

"Tensor renormalization group study of (1+1)-dimensional O(3) nonlinear sigma model w/ and w/o finite chemical potential "

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Plan of talk

- Introduction to Tensor Renormalization Group(TRG)
- Application of TRG to Quantum Field Theories(QFTs)
- Entanglement Entropy(EE) of (1+1)d O(3) NLSM at $\mu = 0$
- Quantum Phase Transition of (1+1)d O(3) NLSM at $\mu \neq 0$
- Summary and Outlook

TRG vs Monte Carlo

Advantages of TRG

・ Free from sign problem/complex action problem in MC method

 $Z = \int \mathcal{D}\phi \exp(-S_{\text{Re}}[\phi] + iS_{\text{Im}}[\phi])$

- **Computational cost for L^D system size ∝ D × log(L)**
- **EXECT** Direct manipulation of Grassmann numbers 1 tion of G $\frac{1}{2}$ *x* (*x* + *x*) $\frac{d}{dx}$ *x* + *x*
- Direct evaluation of partition function Z (density matrix ρ) itself

Applications in particle physics: ,
J = *z* = b = 2 *x,µ* −π

> Finite density QCD, QFTs w/ θ-term, Lattice SUSY etc.
condensed matter physics *U*

Also, in condensed matter physics *Ti,j,k,l* [⇒] *^T{j,k},{l,i}* ⁼ ' *U*Λ*V ^t {j,k},{l,i}* ⁼)

Hubbard model (Mott transition, High Tc superconductivity) etc. *m* (*S*1)*{j,k},m* (*S*3)*{l,i},m*

2d models

TRG Approaches to QFTs (1)

w/ sign problem

Real Φ^4 theory:

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP05(2019)184

Complex φ4 theory at finite density:

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP02(2020)161

U(1) gauge theory w/ θ-term:

YK-Yoshimura, JHEP04(2020)089

Schwinger(2d QED), Schwinger w/ θ-term:

Shimizu-YK, PRD90(2014)014508, 074503, PRD97(2018)034502

N=1 Wess-Zumino model (SUSY):

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP03(2018)141

O(3) NLSM at μ =0 and $\mu\neq0$:

Luo-YK, JHEP03(2024)020, arXiv:2406.08865

U(1) gauge-Higgs model w/ θ-term under Luscher's admissibility condition : Akiyama-YK, arXiv:2406.08865 \Rightarrow talk by Akiyama at 14:55 on Fri.

Application to various models w/ sign problem, Development of calculational methods for scalar, fermion and gauge fields

TRG Approaches to QFTs (2)

w/ sign problem

3d models

Z₂ gauge-Higgs model at finite density: Akiyama-YK,JHEP05(2022)102 Real φ4 theory:Akiyama-YK-Yoshimura, PRD104(2021)034507 Z₂gauge theory at finite temperature: YK-Yoshimura, JHEP08(2019)023

4d models

Ising model:Akiyama-YK-Yamashita-Yoshimura, PRD100(2019)054510 Complex φ4 theory at finite density:

Akiyama-Kadoh-YK-Yamashita-Yoshimura, JHEP09(2020)177 NJL model at finite density:

Akiyama-YK-Yamashita-Yoshimura, JHEP01(2021)121 Real φ4 theory:Akiyama-YK-Yoshimura, PRD104(2021)034507 Z₂ gauge-Higgs model at finite density: Akiyama-YK, JHEP05(2022)102 Z₃ gauge-Higgs model at finite density: Akiyama-YK, JHEP10(2023)077

⇒ Research target is shifting from 2d models to 4d ones

TRG Approaches to QFTs (3) Γ the theoretical prediction of Γ **Previous Monte Carlo Simulation Monte Carlo Simulation Carlo Simulat** ϵ and ϵ is the same conclusion ϵ

d the two sign problem

APPENDIX: GRASSMANN TENSOR FOR

Condensed matter physics $\mathcal{D}_\mathcal{D}$, we repeat the calculation changing D. The calculation changing $D_\mathcal{D}$

