_{x\in e,M} \kappa_M\, \bar{\xi}_x P(e_M)\xi_{x+\hat{M}}$$

$\nu = 1,2,3,4$

PP, PA, AA, AP in (σ_1,σ_2)

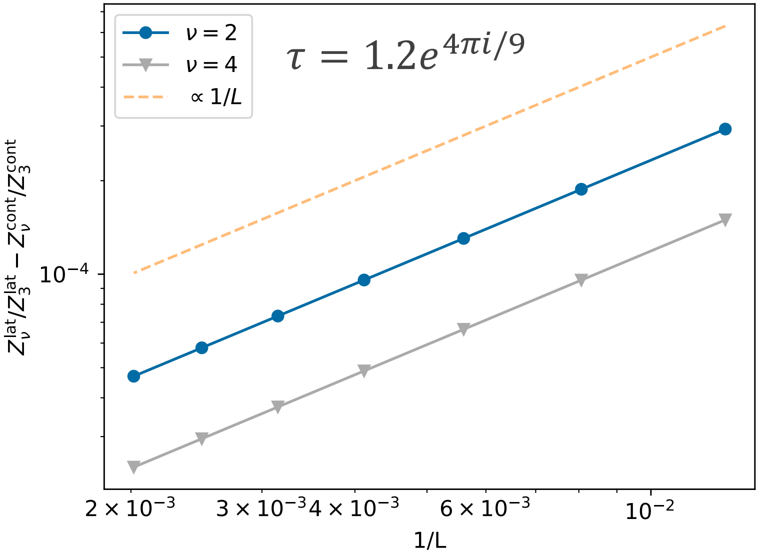
$$P(e_M) \equiv \frac{1}{2}(1-e_M^\alpha\gamma_\alpha) = \begin{bmatrix}1\\ -e^{i\alpha_M}\end{bmatrix}\begin{bmatrix}1 & -e^{-i\alpha_M}\end{bmatrix}$$

$$\gamma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \gamma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\bar{\xi} \equiv \xi^T \mathcal{C}, \quad \mathcal{C} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

- $Z_\nu^{\text{lat}}(\tau_1,\tau_2;L)$ approaches $Z_\nu^{\text{cont}}(\tau_1,\tau_2)$ as $L\to\infty$ with a diverging const:

$$Z_\nu^{\text{lat}}(\tau_1,\tau_2;L) = \mathcal{N}(\tau_1,\tau_2;L)\, Z_\nu^{\text{cont}}(\tau_1,\tau_2;L)$$



- With the classical small-a expansion:

$$S_\nu^{\text{lat}} \to S_\nu^{\text{cont}}$$

$$S_\nu^{\text{cont}} = \frac{1}{4\pi}\int d^2x\, \bar{\psi}\,\gamma_\alpha\partial_\alpha\psi$$

$$= \frac{1}{4\pi}\int d^2z\,(\eta\bar{\partial}\eta + \tilde{\eta}\partial\tilde{\eta})$$

$$\xi(x) = \sqrt{s/(2\pi)}\,\psi(x) \qquad s \equiv \frac{\sum_M |\ell_M^*|}{2}$$

$$\eta(z) = \psi_1(x), \tilde{\eta}(\bar{z}) = -i\psi_2(x)$$