Grassmann Tensor Renormalization Group for $N_f = 2$ Schwinger model with a θ term

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Based on the work with
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(in preparation)

Why is a θ term important?

What is a θ term?

$$Z_{YM} = \int \mathcal{D}A \, \mathrm{e}^{-S_{YM}} \supset \sum_n \mathrm{e}^{in\theta}$$

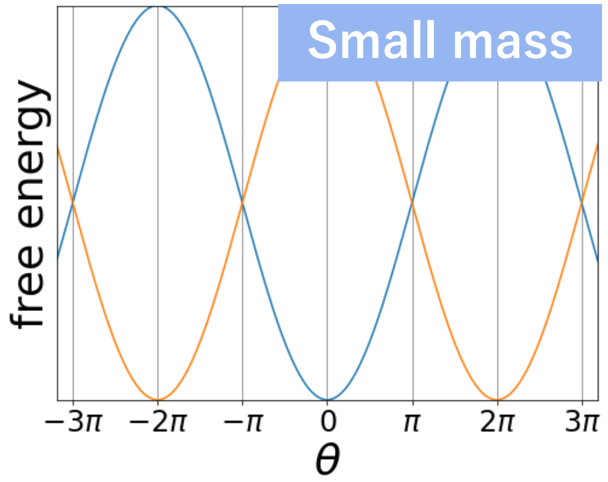
- A topological term in 4d **QCD** or Yang-Mills theory.
- Related to the instanton number.

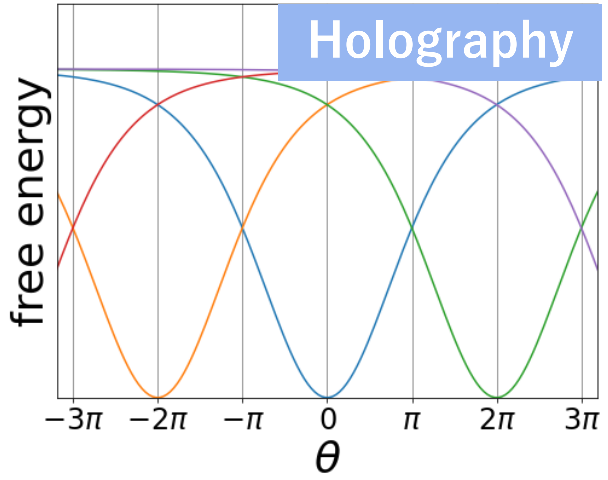
Strong CP problem

- In our world, $\theta < 10^{-10}$. (from neutron EDM experiments)
- Why is it too small? (**Strong CP problem**)

Axion [Peccei, Quinn 1977]

- Axion $\simeq$ scalar field which couples to the **QCD** like θ.
- Candidate for a dark matter
- Application to inflation; Natural inflation [Freese et al. 1990]
- Axion potential = **θ dependence of the free energy**
  ($\theta = 0 \pmod{2\pi}$) is the stable point)





How to calculate QCD with the θ term

Monte Carlo method: the sign problem

$$Z_{YM} = \int \mathcal{D}A \, \mathrm{e}^{-S_{YM}} \supset \sum_n \mathrm{e}^{in\theta}$$

- With finite θ, the partition function includes imaginary part.

→ The Monte Carlo simulation does not work well.

- There are some studies by the Monte Carlo.
  e.g.) 4d $SU(2)$ YM theory [Kitano et al. 2102.08784]
- But, $\theta = \pi$ point is tough…

Tensor network methods do not have the sign problem!

- It is hard to use tensor network methods for 4d QCD.
- However, tensor networks work well for 2d theories.

→ We calculate the Schwinger model (2d toy model of the QCD) by tensor renormalization group (TRG). [Levin, Nave 2007]

Plan

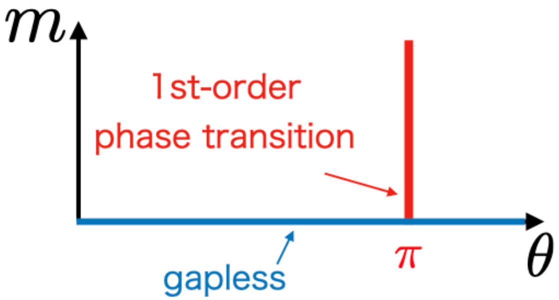
1. Introduction (2)
   - Why is the θ term important?
   - How to calculate QCD with θ
2. Schwinger model (2)
   - What is the Schwinger model?
   - θ dependence of the free energy
3. TRG (3)
   - TRG
   - Lattice action
4. Results (7)
   - Our set up
   - 2π periodicity
   - Large mass limit
   - Small mass limit
   - Intermediate mass
5. Conclusion (1)

What is the Schwinger model? [Schwinger 1962, Coleman 1976, ⋯]