Similarity btw Hubbard models and NJL ones Action consisting of hopping terms and 4-fermi interaction term fitting startings. The solid line shows the fitting result with the fitting result with the fitting result with the fitting results of the fitting results of the fitting results of the fitting results of the fitting result sional Hubbard model, whose action is given by

$$
S = \sum_{n \in \Lambda_{d+1}} \epsilon \left\{ \bar{\psi}(n) \left(\frac{\psi(n+\hat{\tau}) - \psi(n)}{\epsilon} \right) - t \sum_{\sigma=1}^d \left(\bar{\psi}(n+\hat{\sigma}) \psi(n) + \bar{\psi}(n) \psi(n+\hat{\sigma}) \right) + \frac{U}{2} (\bar{\psi}(n) \psi(n))^2 - \mu \bar{\psi}(n) \psi(n) \right\}
$$

First principle calculation at finite density **and internal sease the following decomposition** (1+1)d Hubbard model:Akiyama-YK, PRD104(2021)014504 (2+1)d Hubbard model: Akiyama-YK-Yamashita, PTEP2022(2022)023I01

In this talk we focus on $(1+1)d$ O(3) NLSM Entanglement Entropy (EE) at $\mu = 0$ Direct evaluation of partition function Z (density matrix ρ) itself Quantum phase transition at $\mu \neq 0$ Free from sign problem/complex action problem Determination of dynamical critical exponent *z*

EE of $(1+1)$ d O(3) NLSM at $\mu = 0$ **S** and KL of (1+1)d O(3) NLSM at $\mu = 0$ \sum **Decute is the bond of the precision** of $\mu = 0$ \bigcap ² NII *x*0*x*⁰ 0*y*0*y*⁰ $^{\circ}$ ⁿ−1¹ $= 0$

 $\mathsf{Luo\text{-}YK, JHEPO3(2024)020}$ the TRG method. The tensor network representation is given by $\mathsf{Luo}\text{-}\mathsf{YK}$, $\mathsf{JHEPO3}(2)$ we can obtain the tensor network representation of the O(3) NLSM on the Site λ

(1+1)d lattice O(3) NLSM (asymptotic free)

$$
Z = \int \mathcal{D}[s]e^{-S}
$$

$$
S = -\beta \sum_{n \in \Lambda_{1+1}, \nu} s(n) \cdot s(n + \hat{\nu})
$$

$$
\boldsymbol{s}^T(\Omega) = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi) \Omega = (\theta, \phi) \quad , \ \theta \in (0, \pi], \ \phi \in (0, 2\pi].
$$

 (θ, ϕ) is discretized w/ Gauss-Legendre quadrature \rightarrow TN representation $\frac{1}{2}$ (θ, ϕ) is discretized w/ Gauss-Legendre quadrature \rightarrow TN representation (θ, ϕ) is discretized w/ Gauss-Legendre quadrature \rightarrow TN representation

Entanglement Entropy(EE) **Example 19** and substituted to the substitute the substitute the substitute the density matrix of the substitute of the substitute of the substitute of the substitute of the density matrix of the ⌦ = (✓*,*) *,* ✓ ² (0*,* ⇡]*,* ² (0*,* ²⇡]*.* (2.3) where $\overline{}$ is the O(3) Haar measure, whose expression is given later.

