Kernels and integration cycles in complex Langevin simulations

Michael Mandl

with Michael W. Hansen, Dénes Sexty and Erhard Seiler

LATTICE 2024 - 29/07/2024

FWF Austrian Science Fund

UNI GRAZ

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Wednesday @ 11:55 Tuesday @ 15:05

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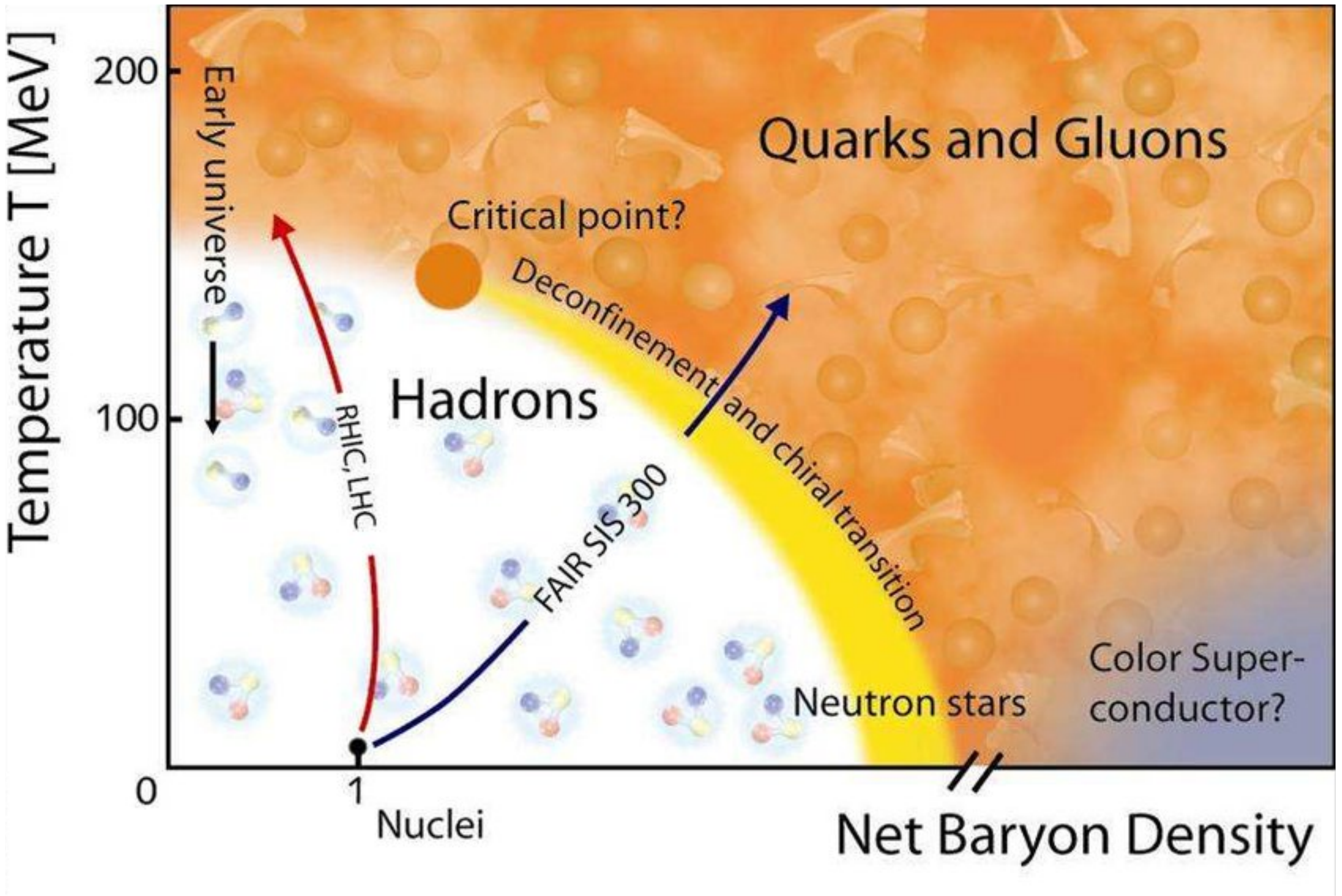
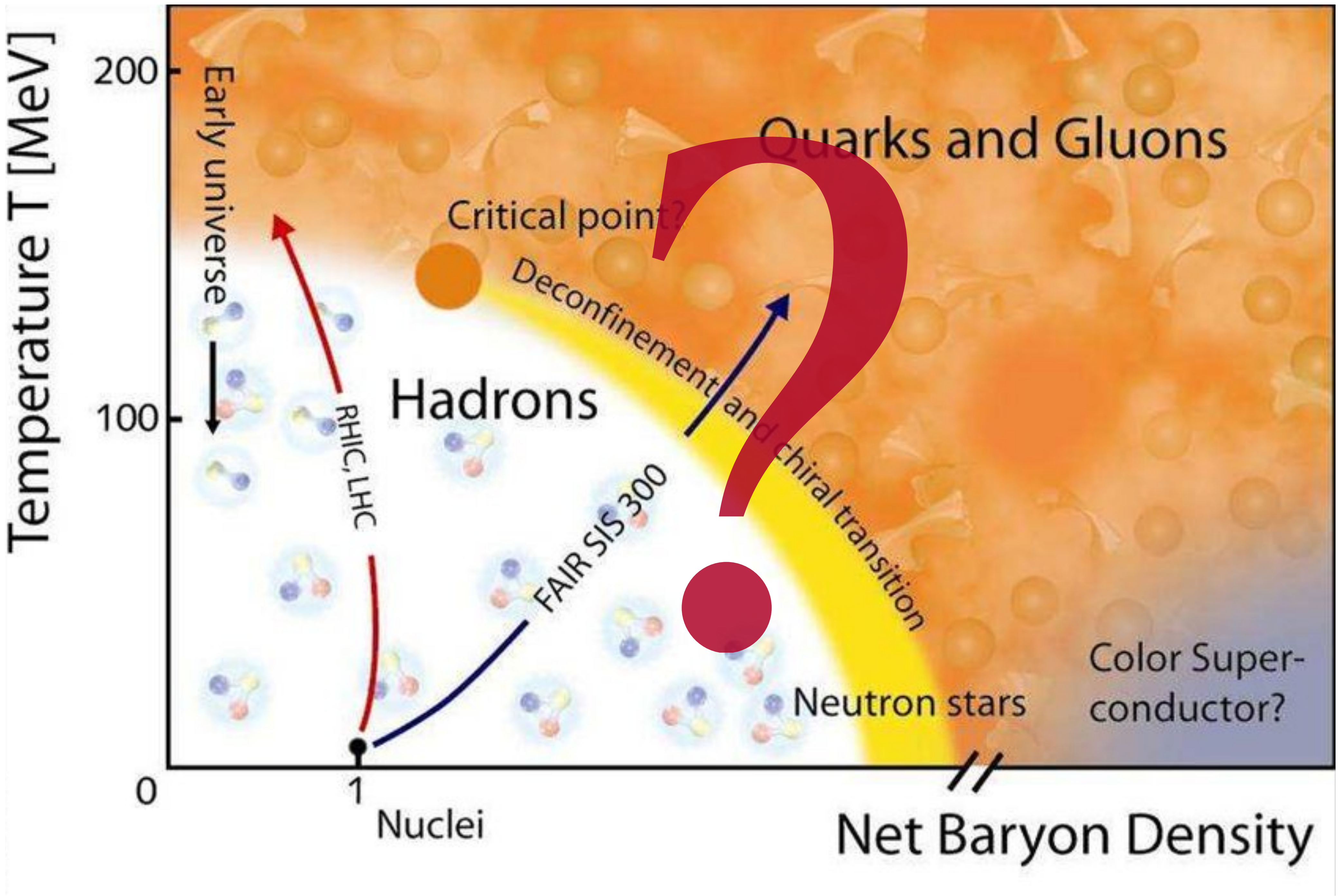


Temperature T [MeV]
200
100
0
Early universe
Quarks and Gluons
Critical point?
Deconfinement and chiral transition
Hadrons
RHIC, LHC
FAIR SIS 300
Neutron stars
Color Super-
conductor?
1
Nuclei
Net Baryon Density

Motivation

- QCD at finite density poorly understood.



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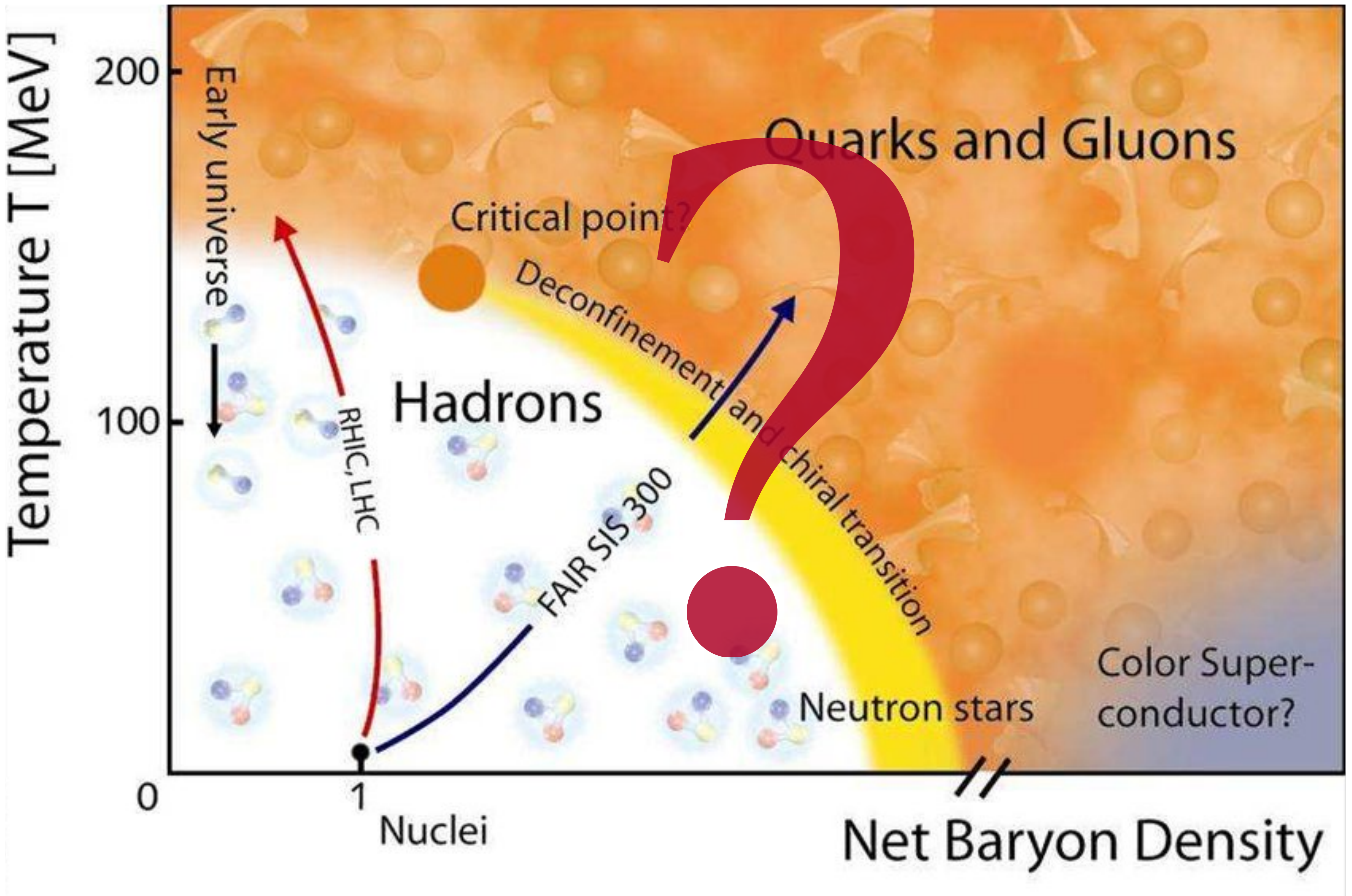

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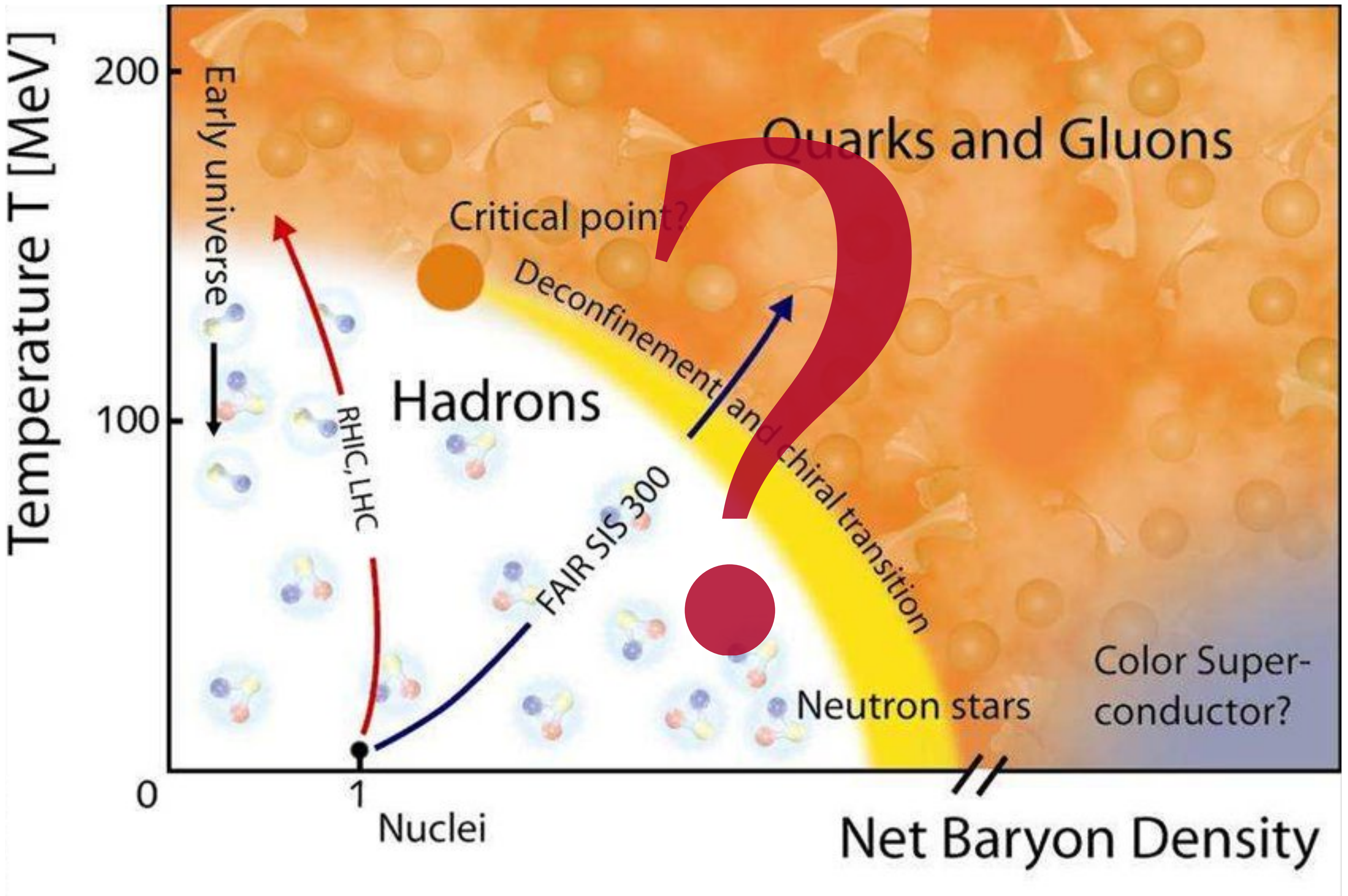

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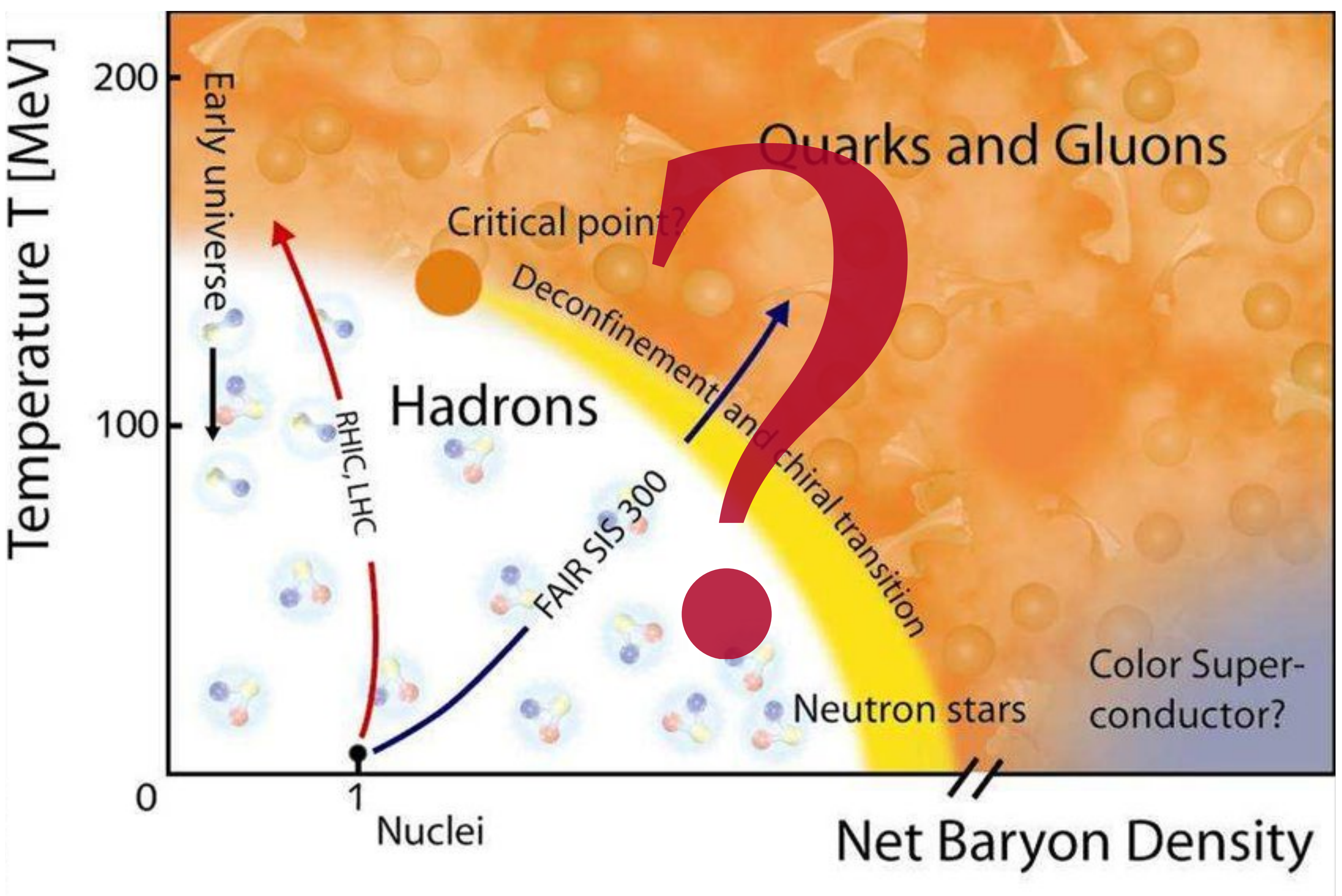

