Lattice study of RG fixed point based on gradient flow in 3D $O(N)$ sigma model

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Introduction

- Renormalization group (RG) is characterized by scaling relation
- Beyond perturbation theory, nonperturbative construction of quantum field theory
  - Nontrivial fixed point, anomalous dimensions

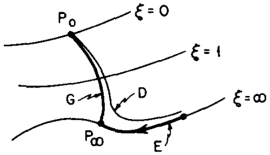
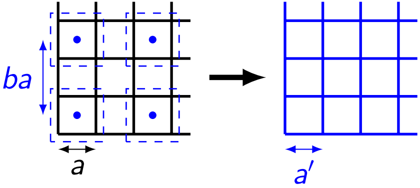


Fig. 12.6. Renormalization group trajectories near two fixed points. The curve D begins close to the critical surface. As the origin of D approaches the critical surface, the trajectory D approaches the trajectories E and G. For this example G is $S(\infty)$.

[Wilson–Kogut '74]

- Condensed matter lattice model: Spin blocking



- Nonperturbative field theory: Requires a momentum cutoff... Crucial problem because it is incompatible with gauge symmetry

Gradient flow and block spin transformation?

- Gradient flow [Narayanan–Neuberger, Lüscher]:
  an evolution of physical quantities along a *fictitious* time
- It looks like gradient flow acts as "coarse-graining" [Lüscher ...]
- E.g., Yang–Mills gradient flow equation

$$\partial_t B_\mu(t,x) = -\left.\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu}\right|_{A_\mu \to B_\mu} = D_\nu G_{\nu\mu}(t,x), \quad B_\mu(0,x) = A_\mu(x)$$

  where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$
- Diffusion eq. $\partial_t \phi(t,x) = \partial_\mu^2 \phi(t,x)$ w/ diffusion length $x \sim \sqrt{8t}$
- Blocked field is given as

$$\varphi_b(x_b) \sim \sum_\epsilon \varphi(x+\epsilon)$$

  - $b \sim \sqrt{8t}$?
- Solution of $\partial_t \phi(t,x) = \partial_\mu^2 \phi(t,x)$

$$\phi(t,x) \sim \int_y e^{-(x-y)^2/4t} \varphi(y)$$

- *Can we define gauge-invariant Wilson's RG by gradient flow?*

Wilsonian RG flow [Makino–OM–Suzuki '18]

- Scaling relation under Wilsonian RG (running coupling $g_i(\xi)$)

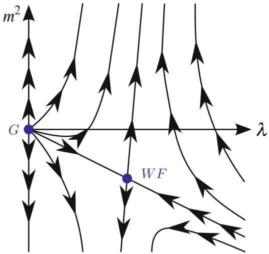
$$\left\langle \mathcal{O}_1(e^{2\xi}t_1, e^{\xi}x_1)\dots\mathcal{O}_N(e^{2\xi}t_N, e^{\xi}x_N)\right\rangle_{\{g_i\}}$$
$$= Z(\xi)\,\langle \mathcal{O}_1(t_1,x_1)\dots\mathcal{O}_N(t_N,x_N)\rangle_{\{g_i(\xi)\}}$$

- Define new scheme as $g^2_{\mathrm{GF}}(\mu) = \langle \mathcal{O}(t)\rangle_{\mu=1/\sqrt{8t}} \overset{\text{pert.}}{\sim} g^2 + O(g^4)$
  - cf. Numerical determination of α_S in QCD [ALPHA '17]
- Assumption as one-to-one map $g_i(\xi) \mapsto \langle \mathcal{O}_i(t)\rangle = \mathcal{R}_i[\{g_j(\xi)\}]$

Gradient flow $\quad \left\langle \mathcal{O}_j(e^{2\xi}t)\right\rangle_{\{g_i\}} = \langle \mathcal{O}_j(t)\rangle_{\{g_i(\xi)\}} \quad$ Running

[Makino–OM–Suzuki '18]

- Gradient flow is gauge covariant!
  A gauge-invariant construction of RG flow
  - Banks–Zaks fixed point in (2-loop approx) N_f-flavor QCD
  - Wilson–Fisher fixed point in Large N 3D $O(N)$ linear sigma model