Whole system (V=2L \times Nt) is divided to subsystems A, B (V_A,V_B=L \times Nt) $P(\text{mean n})$ Rényi(n-th) *Mhole system (V=2LX Nt) is divided to subsy Z* α **D** $= -\text{Tr}_A \rho_A \log(\rho_A)$ $S_A^{(n)}$ $=\frac{1}{Z}\text{Tr}_B[T\cdots]$ sin(✓*p*)*d*✓*pd^p .* (2.5) $\rho_A = \frac{1}{Z} \text{Tr}_B[T\cdots T] \qquad \qquad \rho_A \; - \; 1-n$ \overline{D} = \overline{D} *p*=1 $S_A = -\text{Tr}_{A} \rho_A \log(\rho_A)$ (n) l $\mathcal{L}^p(\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal{U}|\mathcal$ subsystem a, in which $\overline{M/h}$ *Z I***B** Y *ZD**X**Nt***) is divided to subsystems A, B (V₄,V_B=L** \times **Nt)** $S_A = -\text{Tr}_A \rho_A \log(\rho_A)$ $S_A^{(n)} = \frac{\ln \text{Tr}_A \rho_A^n}{1 - n}$ *S^A* = Tr*A*⇢*^A* log(⇢*A*)*.* (2.14) Figure 2 depicts the calculation procedure of the *n*th-order R´enyi entropy defined by $\ln \text{Tr}_A \rho_A^n$ $1 - n$ \overline{v} and \overline{v} is equalent and \overline{v} and \overline{v} and \overline{v} is equalent and \overline{v} $S_A^{\langle n \rangle} = -AFA^{\rho}A^{-\rho}S(\rho A)$ $S_A^{\langle n \rangle} = \rho_A =$

Calculation of EE

Spatial size (L) dependence of EE

Luo-YK, JHEP03(2024)020

von Neumann EE 2nd Rényi EE

 $S_A \sim \frac{C}{3} \ln(\xi)$ (c: central charge, ξ : correlation length) $S_A^{(2)} \sim \frac{C}{3} (1 + \frac{1}{2}) \ln(\xi)$

$$
S_A^{(2)} \sim \frac{c}{3} \left(1 + \frac{1}{2} \right) \ln(\xi)
$$

Convergence at $L \gg \xi$ is confirmed

von Neumann vs Rényi other hand, the interpolations of the R´enyi entropy at *n* = 1*/*2*,* 2*, ...,* 4 and *n* = 1*/*2*,* 2*, ...,* 11 with the third and sixth order polynomial functions, respectively, $r_{\rm eff}$

Luo-YK, JHEP03(2024)020 pink and purple curves in Fig. 12, give consistent results with the entanglement $\frac{1}{2}$

Comparison btw von Neumann $EE(n=1)$ and Rényi $EE(n\neq 1)$

 $S_A^{(2)}$ is not a good approximation of von Neumann EE(n=1) Difficult to extrapolate $S_A^{(n)}(n \ge 2)$ to n=1 at high precision Reliable interpolation to n=1 using $S_A^{(1/2)}$ and $S_A^{(n)}$ (n \geq 2) Figure 12: *n* dependence of *n*th-order R´enyi entropy with *N^t* = 1024 at = 1*.*5. Solid Difficult to extrapolate σ_A ^{ov} (ri \geq Z) to ri-T at riigh precision

Central Charge

Luo-YK, JHEP03(2024)020

Figure 6: θ and θ and θ and θ and θ at θ θ θ θ θ θ $m=\frac{8}{5}$ $\frac{8}{e} \Lambda_{\overline{\mathrm{MS}}} = 64 \Lambda_L = \frac{128 \pi}{a}$ $S = \frac{S}{\sqrt{N}} = \frac{S}{\sqrt{N}} = \frac{S}{\sqrt{N}} = \frac{128\pi}{\sqrt{N}} = 9.8 \text{ N} = 1.070$ entropy control mass gan(two loon): $m = {}^8\Lambda_{\overline{\rm MS}} = 64 \Lambda_L = \frac{128\pi}{\beta} \beta \exp{(-2\pi \beta)}$ c=1 97(9) (linear fit of the data in terms of 1*/D*cut with 1*/D*cut ⁰*.*02. The dependence of *^S*(2) mass gap(two loop):

where the two-loop expression for the two-loop expression for the beta function \mathbf{C} $\begin{array}{l} \hbox{N}{\bf c}}{\bf c} \end{array}$ and $\begin{array}{l} {\bf c}}{\bf c} \end{array}$ at *L* = 128 with *N^t* = 1024 is plotted in Fig. 6 together with *SA*. We extract the central von Neumann EE: $S_A = \frac{3}{3} (2\pi \beta - \ln \beta) + \text{const.}$ c=2.04(4)

n-th Rényi EE:
$$
S_A^{(n)} = \frac{c}{6} \left(1 + \frac{1}{n} \right) (2\pi\beta - \ln \beta) + \text{const.}
$$