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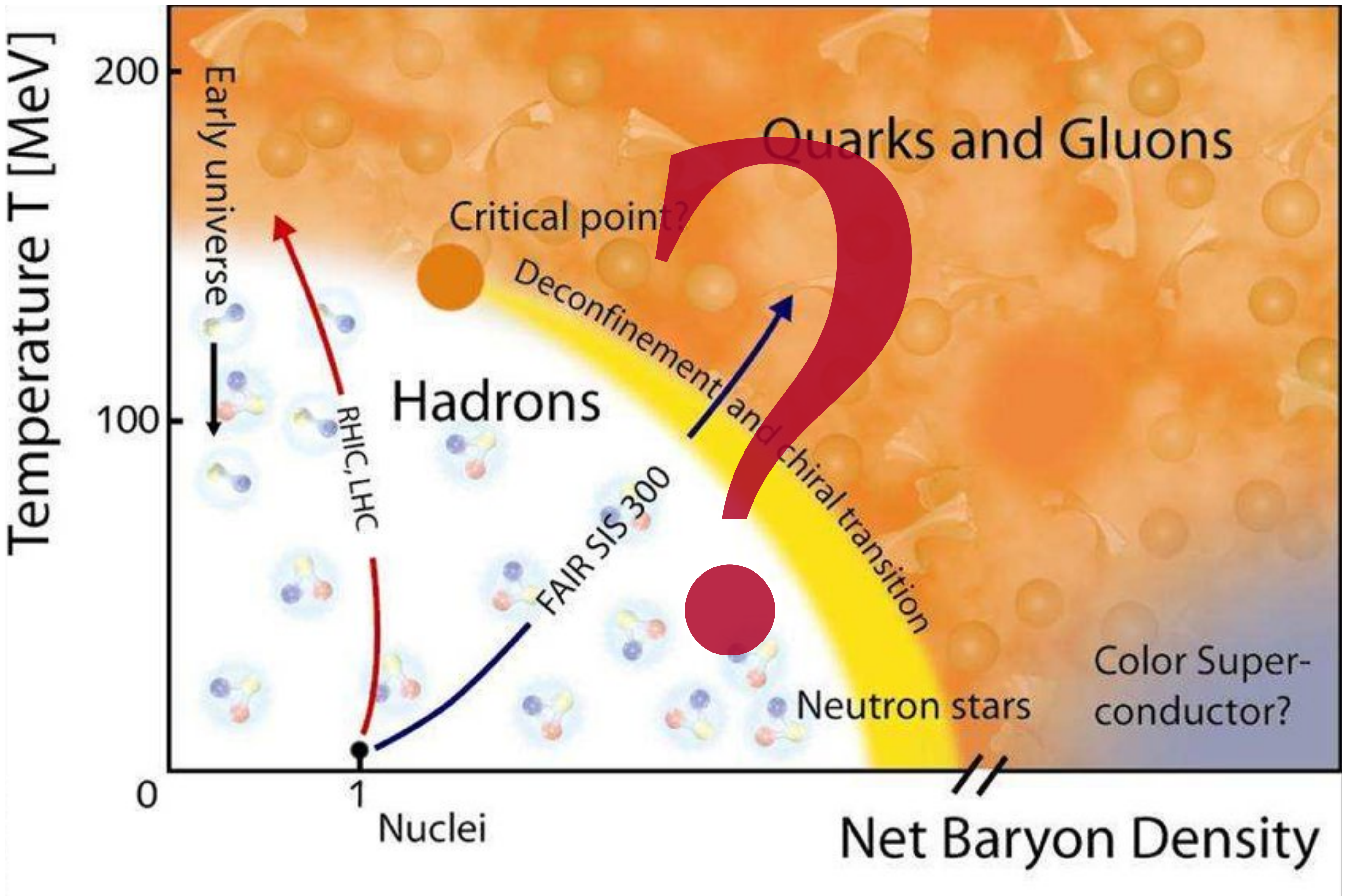

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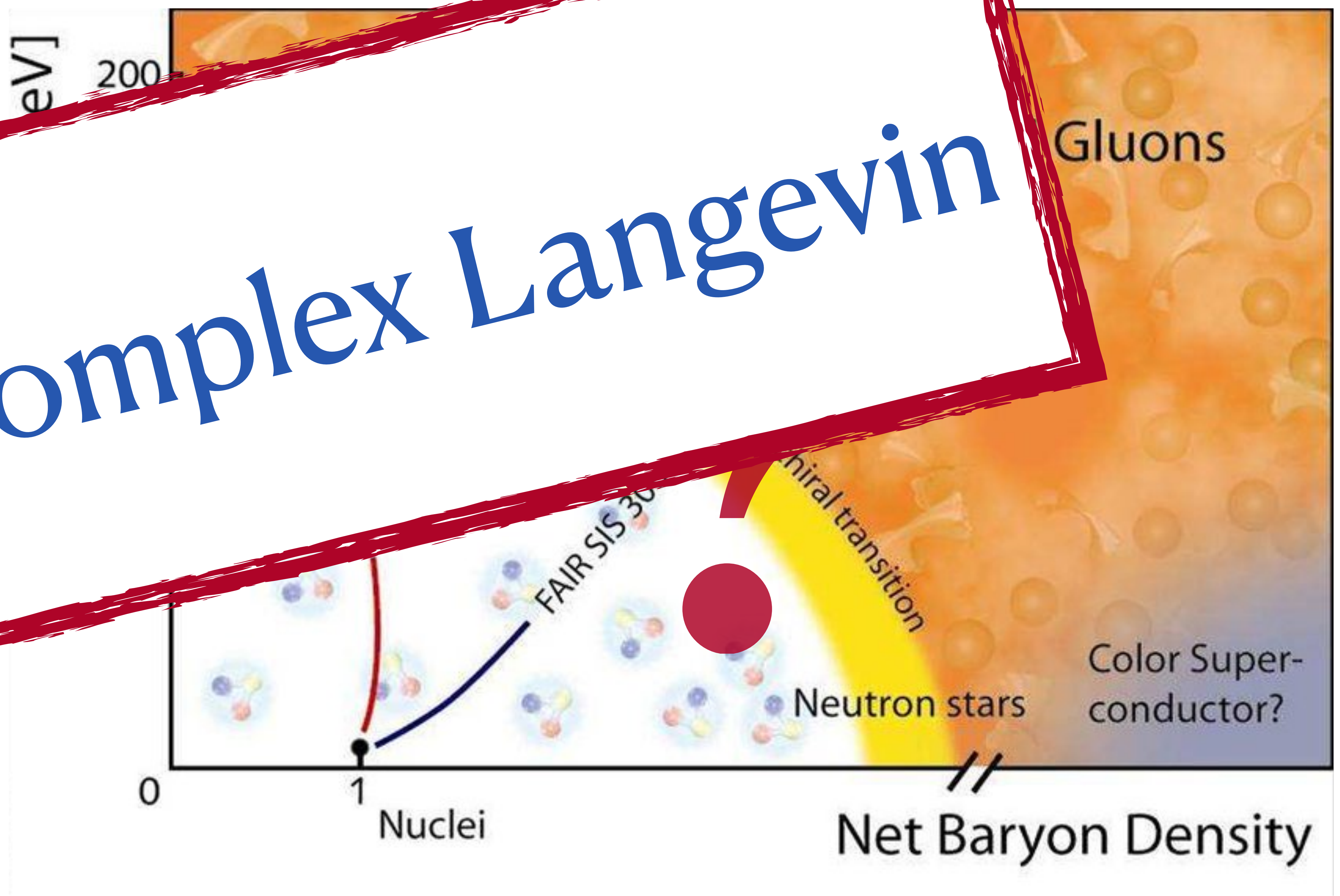

Possible solution: Complex Langevin

The complex Langevin equation

The complex Langevin equation

- Complexify $x \to z = x + \mathrm{i}y$, evolve statistical system in fictitious time direction τ.

The complex Langevin equation

Klauder '83; Parisi '83

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- Obtain target theory $e^{-S(z)}$ in equilibrium limit $\tau \to \infty$.

Drawbacks and pitfalls

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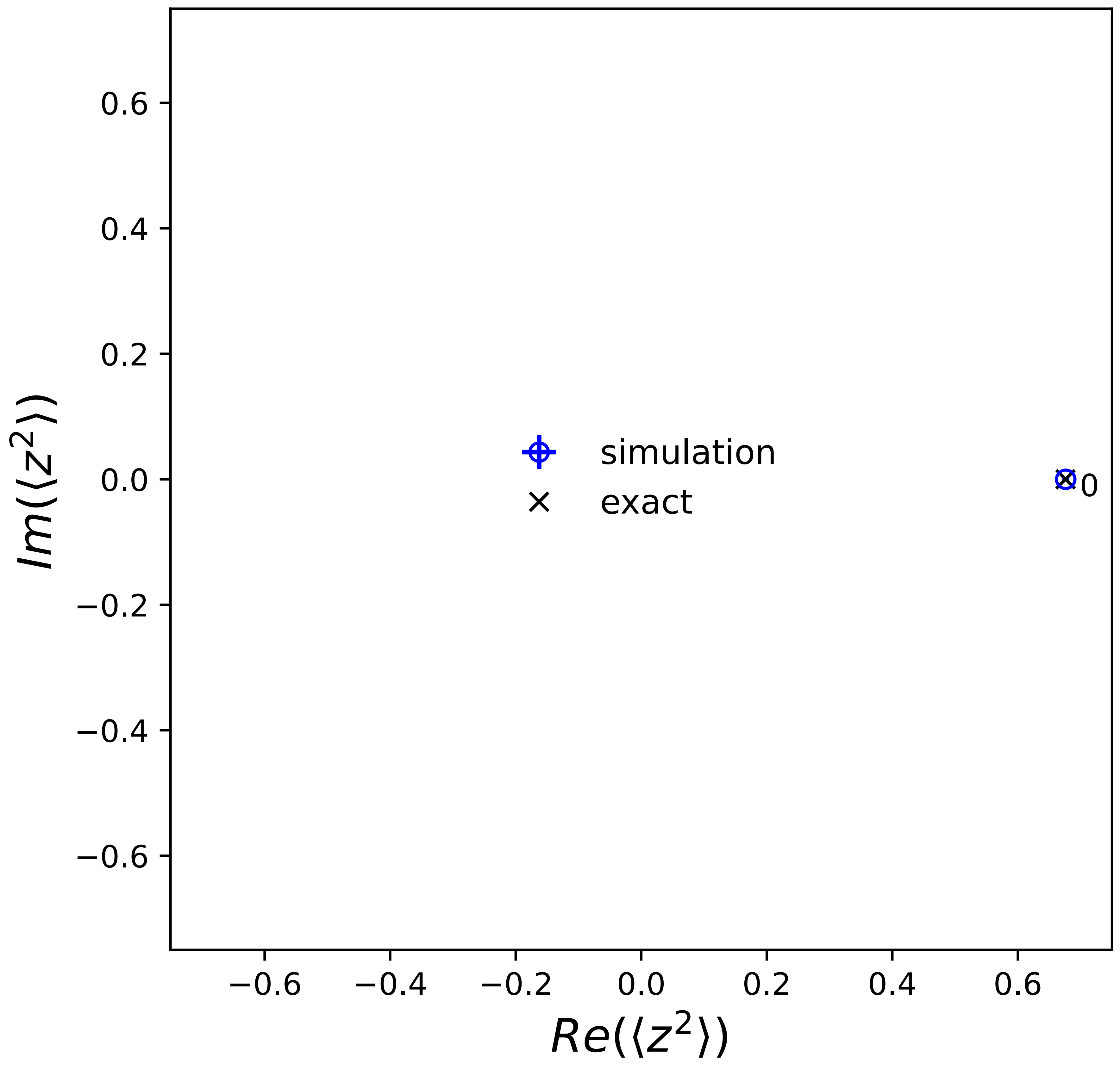

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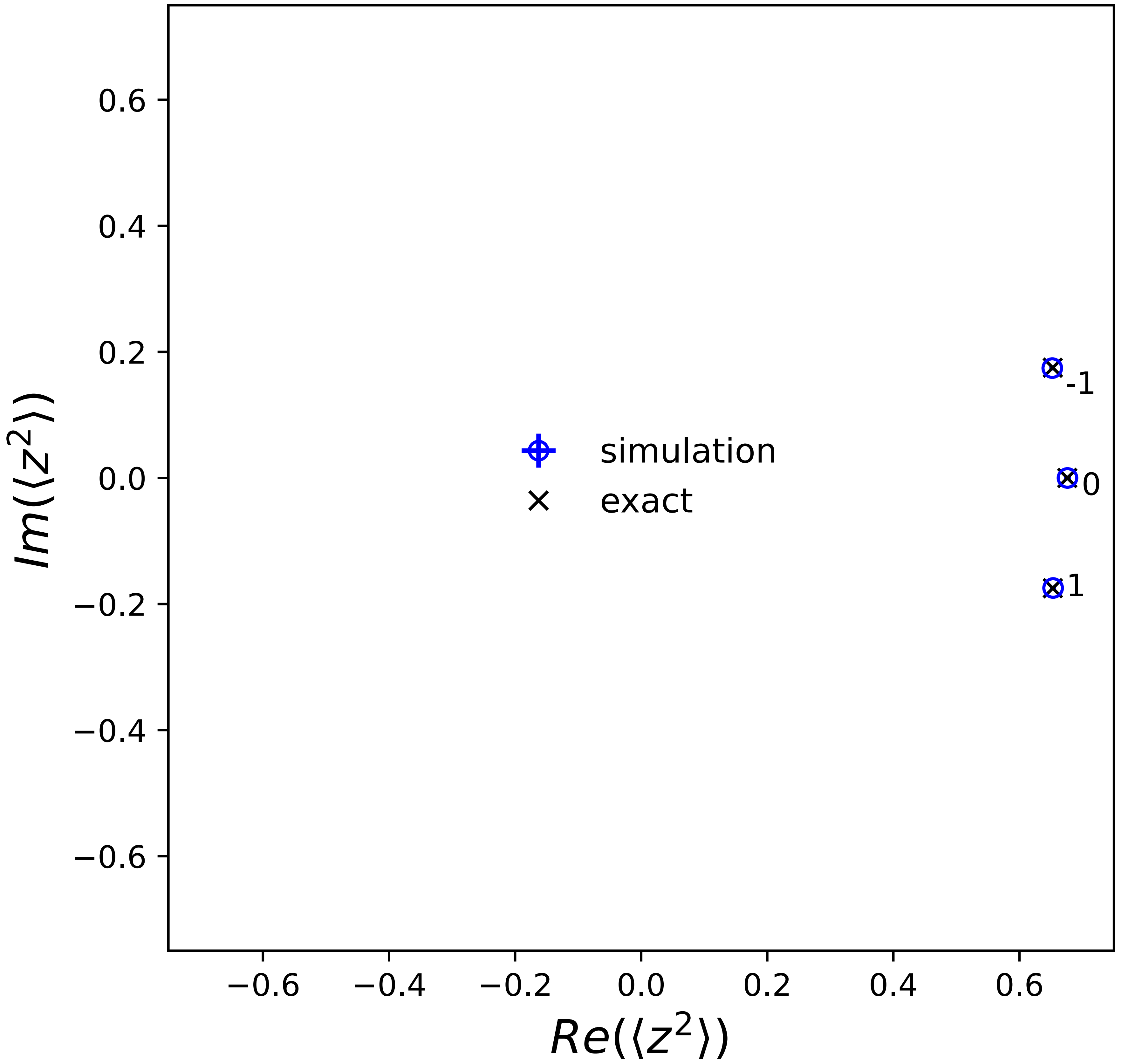

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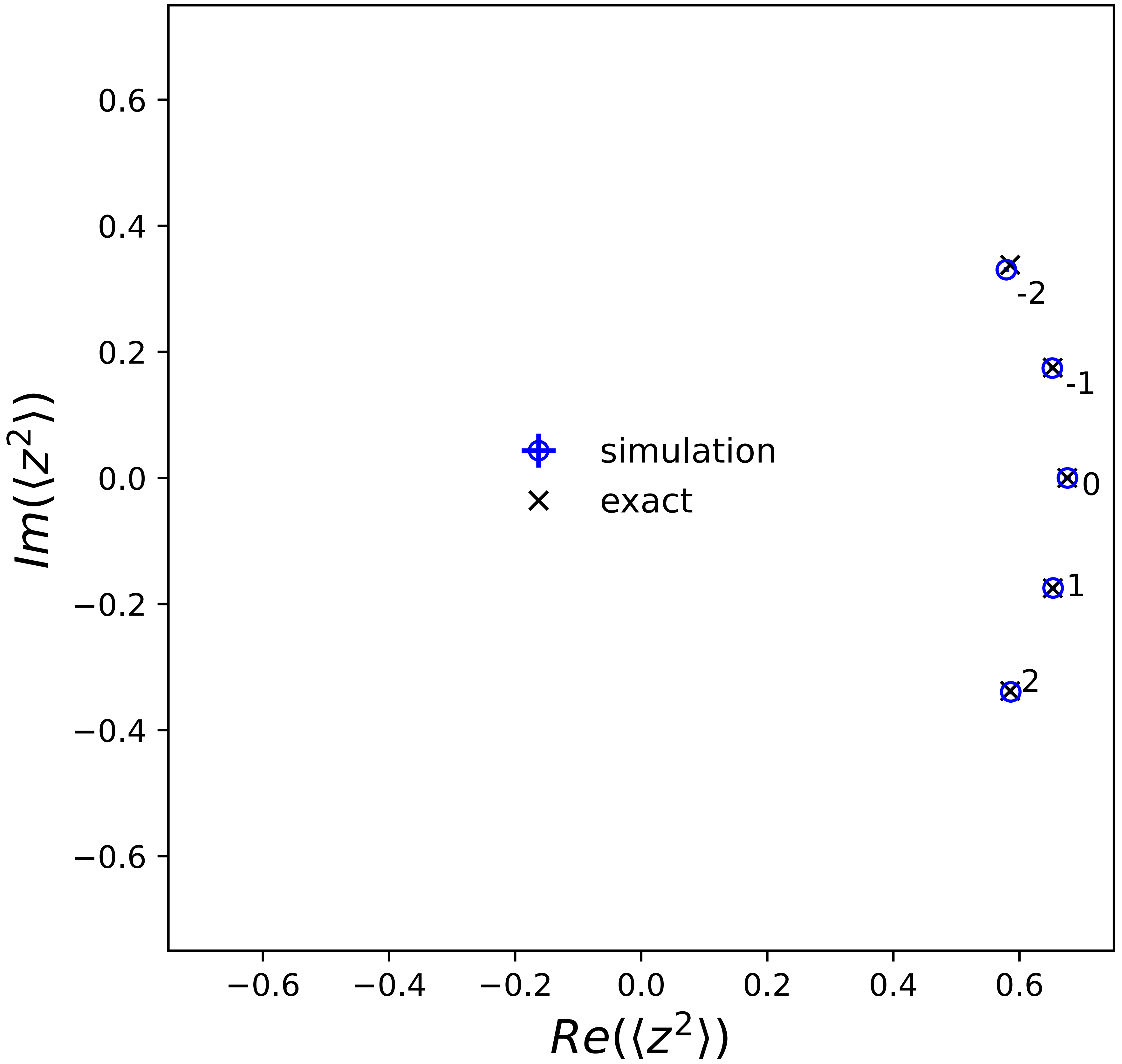