3D $O(N)$ linear sigma model and large N

- Euclidean action ($i = 1, \ldots, N$)

$$S_{\mathrm{E}} = \int d^3x \left\{ \frac{1}{2} [\partial_\mu \phi_i(x)]^2 + \frac{1}{2} m_0^2 \phi_i^2(x) + \frac{1}{8N} \lambda_0 \left[\phi_i(x)^2\right]^2 \right\}$$

$\to$ Wilson–Fisher IR fixed point [72]

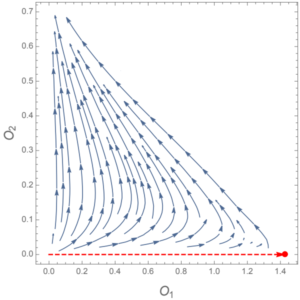
- Solution is well known at $N \to \infty$
- Flow eq. $\partial_t \varphi_i(t,x) = \partial_\mu^2 \varphi_i(t,x)$
- Dimensionless flowed operators:

$$\mathcal{O}_1(t,x) \equiv -\frac{N\left[\varphi_i(t,x)^2\right]^2}{\langle \varphi_j(t,x)^2 \rangle^2} + (N+2) \overset{t\to 0}{\sim} \lambda_0 t^{1/2}$$

$$\mathcal{O}_2(t,x) \equiv \frac{4t\partial_\mu \varphi_i(t,x)^2}{(2\pi)^{1/2} \langle \varphi_j(t,x)^2 \rangle} - \frac{1}{(2\pi)^{1/2}} \overset{t\to 0}{\sim} M t^{1/2}$$

M: physical mass determined by gap equation

- IR fixed point at $M/\lambda_0 \to 0$
  - cf. [Aoki–Balog–Onogi–Weisz '16]

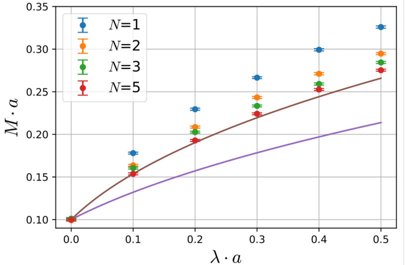


[Makino–OM–Suzuki '18]

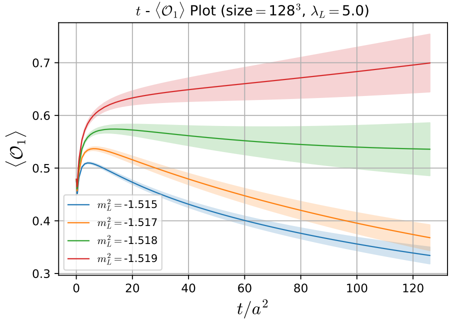
Lattice setup with finite N

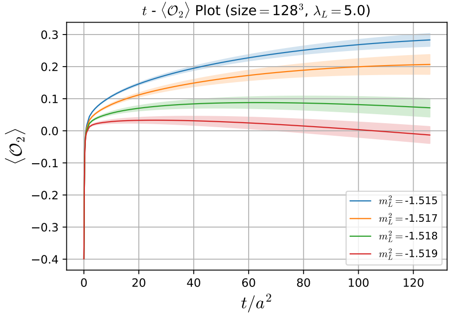
- Parameter set
  - Lattice 64^3, 128^3
  - $N = 1$ (partially 2, 3, 5)
  - Tuning $\lambda_0 a$, $m_0 a$ for small M
- Numerical simulation
  - Overrelaxed heatbath
  - 4th order Runge–Kutta method for gradient flow