 Bruckmann+, PRD99(2019)074501
c 2 by finite size spectrum w/TN

in terms of the constant of the constant μ at μ at μ at μ at μ at μ μ μ μ 1*.*4 1*.*7 with the function of Eq. (3.2), where the condition of ⇠ ⌧ *L* is well satisfied. central charge is determined from β dependence Ueda+, PF We repeat the same calculation for other *n*th-order R´enyi entropy. The *n* dependence central charge is determined from β dependence

Figure 11: *n* dependence of central charge *c* obtained from *n*th-ordr R´enyi (open) and exp (2⇡)*,* (3.1) c=1.97(9) (von Neumann EE) consistent btw different methods D_{m} n-th Rényi EE: $S_A^{(n)} = \frac{3}{6} \left(1 + \frac{1}{n} \right) (2\pi \beta - \ln \beta) + \text{const.}$ c \sim 2 by finite size spectrum w/ TNR c=2.04(4) by Matrix Product State (MPS) Ueda+, PRE106(2022)014104

S and θ *Quantum phase transition at* $\mu \neq 0$ expressions for this work to make this paper self-contained.

Luo-YK, arXiv:2406.08865 *s*(*n*) *· s*(*n* + ˆ⌫)*.* (2.1) **Example 2406.08865** Luo-YK, arXiv:2406.08865

 $(1+1)d$ lattice O(3) NLSM at $\mu \neq 0$ $(1+1)$ _d lottice $O(2)$ NI SM ot $u \neq 0$

$$
Z = \int_{\mathcal{D}} \mathcal{D}[s] e^{-S}
$$

\n
$$
S = -\beta \sum_{n \in \Lambda_{1+1}, \nu} \sum_{\lambda, \gamma=1}^{3} s_{\lambda}(\Omega_n) D_{\lambda \gamma}(\mu, \hat{\nu}) s_{\gamma}(\Omega_{n+\hat{\nu}})
$$

\n
$$
D(\mu, \hat{\nu}) = \begin{pmatrix} 1 \\ \cosh(\delta_{2,\nu}\mu) & -i \sinh(\delta_{2,\nu}\mu) \\ i \sinh(\delta_{2,\nu}\mu) & \cosh(\delta_{2,\nu}\mu) \end{pmatrix} \implies \boxed{\text{complex action}}
$$

\n
$$
s^T(\Omega) = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)
$$

\n
$$
\Omega = (\theta, \phi) \quad , \ \theta \in (0, \pi], \ \phi \in (0, 2\pi].
$$

 (θ, ϕ) is discretized w/ Gauss-Legendre quadrature \rightarrow TN representation Symmetry btw spatial and temporal directions is broken by μ \Rightarrow Spatial correlation length $(\xi) \neq$ Temporal correlation length (ξ_t) $\zeta_t = \xi^z$ with *z* dynamical critical exponent \overline{a} *D*⌦ (θ,ϕ) is discretized w/ Gauss-Legendre quadrature \rightarrow TN representation *p*=1 ith sin(1)^{*d*}_{*p*} *.* (2.5)^{*d*}_{*p*} *.* (2.5)^{*d*} י
I ≠ I7 + nøth i dil *n,*⌫ $e^{i\theta}$ *examples in the partition is determined as* $e^{i\theta}$ *in the partition* $e^{i\theta}$ *is* $e^{i\theta}$ *in* $e^{i\theta}$ *in* $e^{i\theta}$ *in* $e^{i\theta}$ *is* $e^{i\theta}$ *in* $e^{i\theta}$ *in* $e^{i\theta}$ *is* $e^{i\theta}$ *in* $e^{i\theta}$ *in* $e^{i\theta}$ *is* $e^{i\theta}$ *i* $\zeta_t = \xi^2$ with z dynamical critical exponent *<u>i*₂ Symmetry btw spatial and temporal directions is broken by</u> Z $\sqrt{2}$ *w*ith *z* dynamical critical exponent