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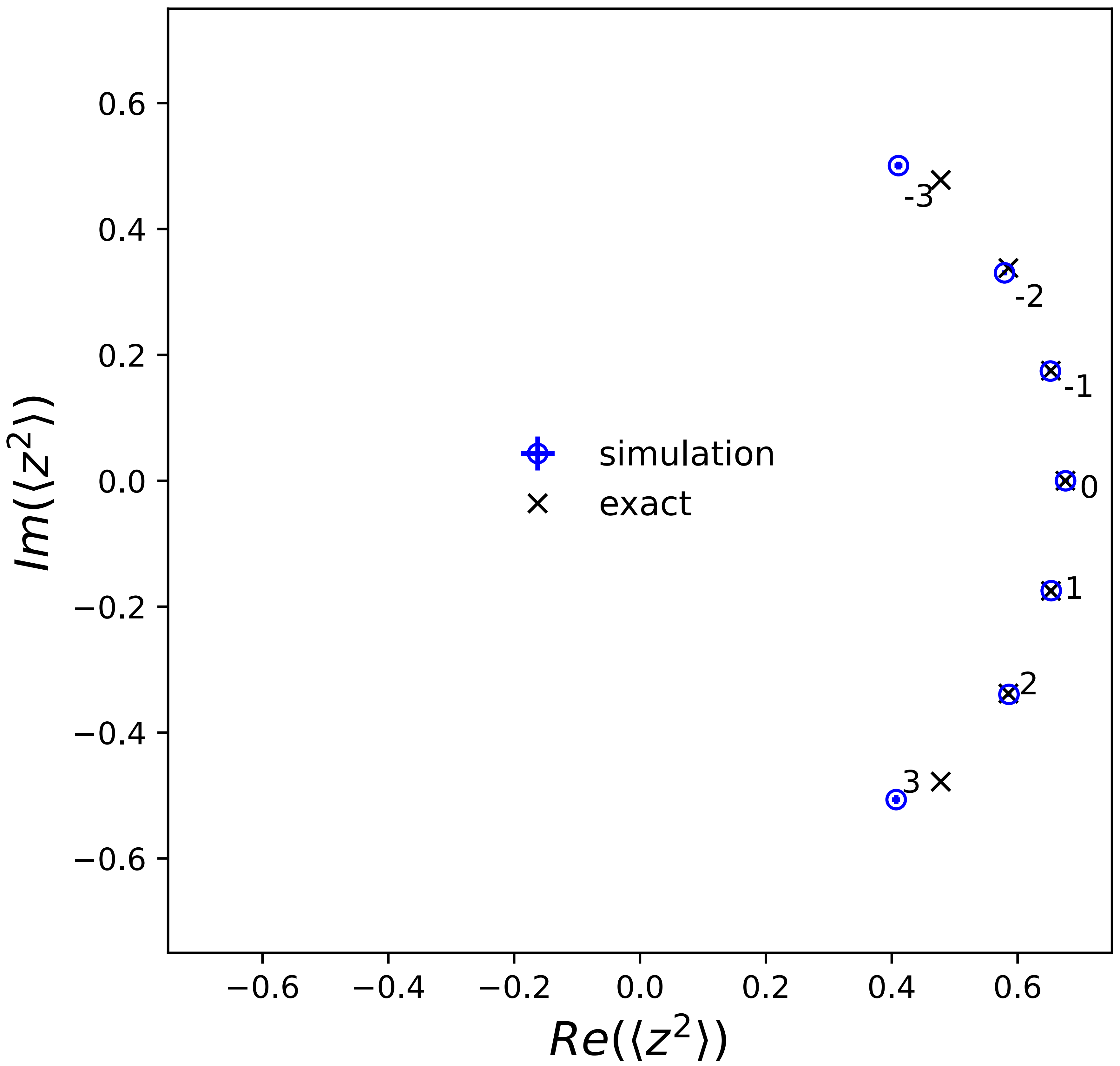

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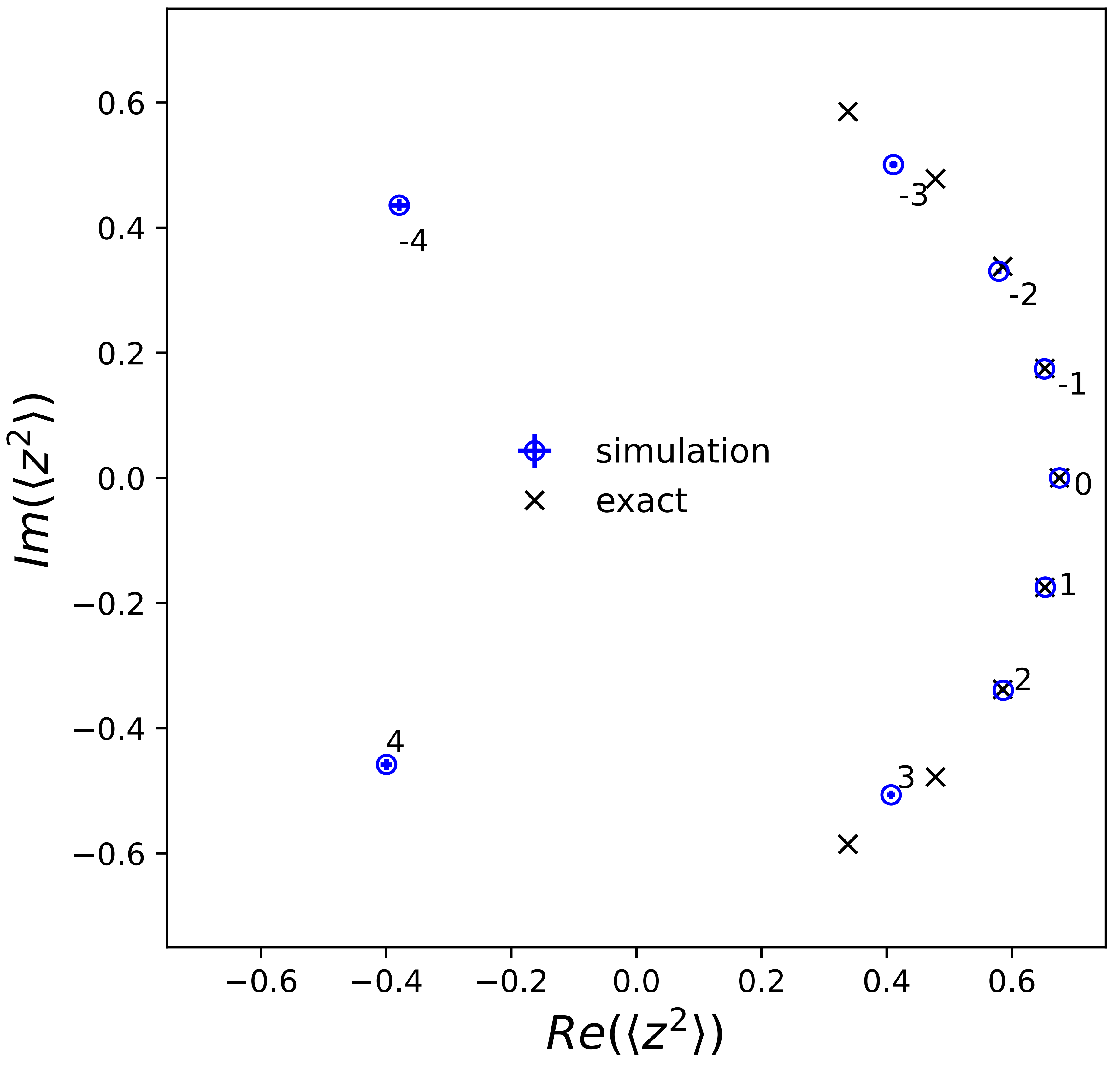

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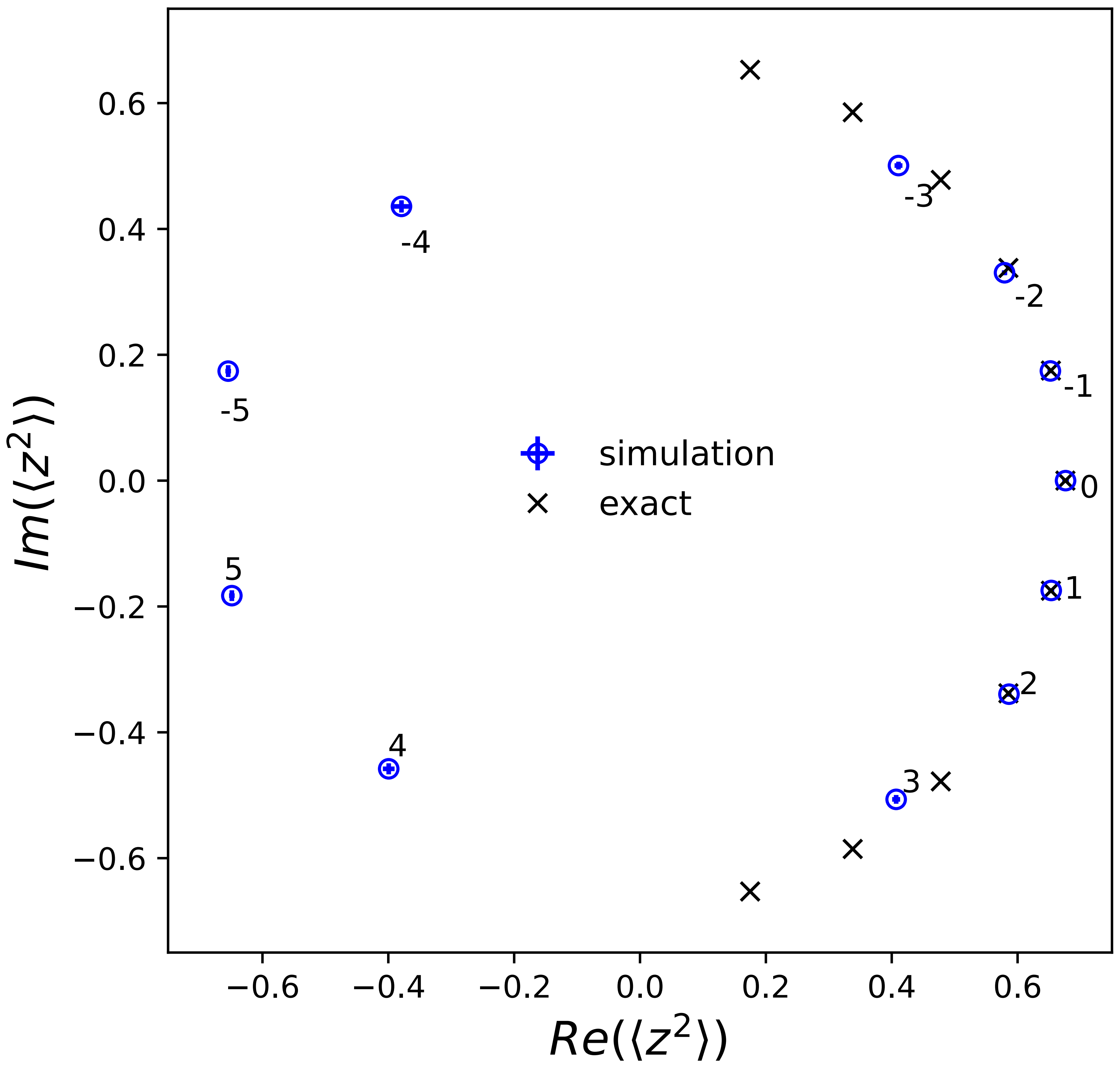

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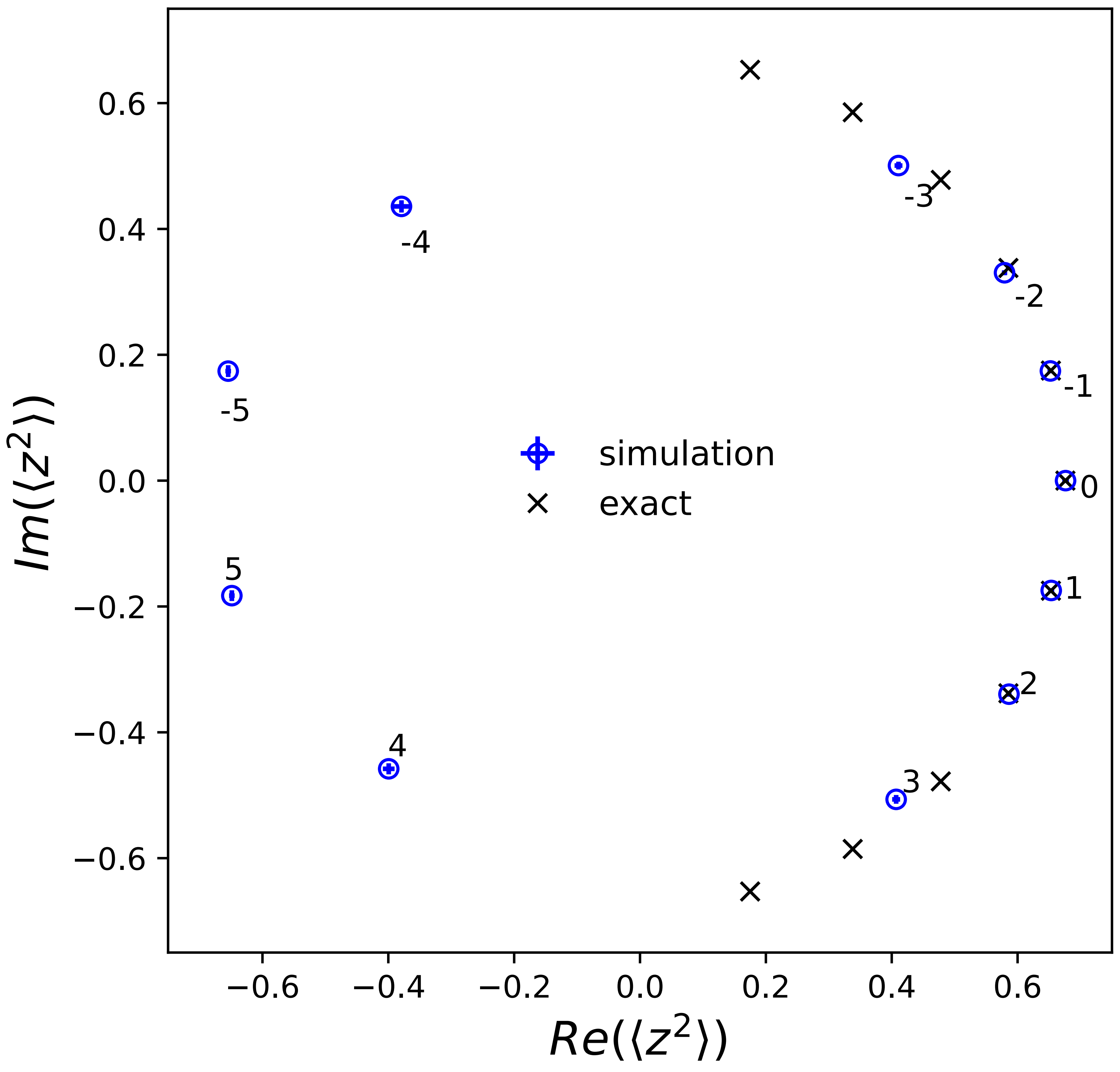

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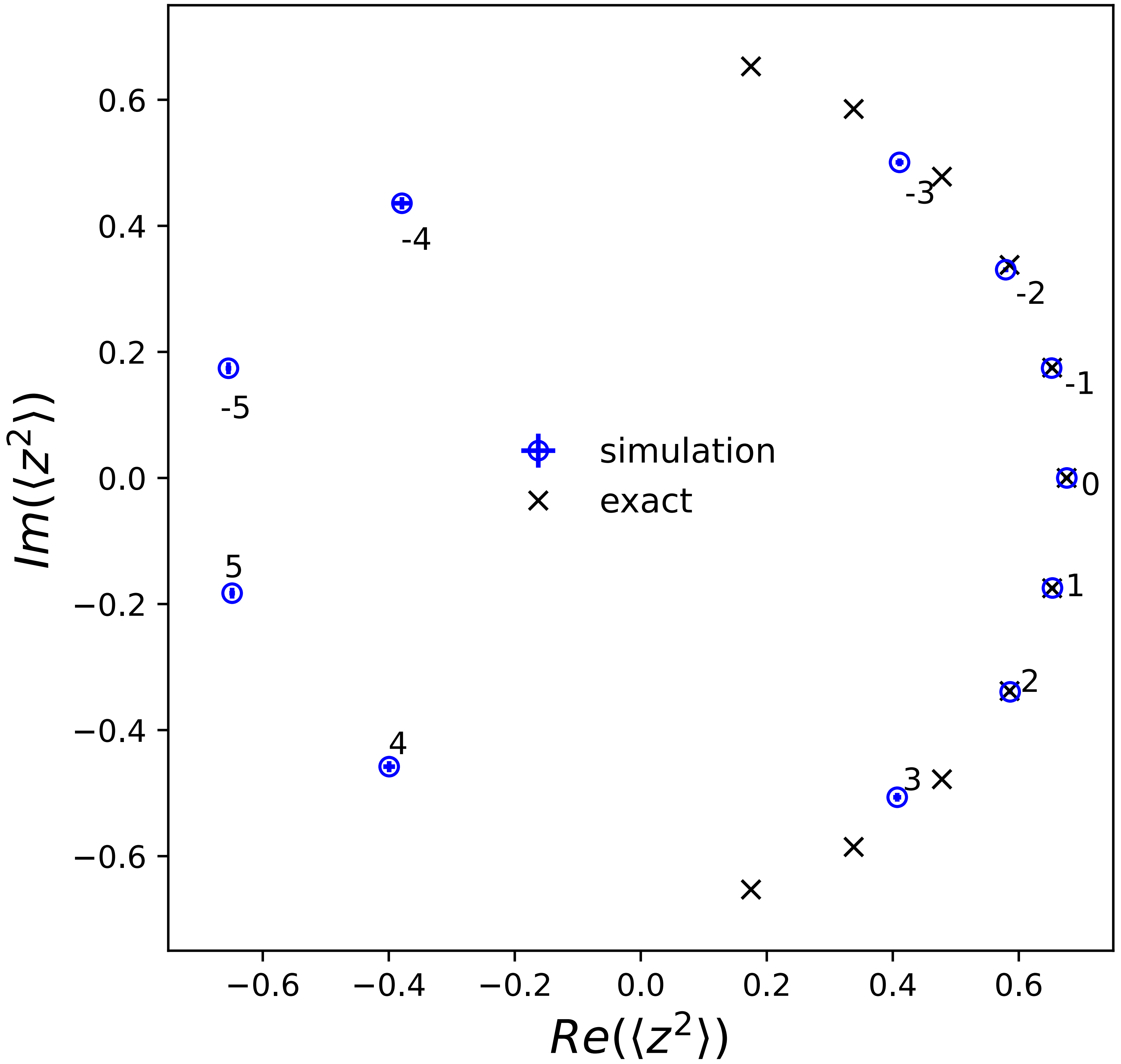

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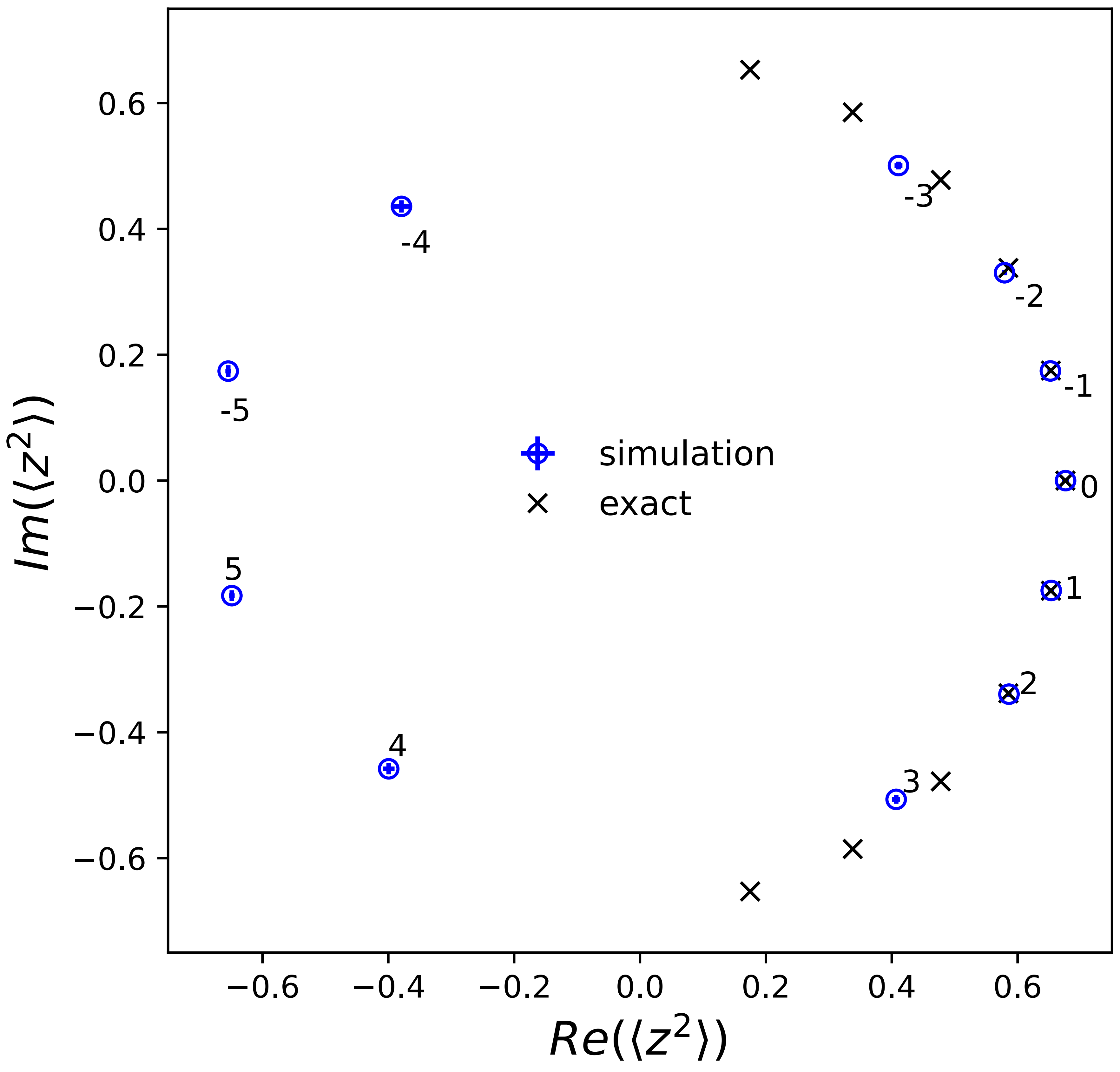