Effective mass M

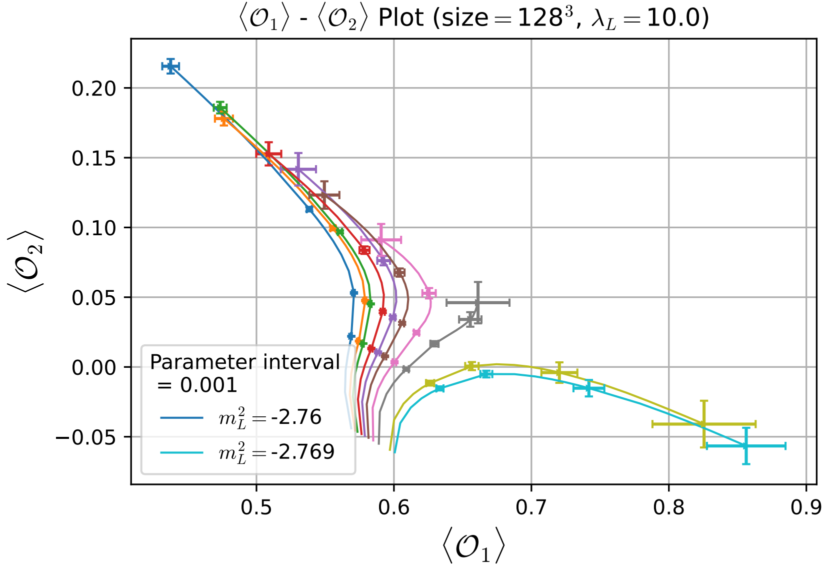


Two curves given by gap eq. at $N \to \infty$ with $\Lambda = \pi/2a$ or π/a





Lattice simulation of RG flow near criticality

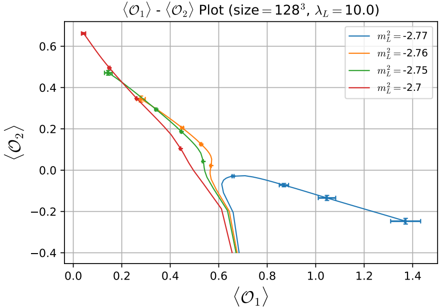


- Discussion: Finite size effect, Gaussian fixed point (following slides)

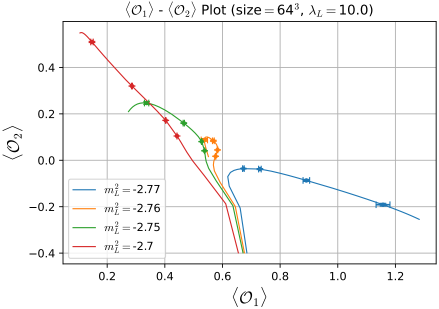
Finite lattice size effect

- Some trajectories in RG flow are intersecting near IR.
- This is because of a finite size effect as follows:

Larger lattice size $L = 128$

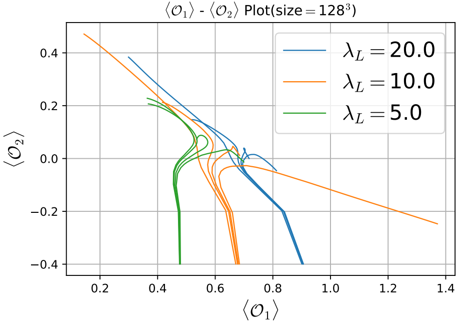


Smaller lattice size $L = 64$



From different “lattice models” to WF fixed point

- Starts from different points at UV (not Gaussian)
  - Not continuum limit (i.e., Gaussian) but just a lattice model.
  - Critical behaviors for each models should be same near WF.



- Small λ: close to Gaussian and far from WF
  - Much bigger box size and reducing errors at large t
- Large λ: far from Gaussian but hopefully close to WF
  - More computational costs when generating configurations

Summary

- Numerical simulation of 3D $O(N)$ linear sigma model
- Depicted RG flow near criticality by using operators evolved by gradient flow
  - Observation of Wilson–Fisher fixed point at IR ($t \to \infty$)
- More understanding on Gradient flow and RG in gauge theory!
  - Manifestly gauge-invariant construction of QFT
  - See Gradient-flow Exact (Functional) RG [Carosso–Hasenfratz–Neil '18, Carosso '19, Sonoda–Suzuki '20, etc.]
  - Application to lattice simulation of gauge theory?
- Gravity? (asymptotic safety and Ricci flow)
- Wilson's philosophy: every quantum system *sub specie aeternitatis* is clarified as a continuum theory near criticality (with infinite distances).