Luo-YK, arXiv:2406.08865

Temporal Correlation Length (1) The definition of the definition of the trace in terms is defined by $\frac{1}{2}$, where $\frac{1}{2}$ ary in the trace in terms in the trace in terms in ter

Luo-YK, arXiv:2406.08865 of the spatial direction of the spatial direction of the legislature $\frac{1}{2}$ in the density matrix in the dens

 ξ_t is obtained from the eigenvalues of density matrix \mathcal{E}_t is obtained from the eigenvalues of density matrix

$$
\xi_t = \frac{N_t}{\ln\left(\frac{\lambda_0}{\lambda_1}\right)}
$$

 λ_0 and λ_1 are the largest and second largest eigenvalues

ex. density matrix on 4×4 lattice *xxyy*⁰ with *T*⇤ the reduced single tensor obtained by HOTRG.

Eigenvalues are calculated on reduced single tensor obtained by HOTRG

Temporal Correlation Length (2)

Luo-YK, arXiv:2406.08865

 ξ_t as a function of coarse-graining steps near μ_c

Plateau behaviors are observed for $L > \xi$ at sufficiently low temperature

Silver Temporal Correlation Length (3) in Fig. 2: the value of the value of the value of \sim 0.1455 and sudden in the subden increase of h

Luo-YK, arXiv:2406.08865 beyond *µ*c. Although the results with *D*cut = 125, 130 and 135 are almost degenerate,

The first successful calculation of dynamical critical exponent with TRG

Summary and Outlook (1)

<u>Entanglement Entropy(EE) of (1+1)d O(3) NLSM at μ =0</u>

- ・ Rényi EE(n=2) is not a good approximation of von Neumann EE(n=1)
- Difficult to extrapolate $S_A^{(n)}(n \geq 2)$ to n=1 at high precision
- ・ Central charge is successfully determined: c=1.97(9) (von Neumann EE)

Quantum Phase Transition of $(1+1)d$ O(3) NLSM at $\mu\neq 0$

- ・ Another evidence that TRG is free from the sign problem
- \cdot μ_c is consistent with mass gap at μ =0
- Dynamical critical exponent is successfully determined: $z=1.96(6)$

Advantages of TRG

・ Free from sign problem/complex action problem in MC method

 $Z = \int \mathcal{D}\phi \exp(-S_{\text{Re}}[\phi] + iS_{\text{Im}}[\phi])$

- **Computational cost for L^D system size ∝ D × log(L)**
- ・ Direct manipulation of Grassmann numbers <u>zinulati</u> *x* tion of G $\frac{1}{2}$ *x* (*x* + *x*) *x* + *x*
- ・ Direct evaluation of partition function Z itself ⎛

T (ϕ*x,*1*,* ϕ*x*+ˆ1*,*2*,* ϕ*x*+ˆ2*,*1*,* ϕ*x,*2) = exp

Current status : Calculations of 4d theories (scalar, fermion, gauge) are possible ⇒ Extension to Abelian (U(1)) and non-Abelian groups (SU(2), SU(3)) Non-perturbative study of Entanglement Entropy of various models **and non-Abelian (** *Ti,j,k,l* [⇒] *^T{j,k},{l,i}* ⁼ ' *U*Λ*V ^t {j,k},{l,i}* ⁼)

[⎝]β cos *px* + *i*

θ

qx

BACKUP

Zero Temperature Limit

Luo-YK, JHEP03(2024)020

Zero temperature limit with large Nt

Converged at $Nt \geq 64$ \Rightarrow Nt=1024 can be regarded as zero temperature limit *D*cut = 130.

Extrapolation of EE to Dcut→∞

Luo-YK, JHEP03(2024)020

Linear extrapolation in terms of 1/Dcut to 1/Dcut \rightarrow 0 (Dcut $\rightarrow \infty$) 1.7. Solid lines denote linear extrapolation. Figure 5: 1*/D*cut dependence of *SA*(*L* = 128) with *N^t* = 1024 at = 1*.*4, 1.5, 1.6 and 1.7.