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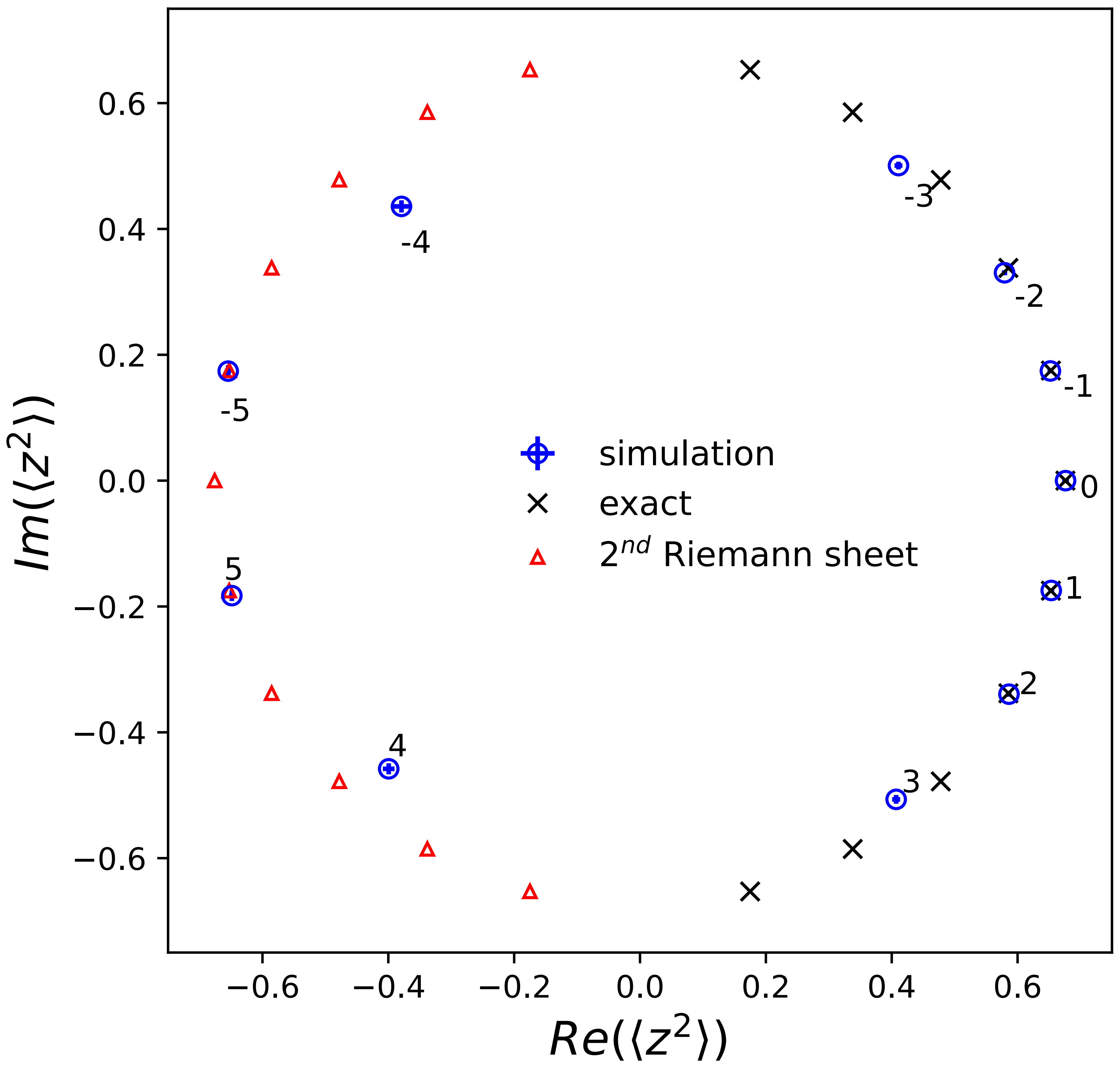

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$$B_{\mathcal{O}(z)}(Y) = \left\langle \Theta\left(Y - |z|\right) L\mathcal{O}(z) \right\rangle$$

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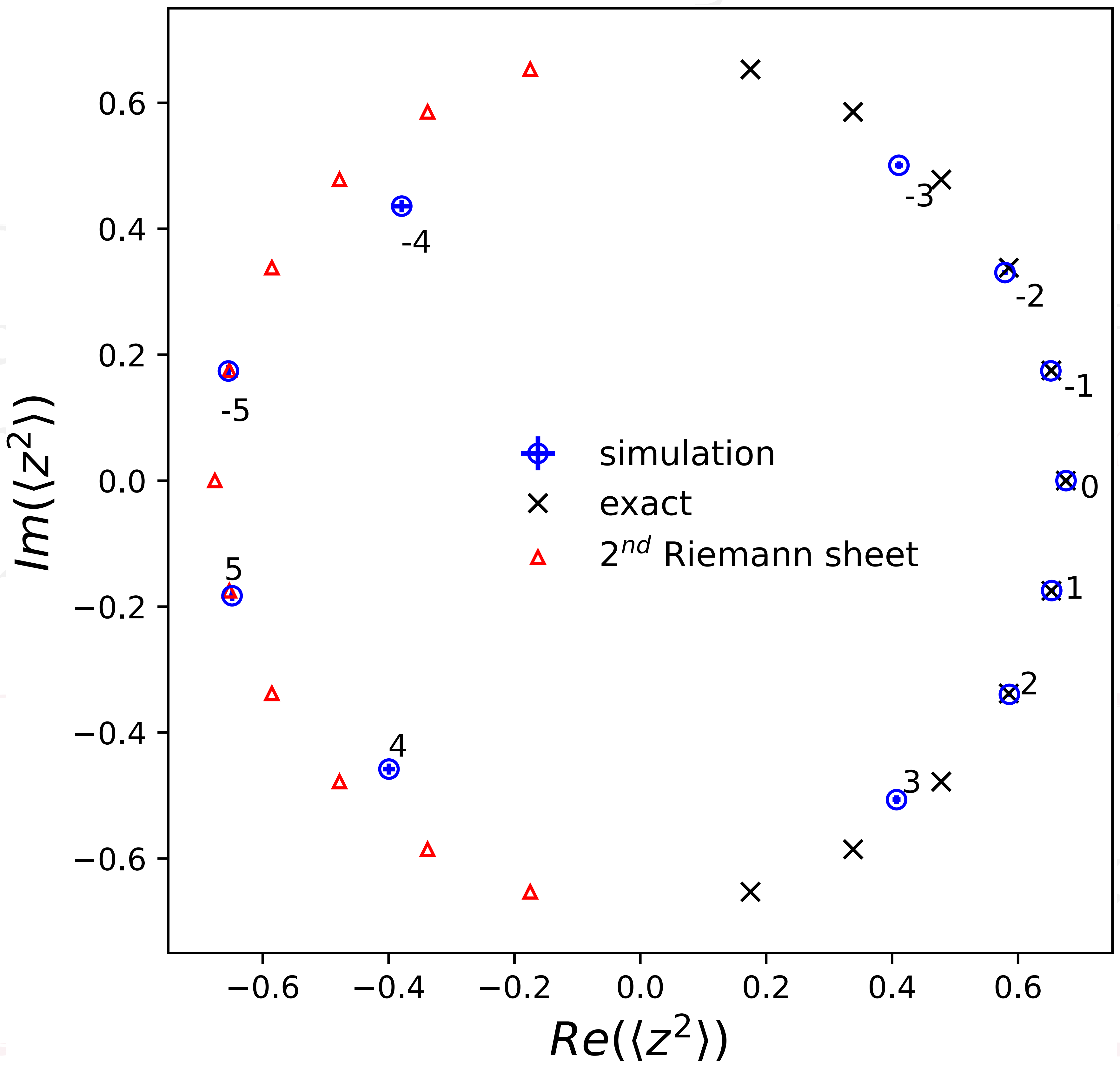

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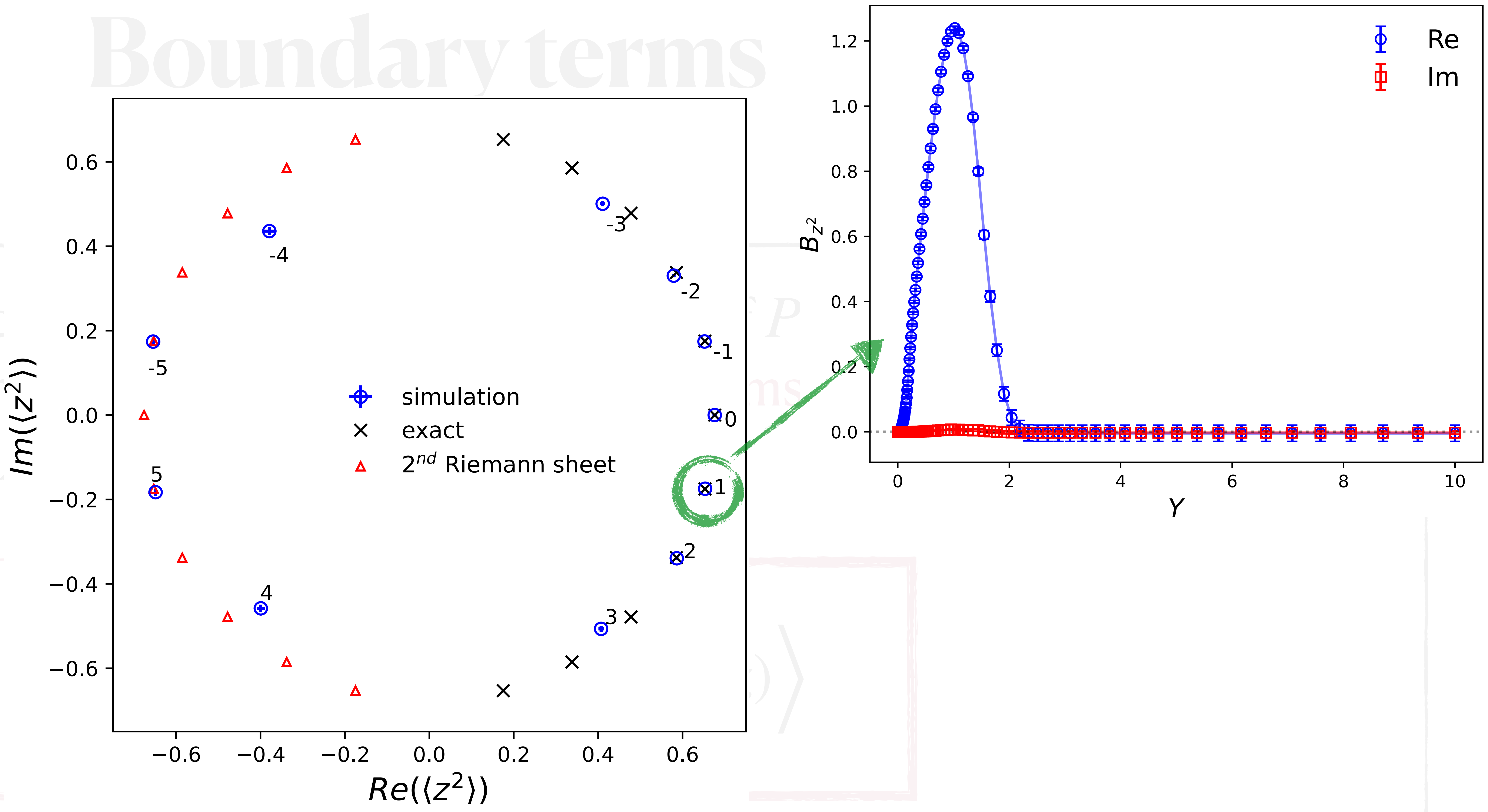

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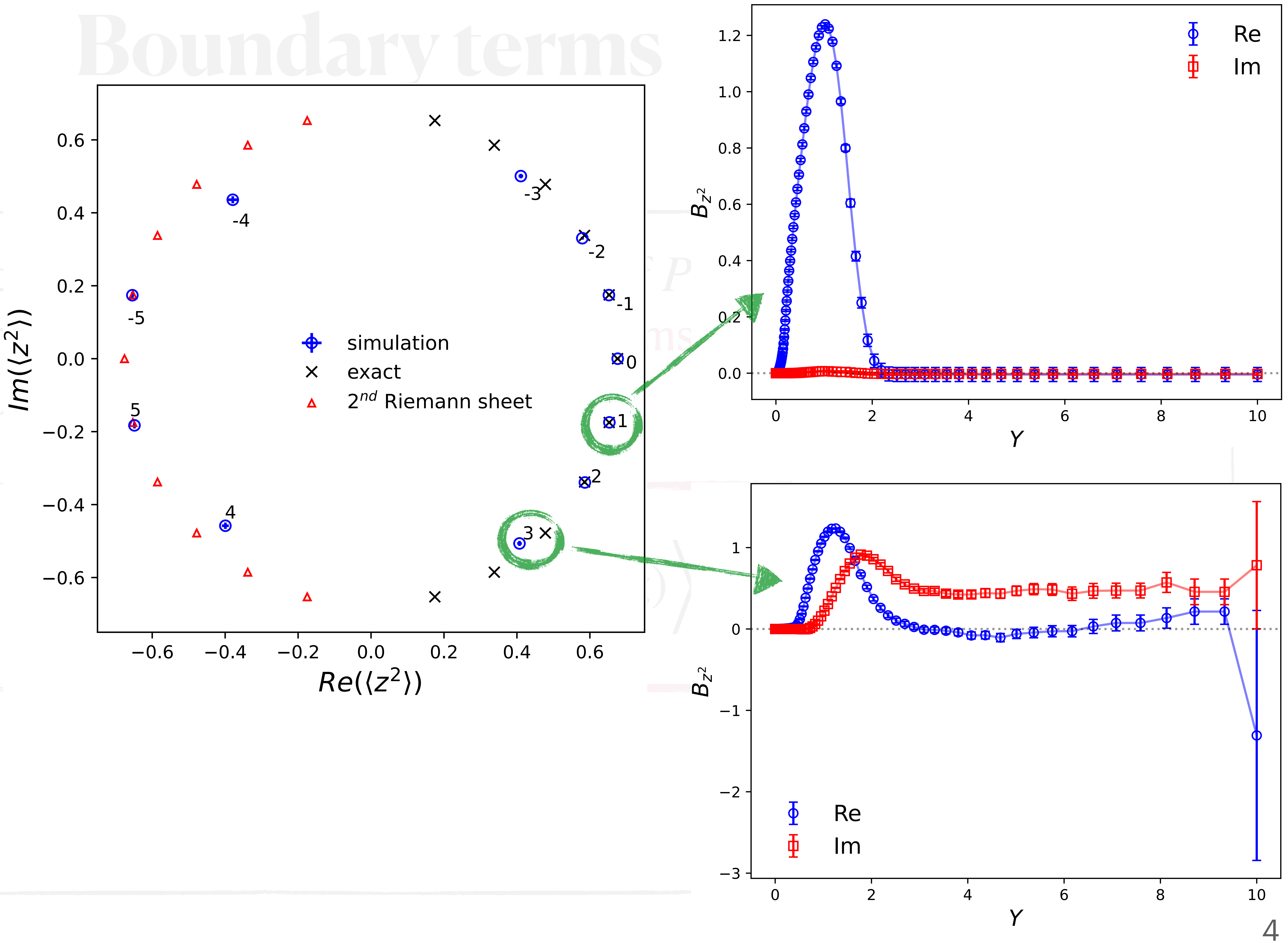

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- Cannot infer correct solutions from vanishing boundary terms.

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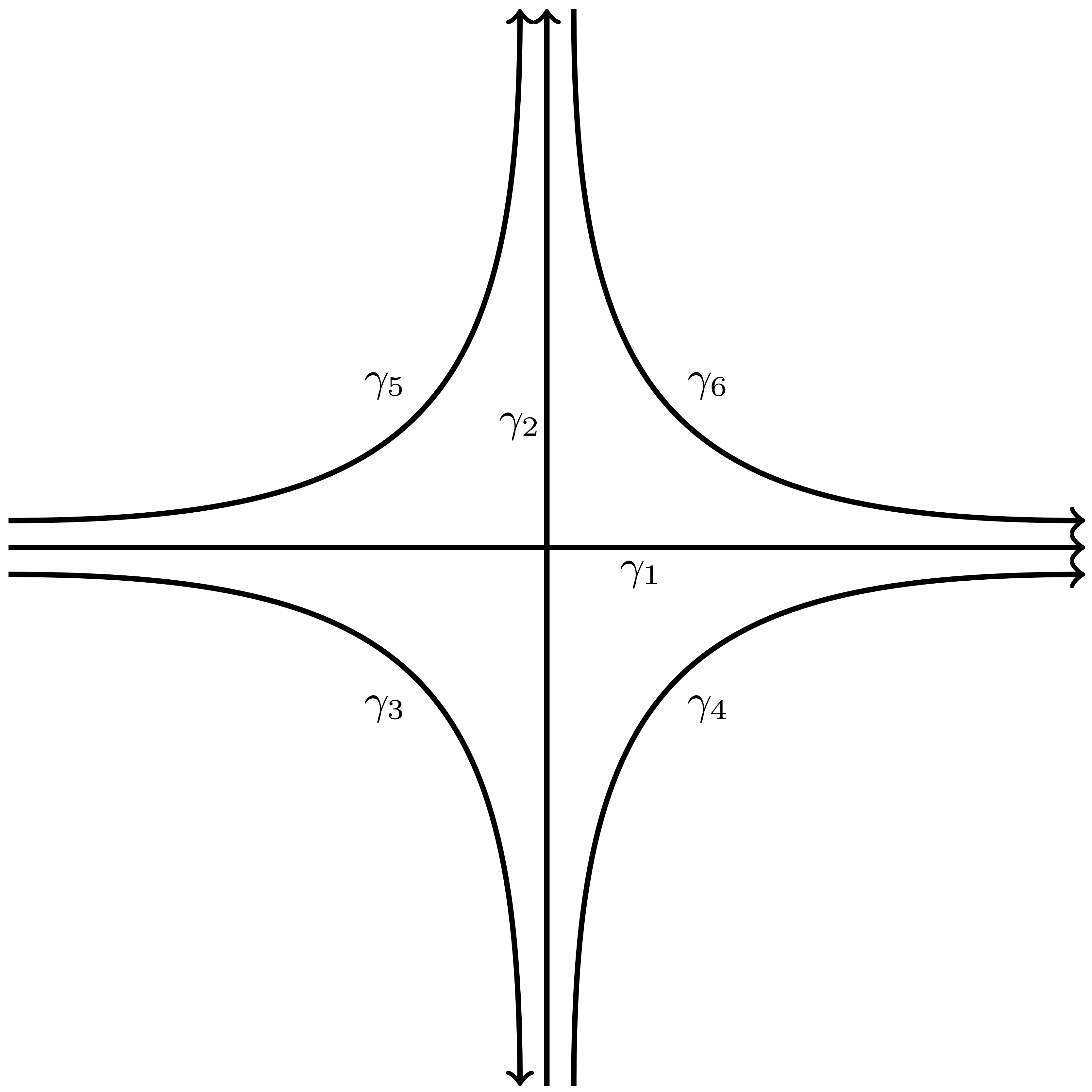

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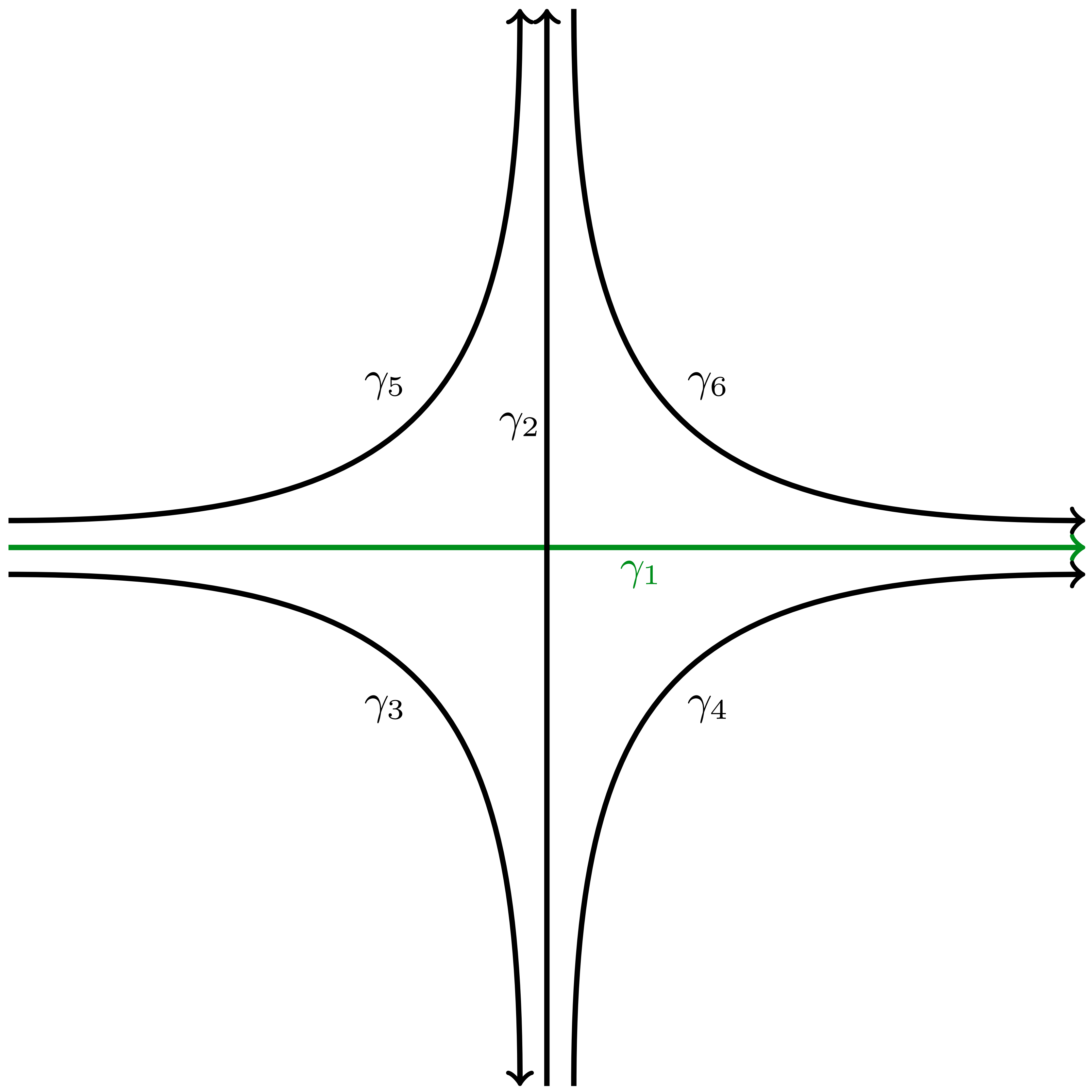

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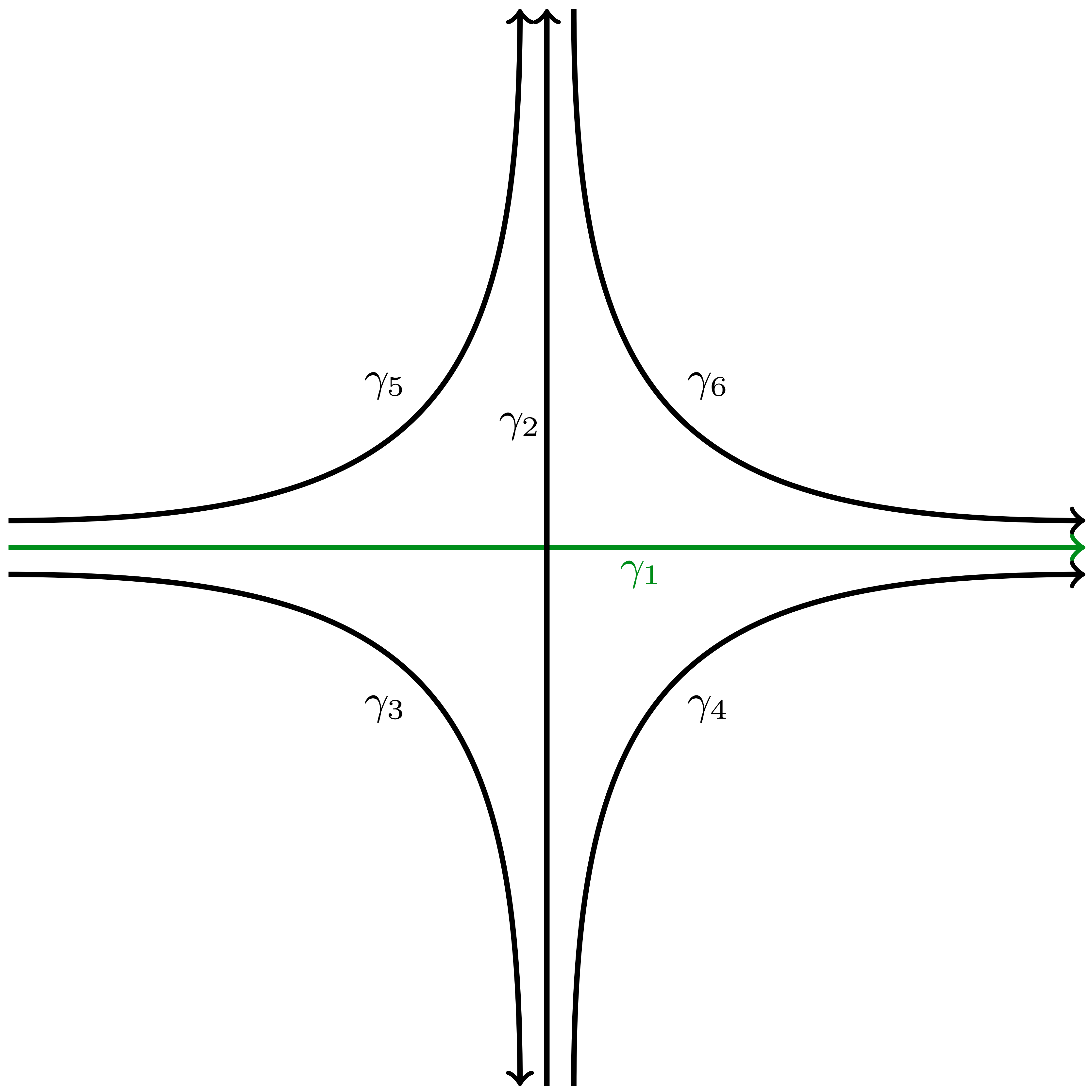
- Vanishing boundary terms only imply that result is linear combination of integration cycles:

$$\langle \mathcal{O} \rangle_{\text{CL}} = \sum_{i=1}^{3} a_i \langle \mathcal{O} \rangle_{\gamma_i}$$

Salcedo, Seiler '19



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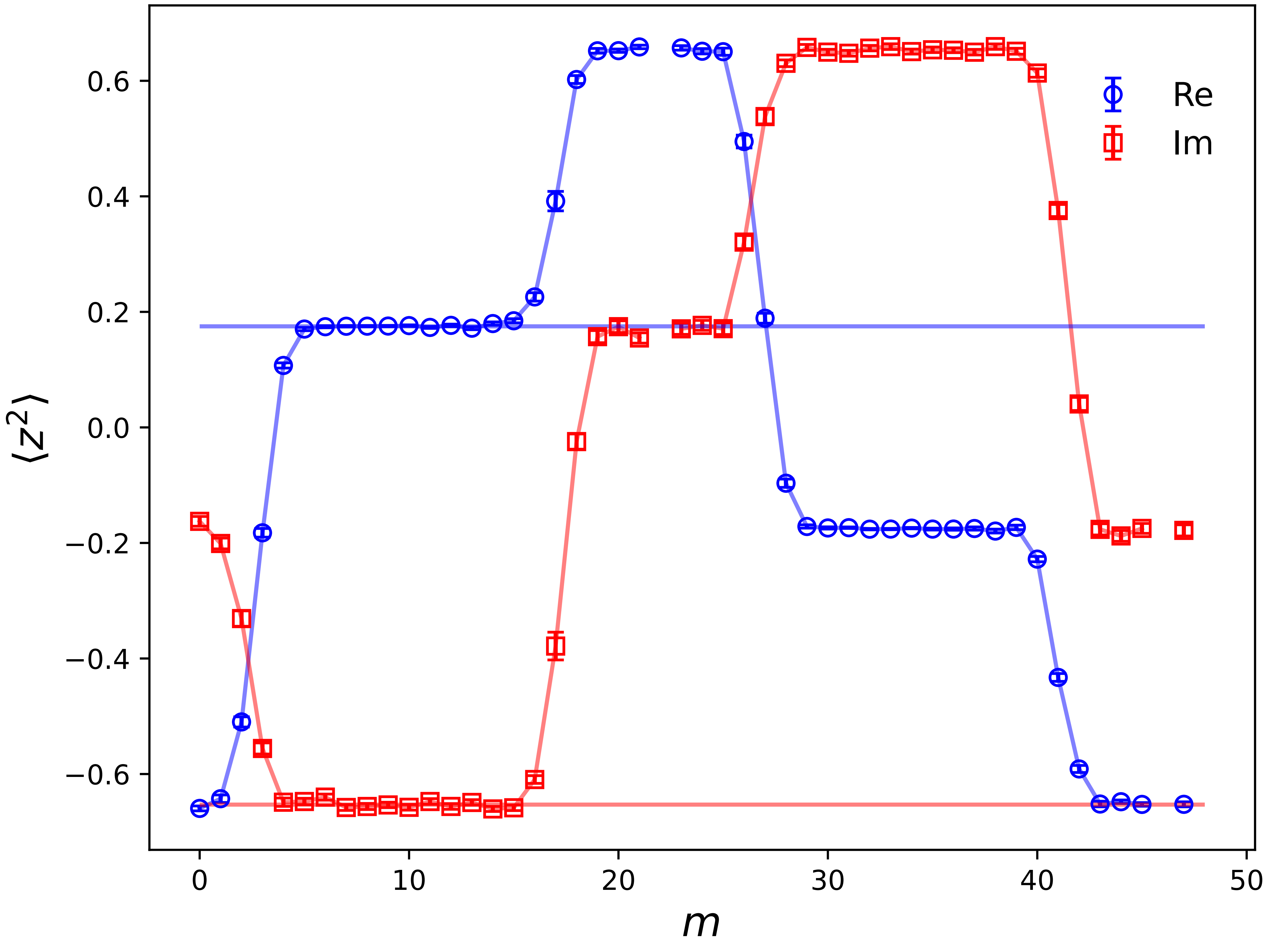

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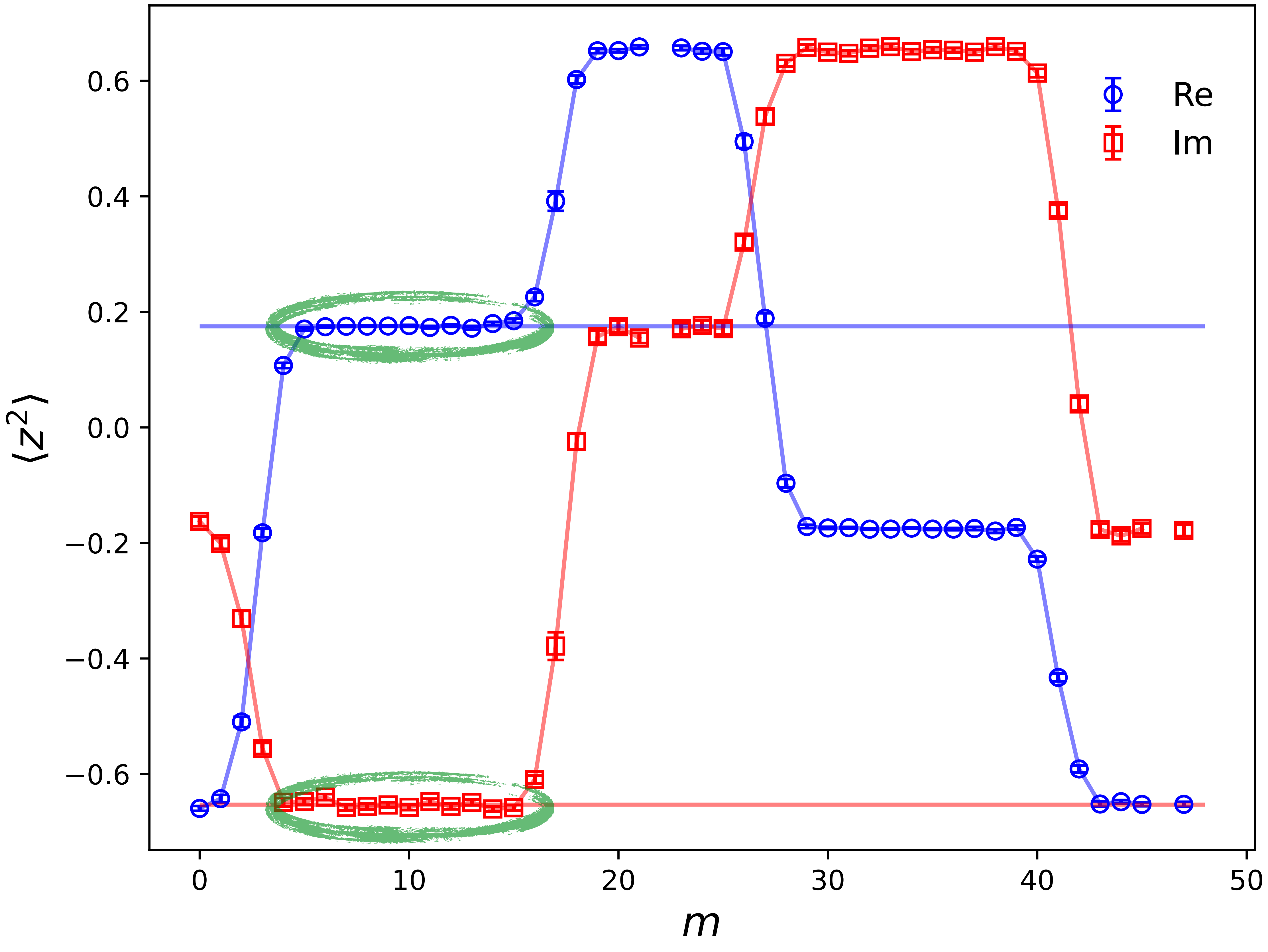

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- Kernel can restore correct convergence.
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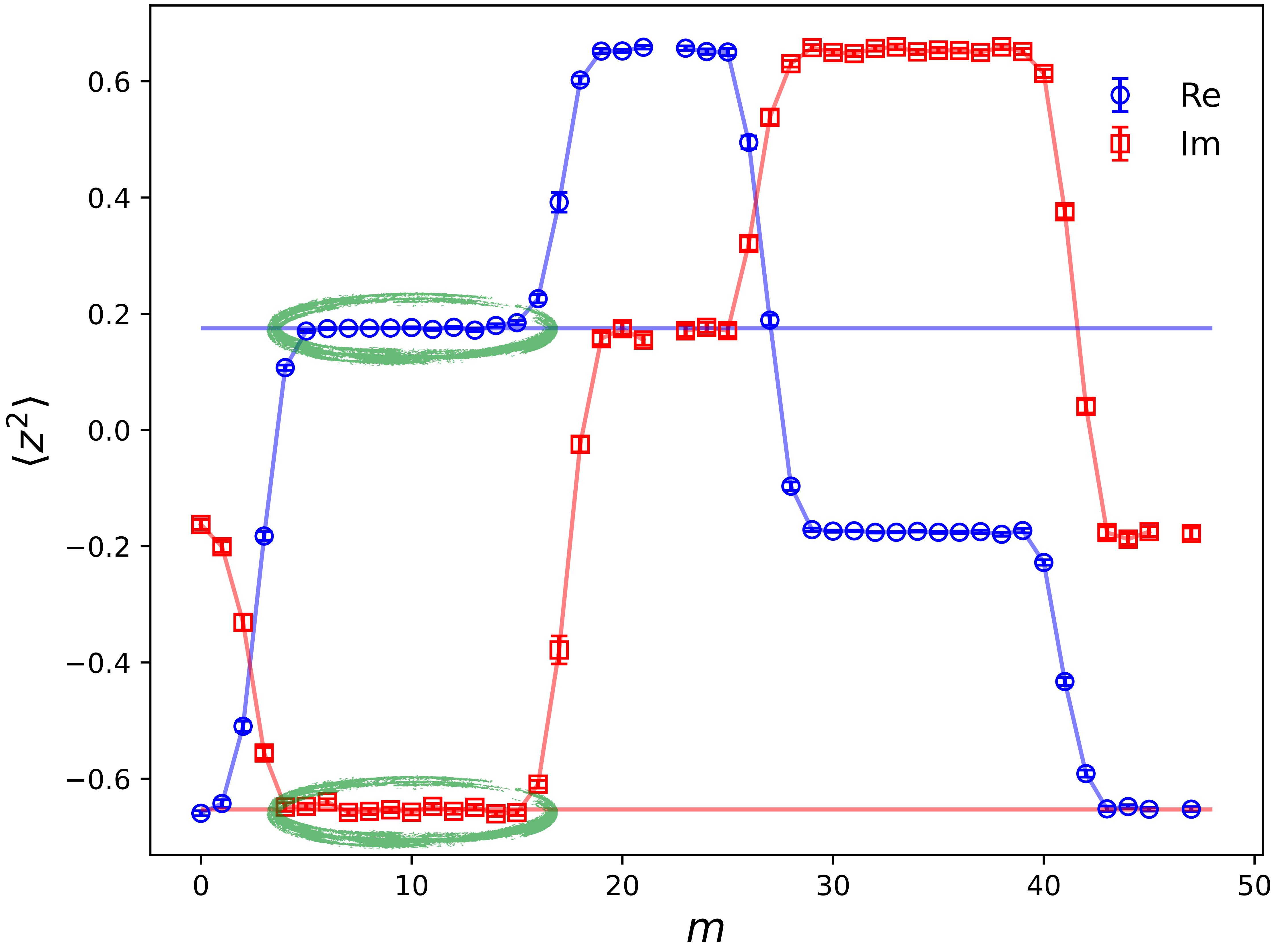

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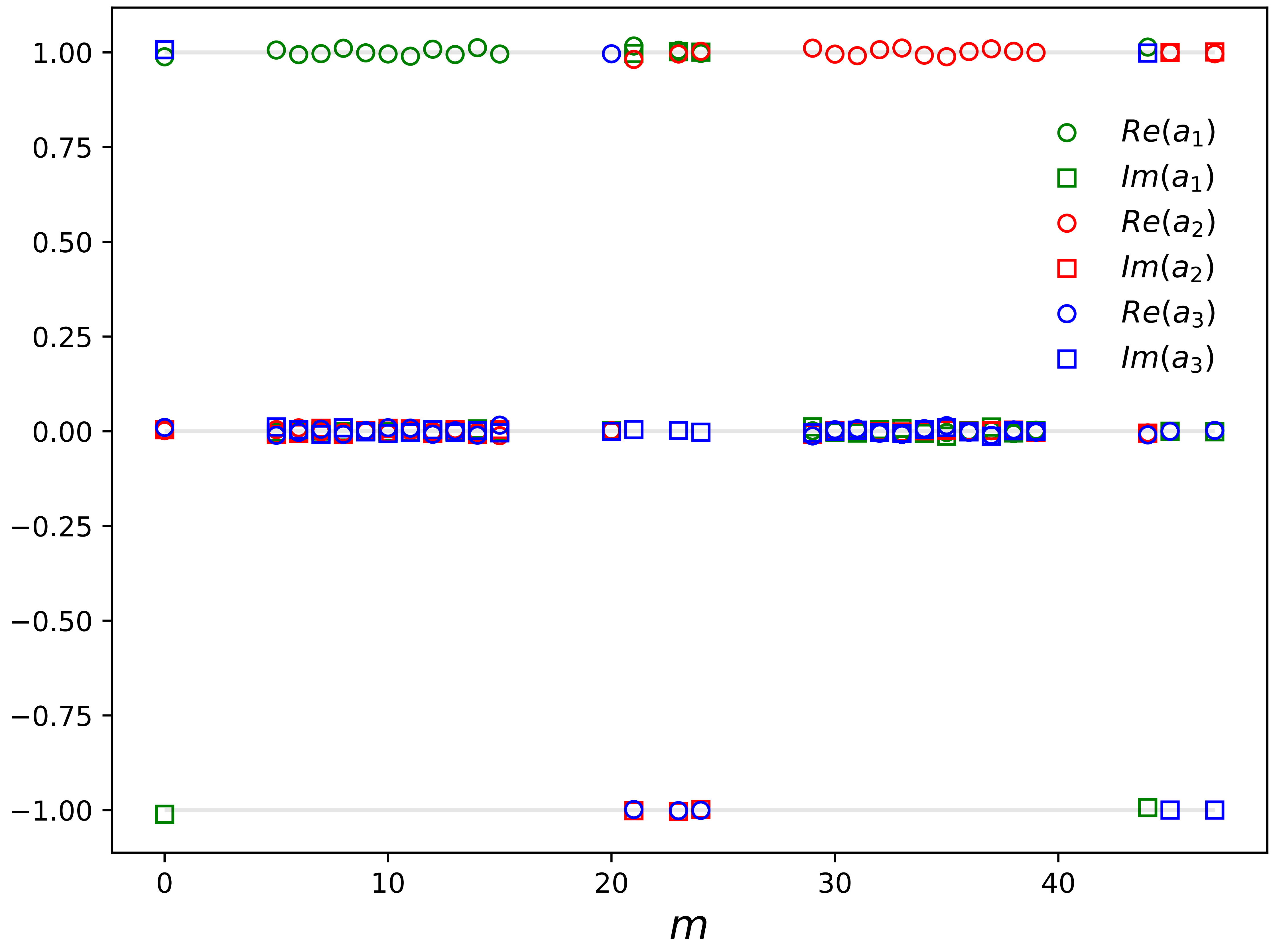

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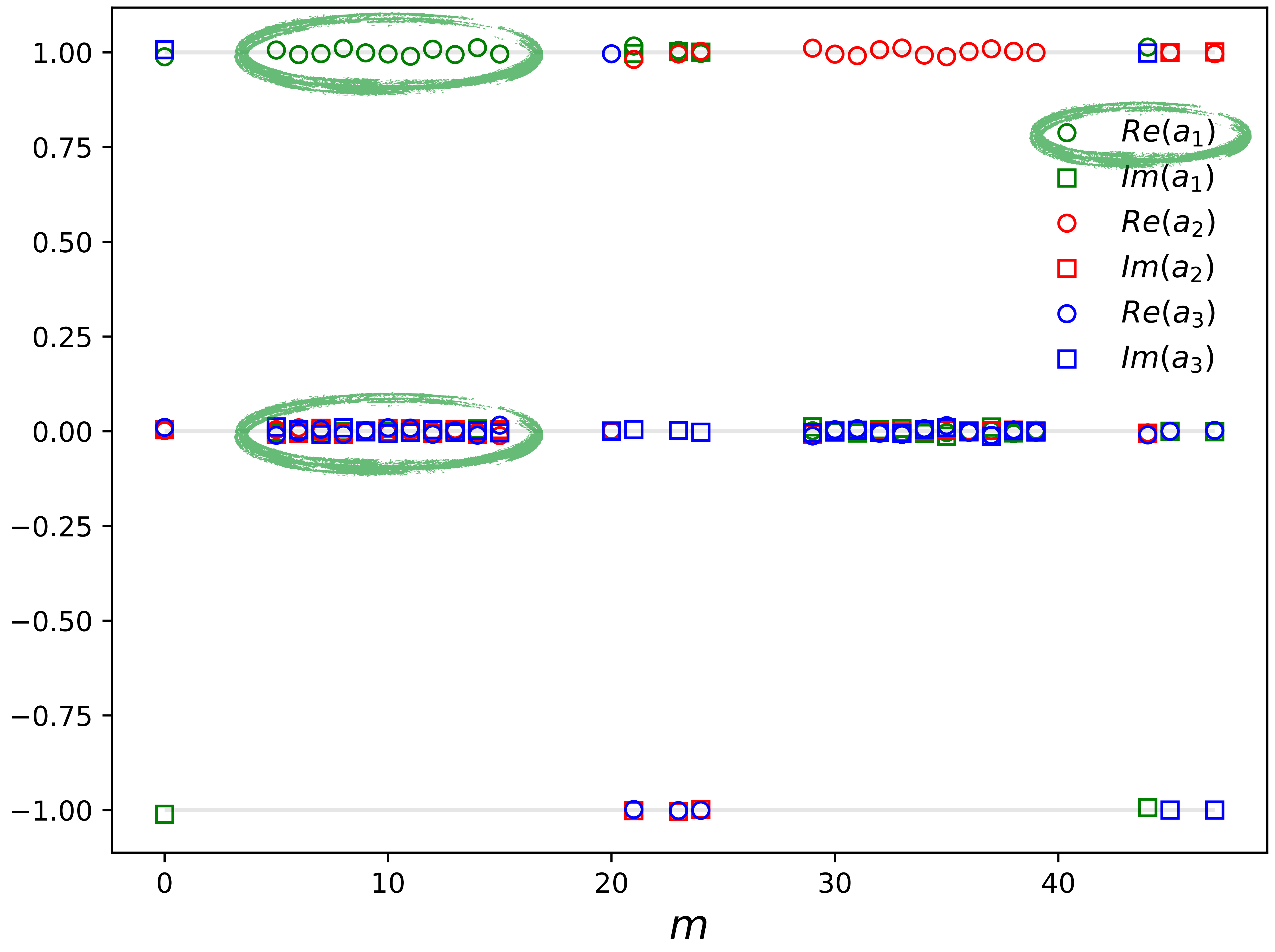

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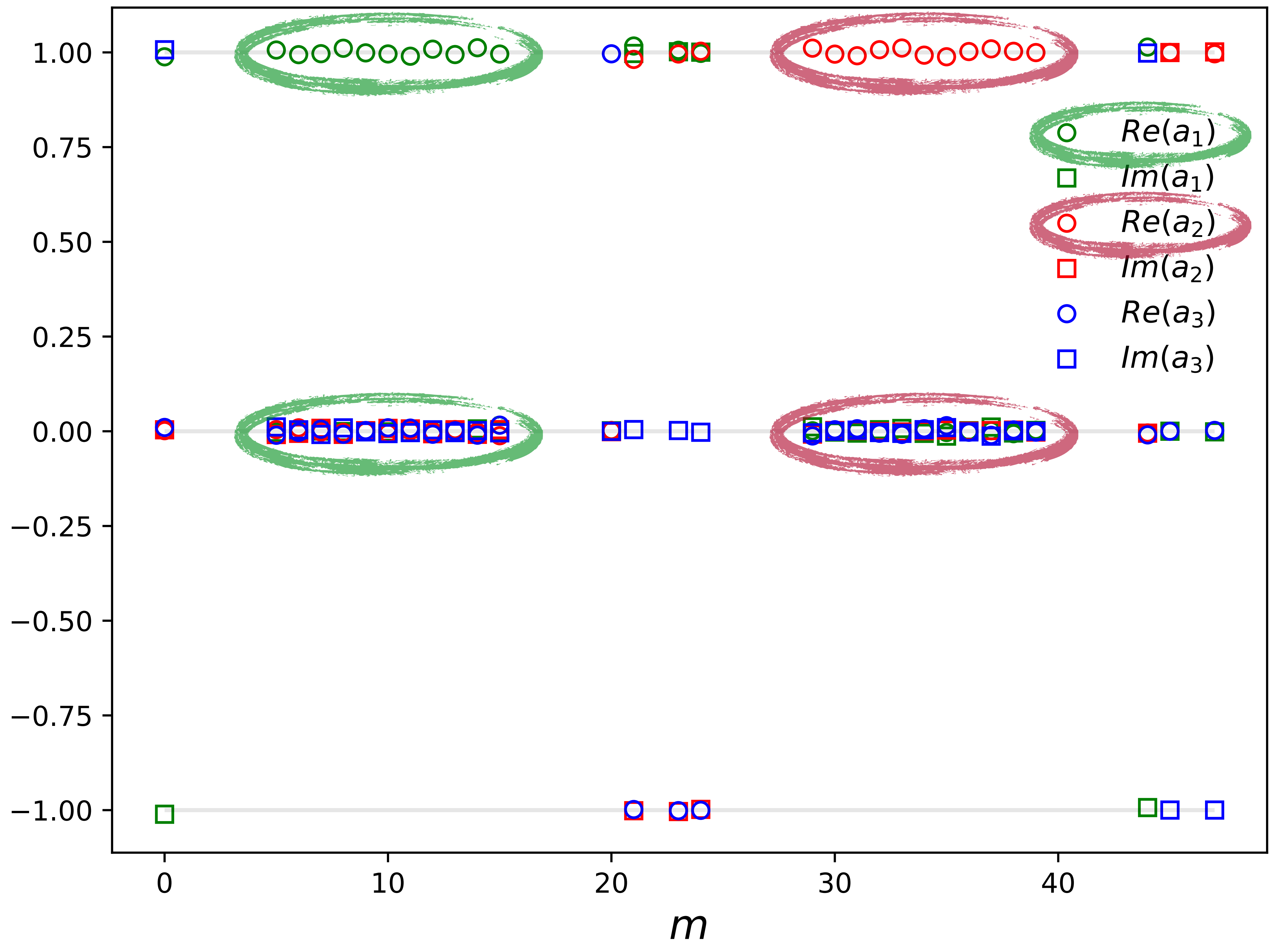

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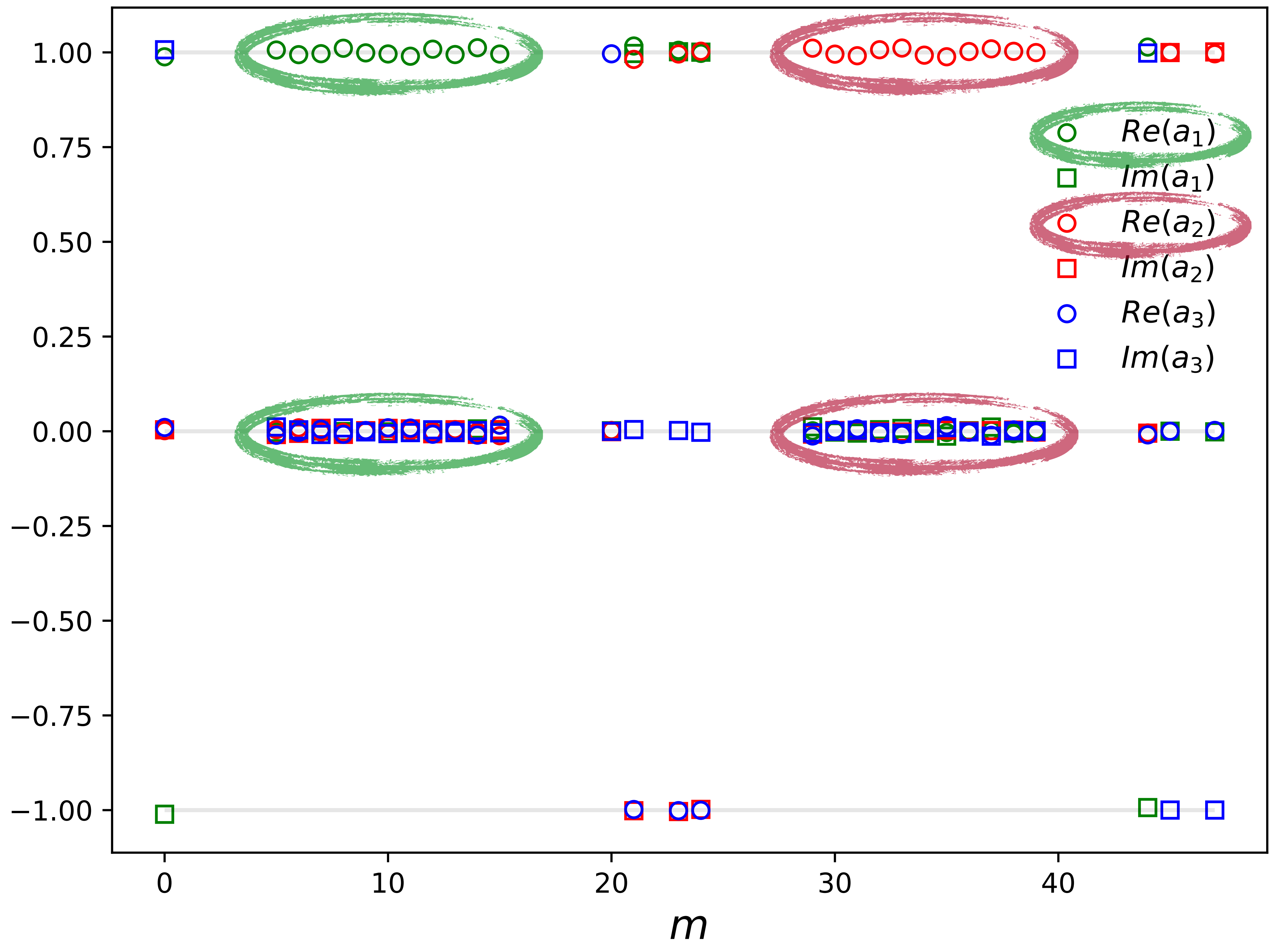

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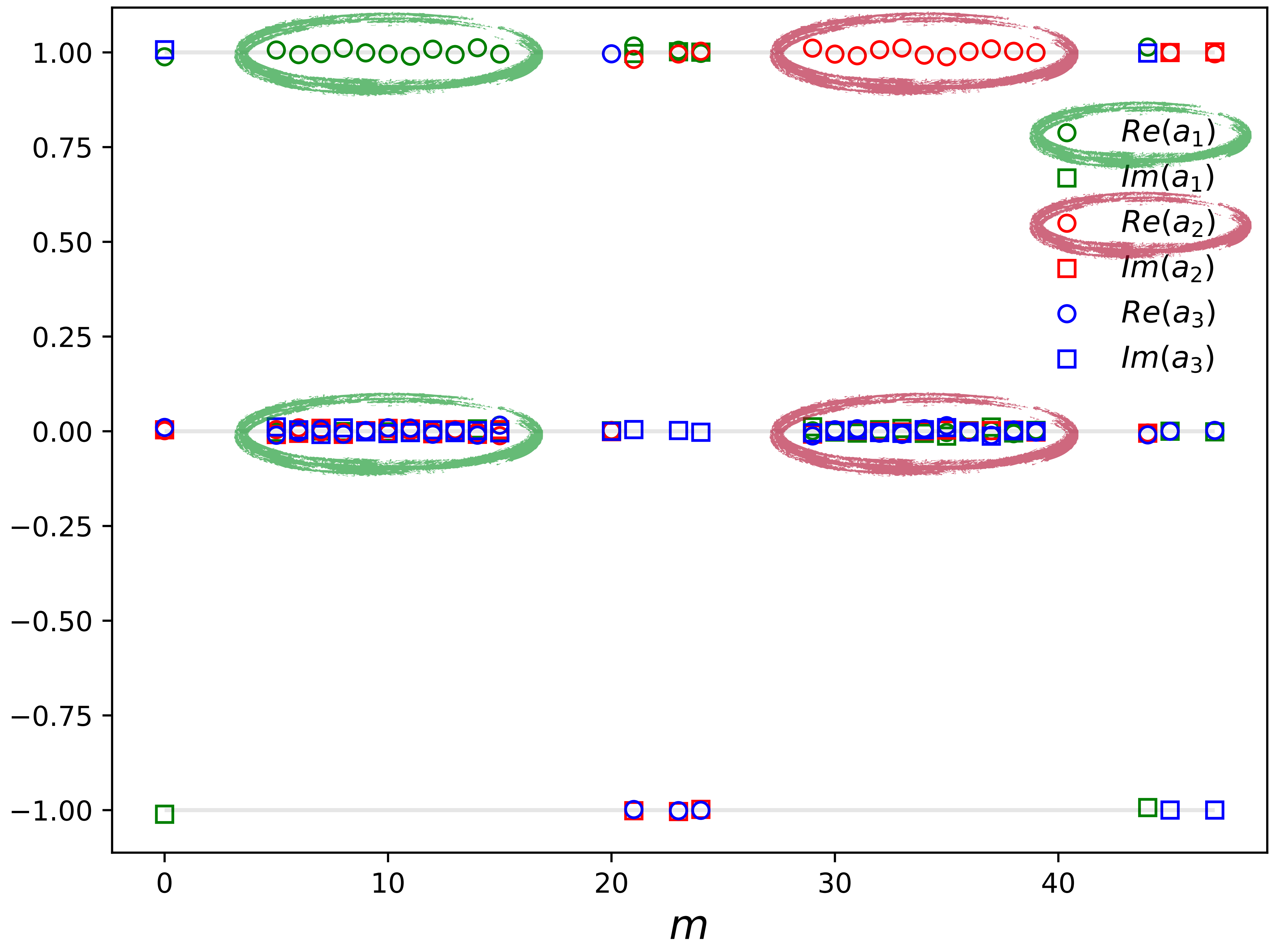

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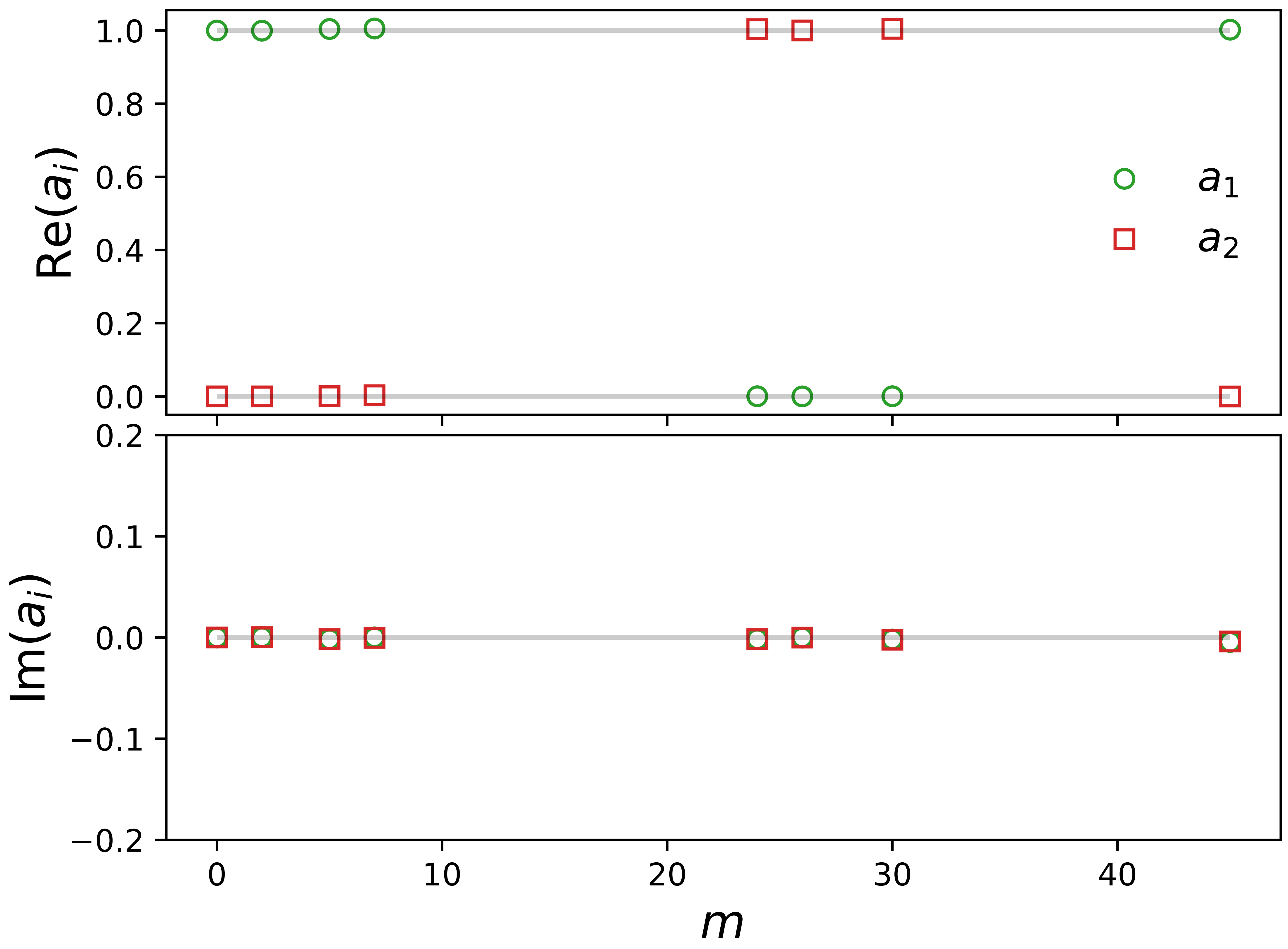

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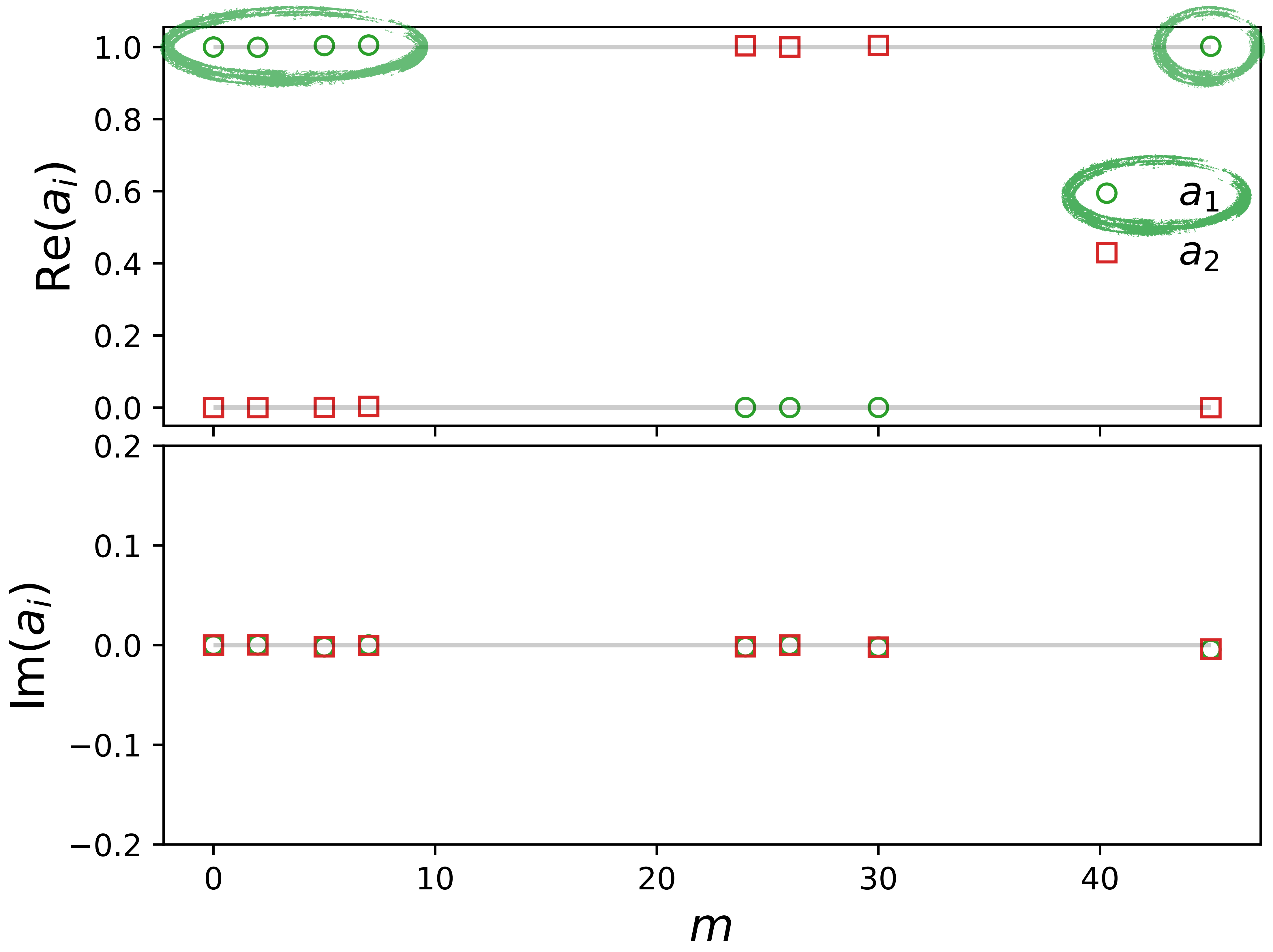

Integration cycles in higher dimensions

- Consider $S(z_1, z_2) = \frac{\lambda}{4}(z_1^2 + z_2^2)^2$.
- $e^{-S(z_1,z_2)}$ has 16 zeros but there are only 2 independent integration cycles.
- $$\langle \mathcal{O} \rangle_{\mathrm{CL}} \stackrel{?}{=} \sum_{i=1}^{2} a_i \langle \mathcal{O} \rangle_{\gamma_i}$$

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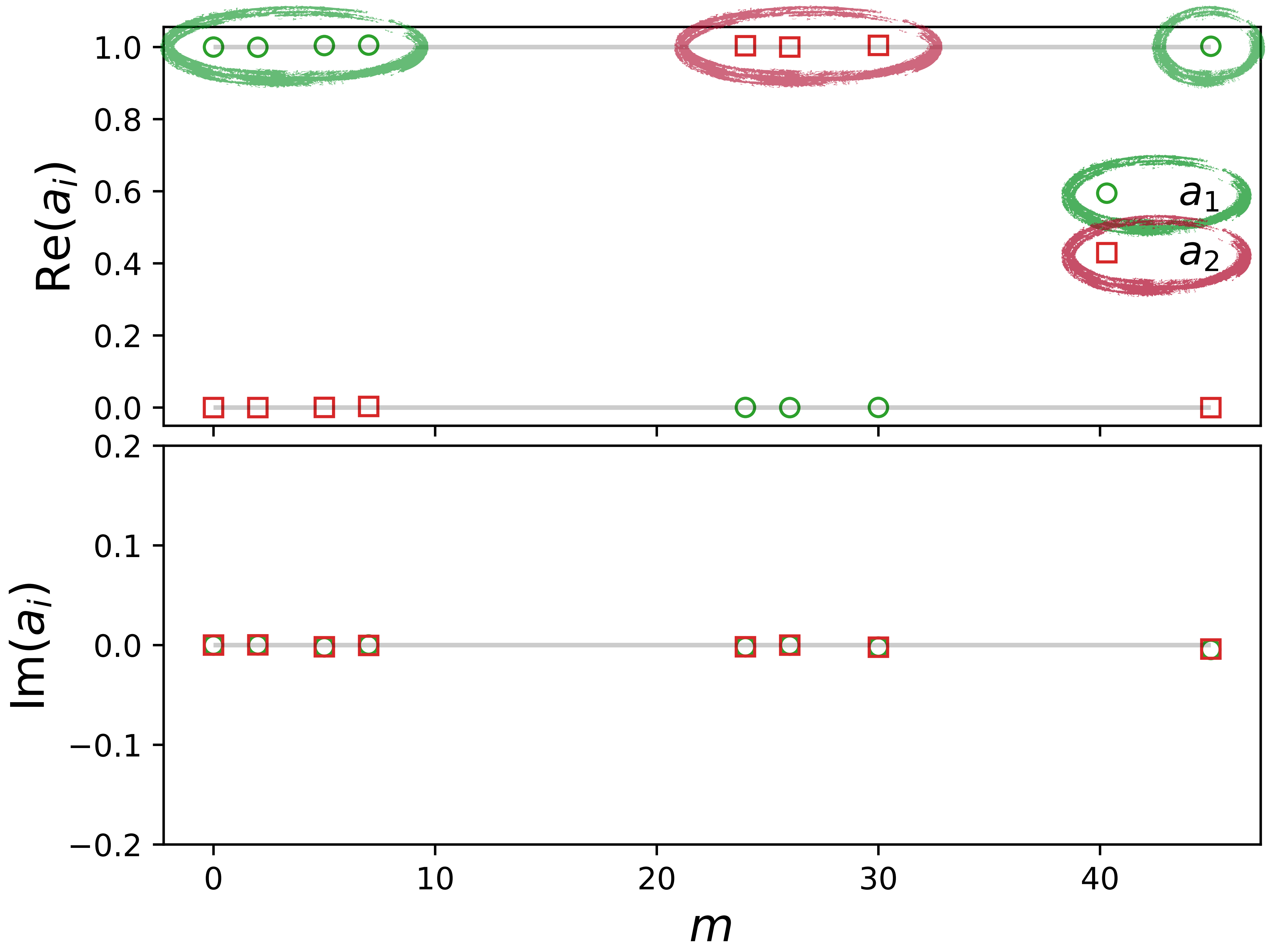

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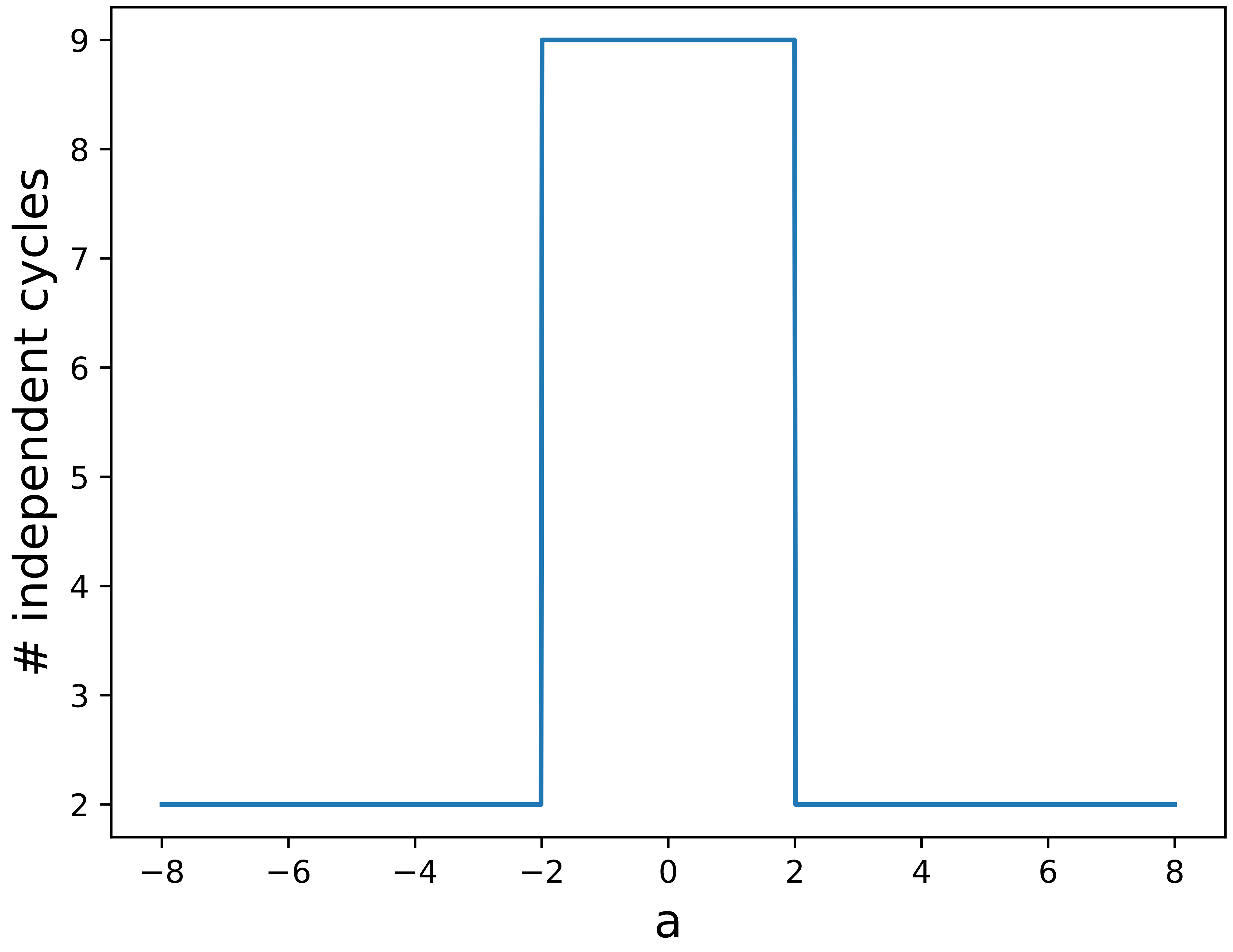

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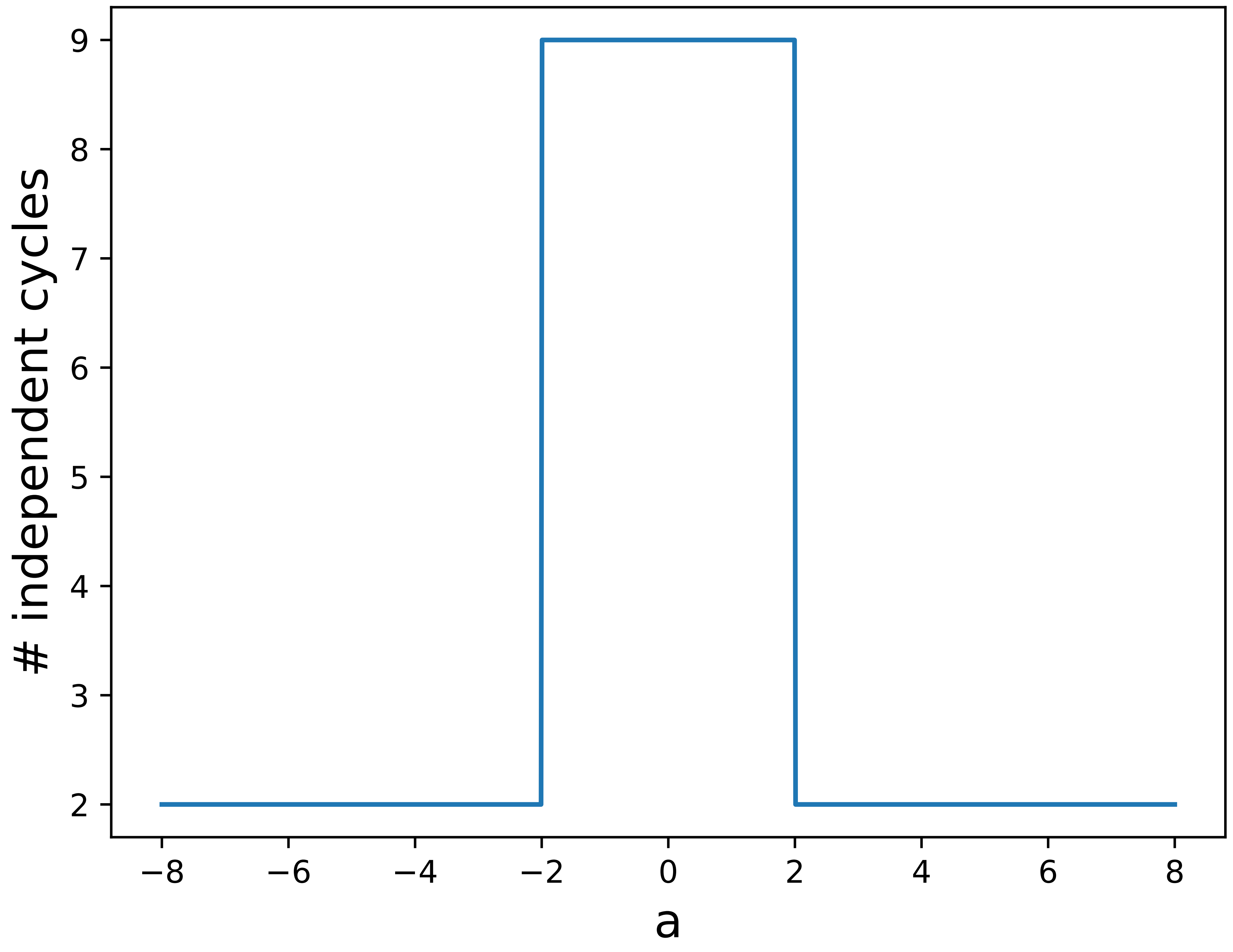

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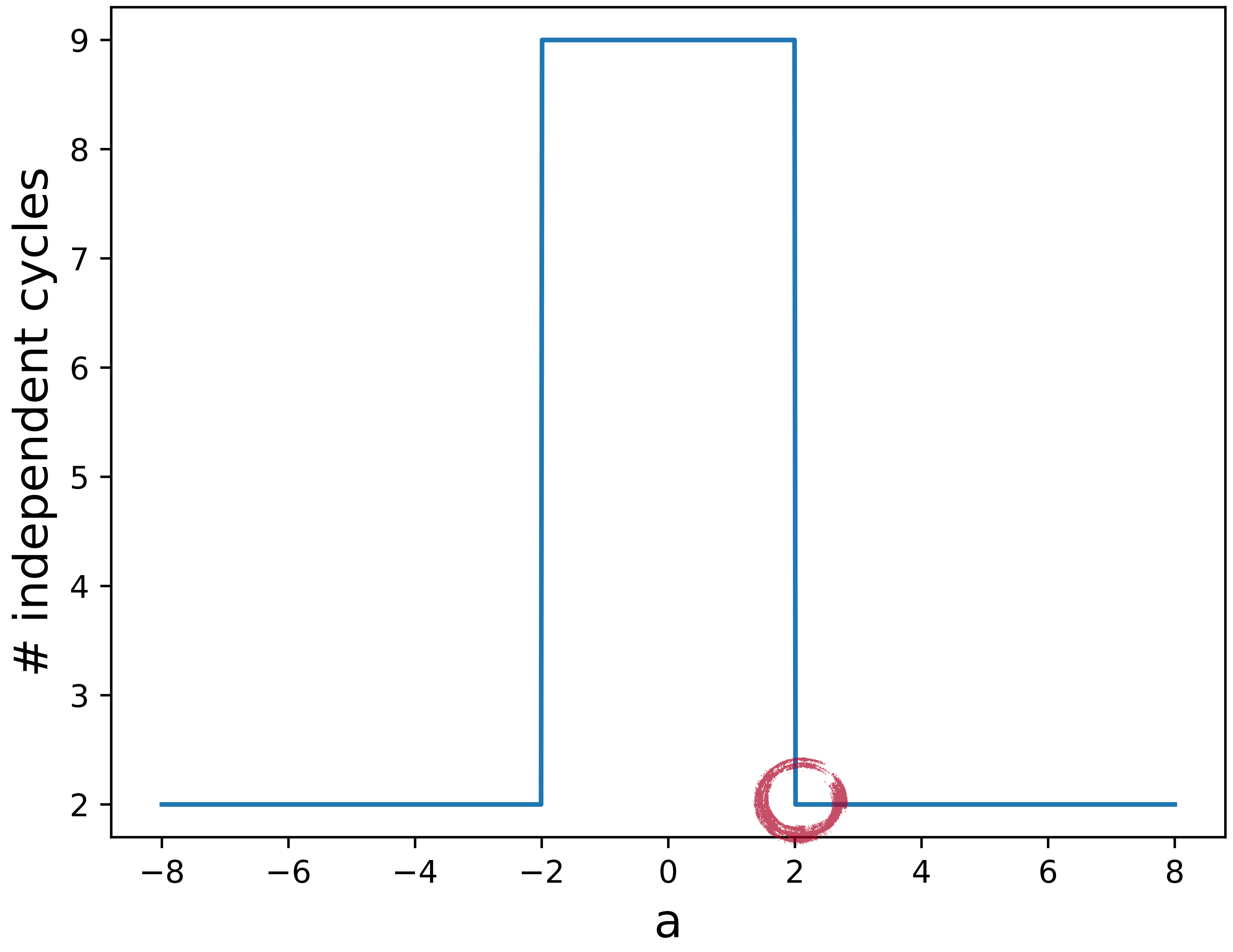
- The O(2)-symmetric point $a = 2$ is "critical".



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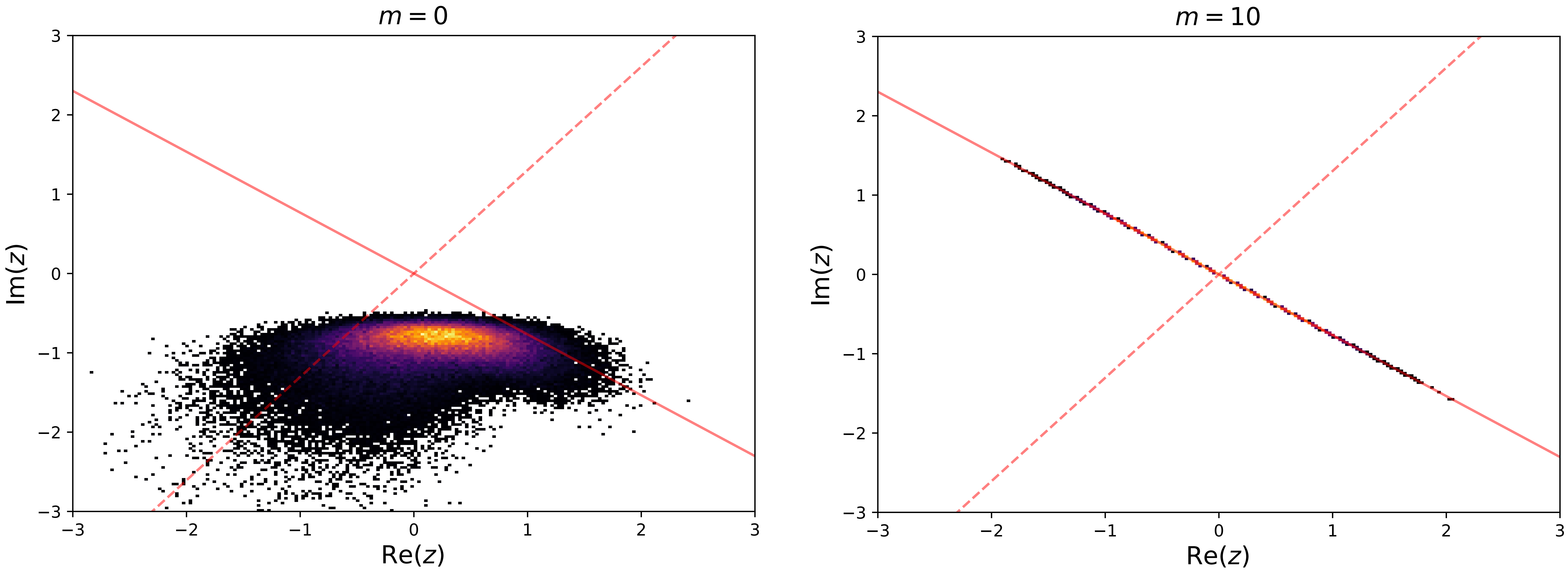
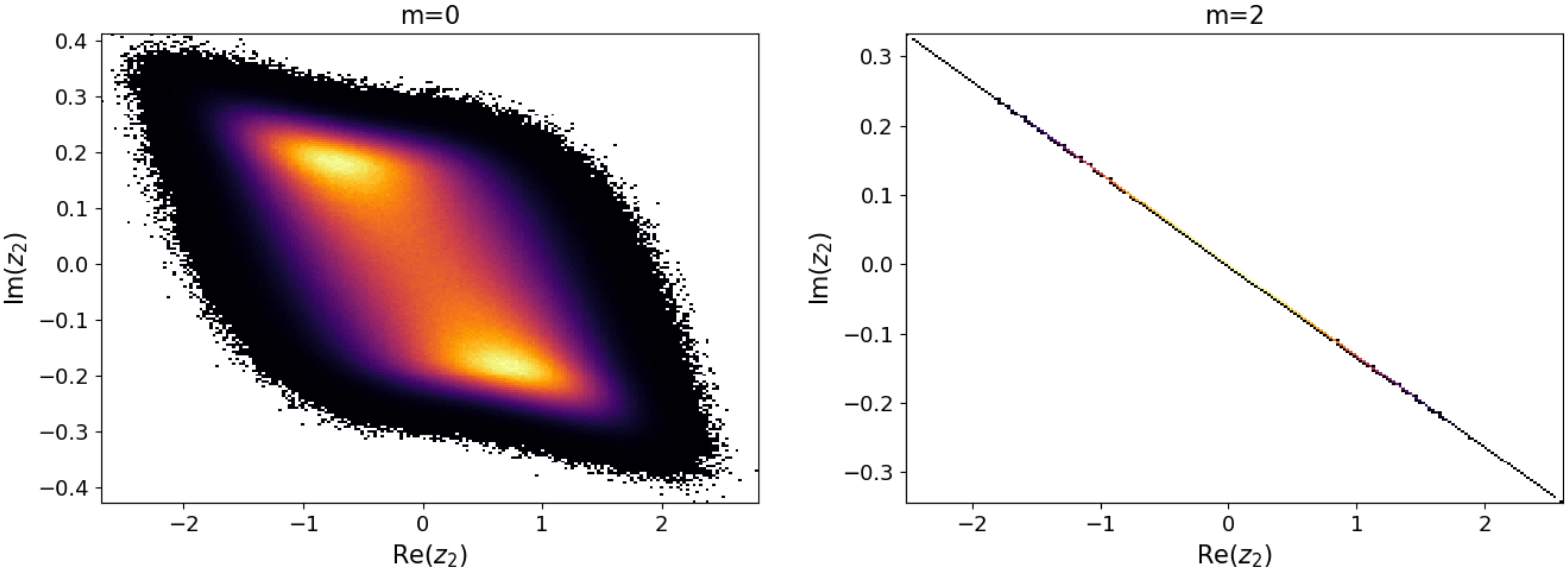
Backup

Effect of a kernel in 1D



Effect of a kernel in 2D



Runaways

$$z \to z - \frac{\partial S(z)}{\partial z}\varepsilon + \sqrt{\varepsilon}\eta$$

- Example: $S(z) = \frac{z^4}{4}$.
- Complexification can introduce runaway trajectories leading to diverging simulation.
- Overcome via adaptive step-size control.

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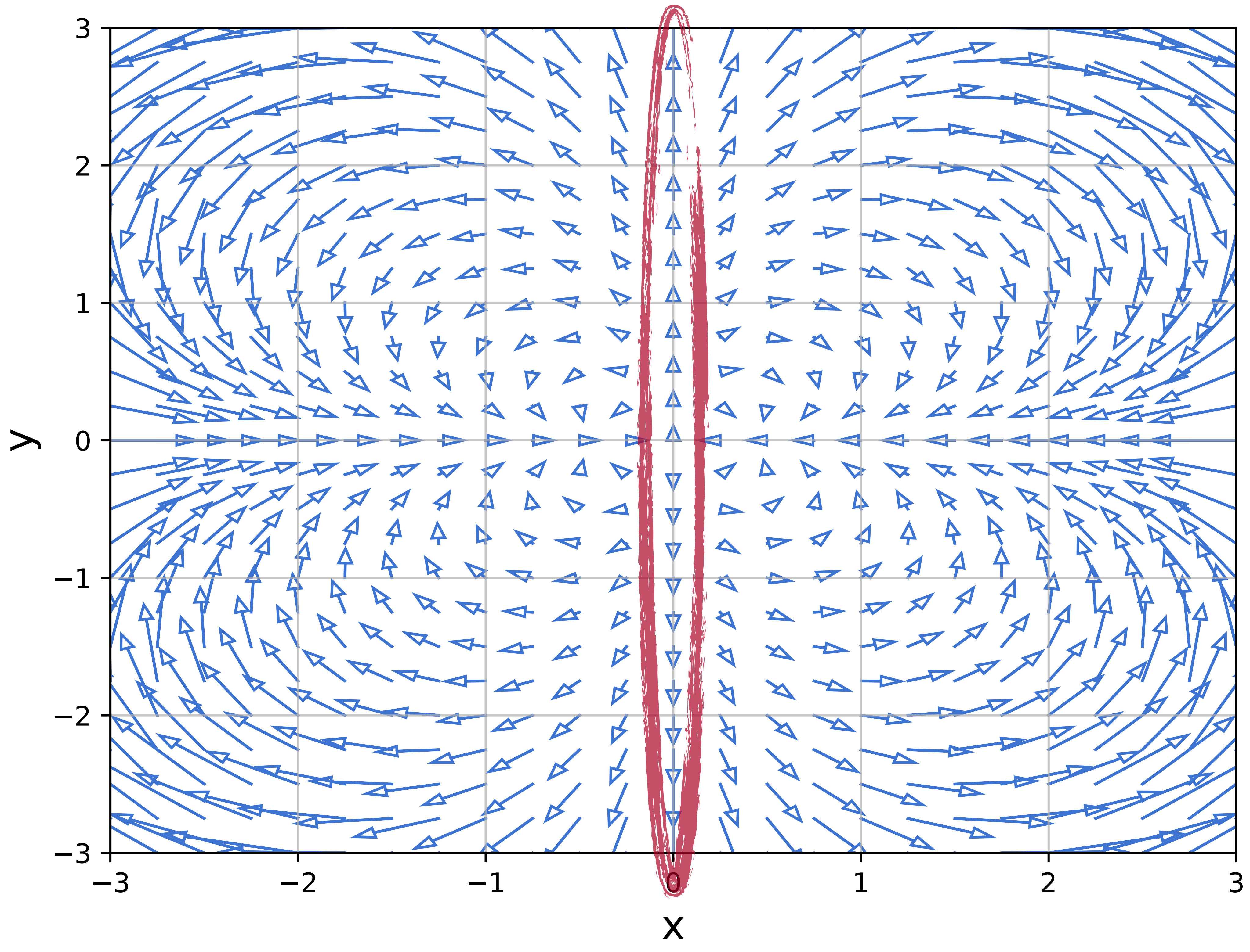
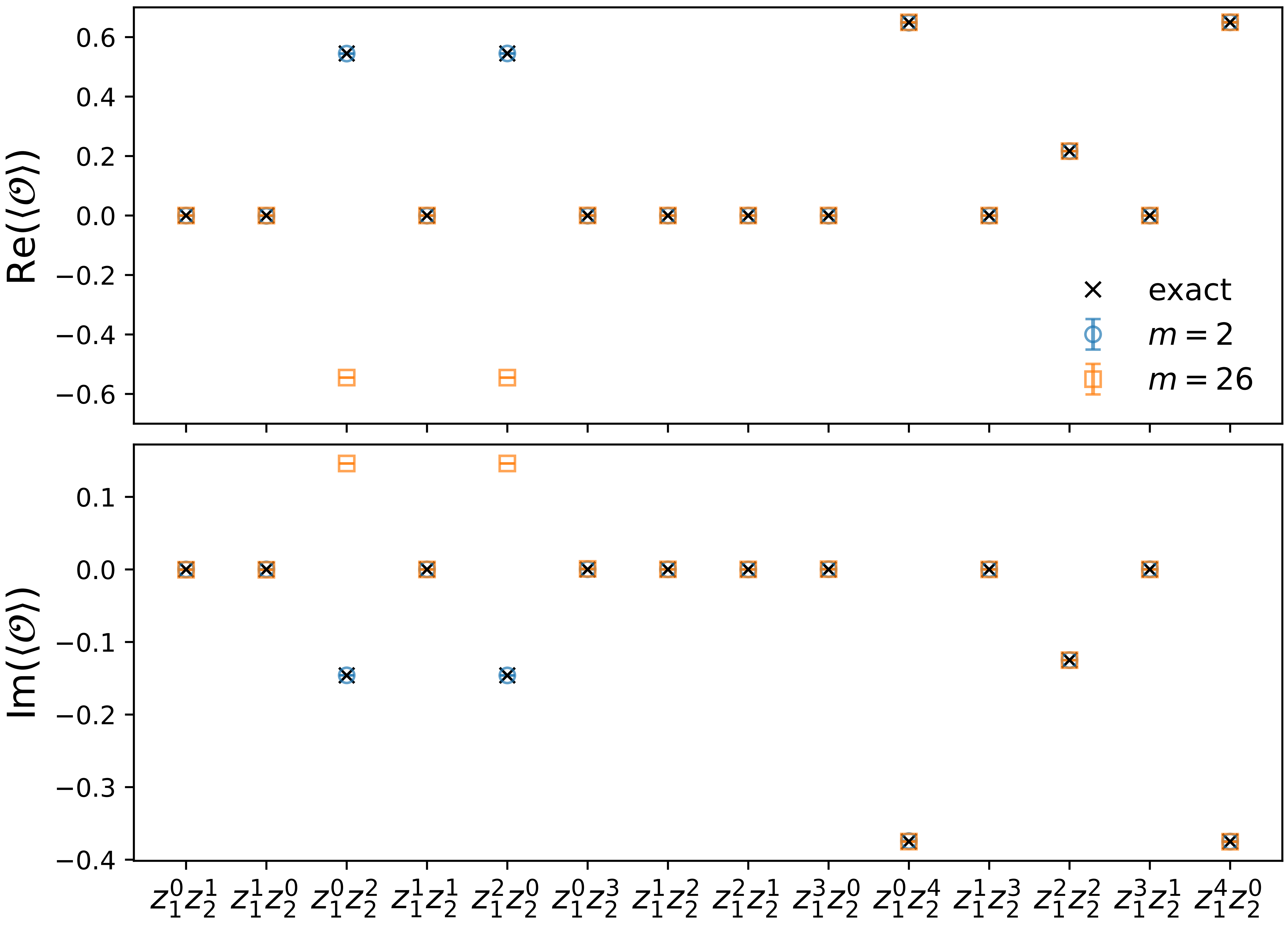


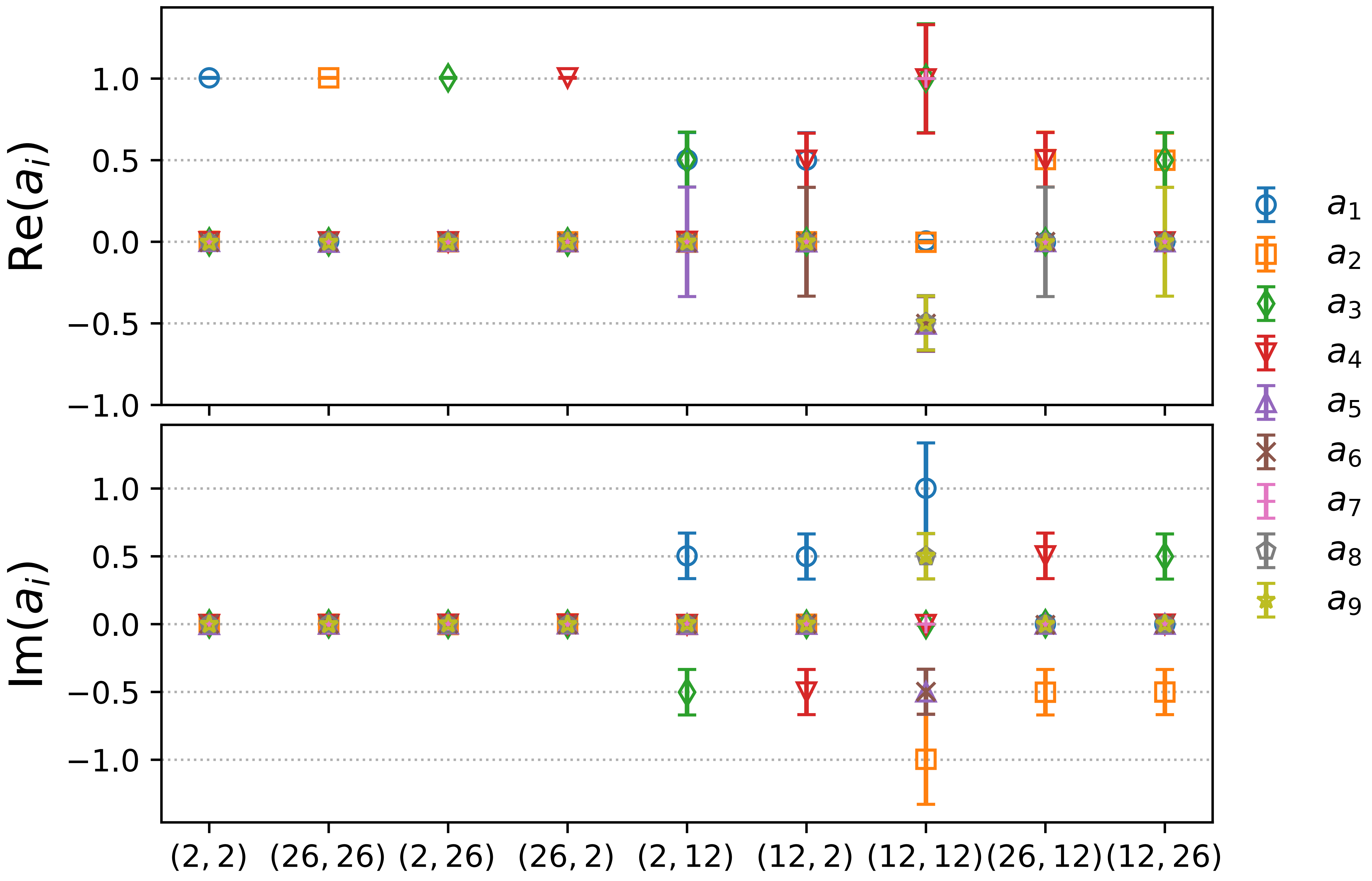
Integration cycles in higher dimensions



Integration cycles in higher dimensions

$$S(z_1, z_2) = \frac{\lambda}{4}(z_1^4 + z_2^4 + a z_1^2 z_2^2)$$

$a = 0$



$a = 1$