Schwinger model = 2d QED

$$S = \int \mathrm{d}^2x \left\{ \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \bar{\psi}\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}A_\mu)\psi + m\bar{\psi}\psi \right\}$$

- $U(1)$ gauge theory + (fundamental) fermions (2dim $U(1)$ gauge theory has a strong coupling.)
- Chiral symmetry @massless : $\frac{U(N_f)_L \times U(N_f)_R}{U(1)_V \times U(1)_A}$ (N_f : Number of flavor)
- θ term, instanton ($\pi_1(U(1)) \simeq \mathbb{Z}$) $\quad \frac{i}{2\pi}\int F = n \in \mathbb{Z}$
- Fermion mass (Here, we take $m \geq 0$)
  - $m = 0$: Chiral symmetry, θ is unphysical (through the $U(1)_A$ anomaly)
  - $m \neq 0$: θ is a physical parameter
- First order phase transition @$\theta = \pi$ (for $N_f \geq 2$)



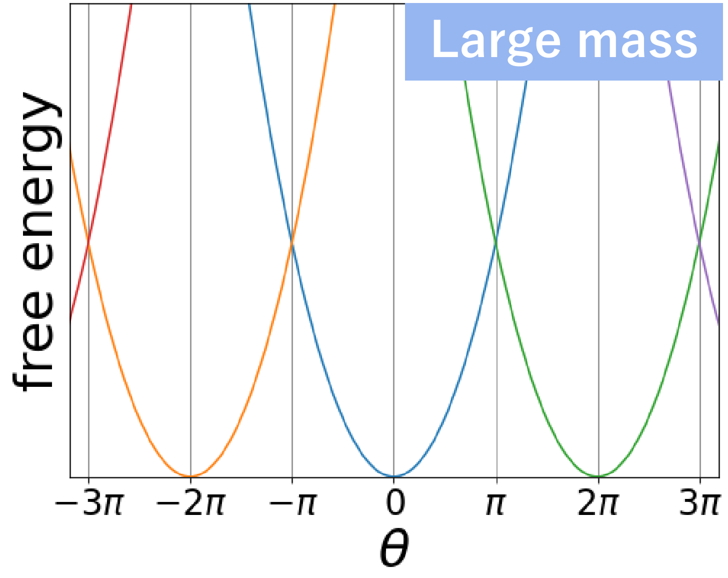
→ Most **QCD**-like theory in 2dim! : How much **QCD** like?

θ dependence of the free energy

Large mass

- $U(1)$ Maxwell theory, gapped
- No propagating d.o.f., we can solve it by hand.
- Free energy :

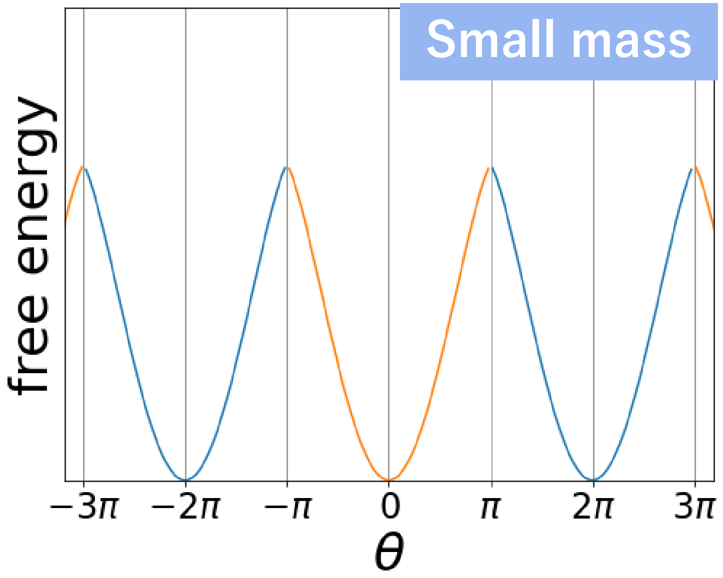
$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \frac{1}{8\pi^2}(\theta - 2\pi n)^2$$



Small mass ($N_f = 2$ case) [Coleman 1976]

- IR EFT is a pion theory ($SU(N_f)_1$ WZW model + mass term)
- The free energy can be calculated by mass perturbation.
- Free energy :

$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \left\{ (e^\gamma)^{\frac{4}{3}} \pi^{-\frac{5}{3}} 2^{\frac{1}{3}} \left(\frac{m^2}{g^2}\right)^{\frac{2}{3}} \cos^{\frac{4}{3}}\left(\frac{\theta - 2\pi n}{2}\right) \right\}$$



→This nature is very similar to the 4d **QCD**!

Unknown part

- The free energy of the **intermediate mass** regime

TRG

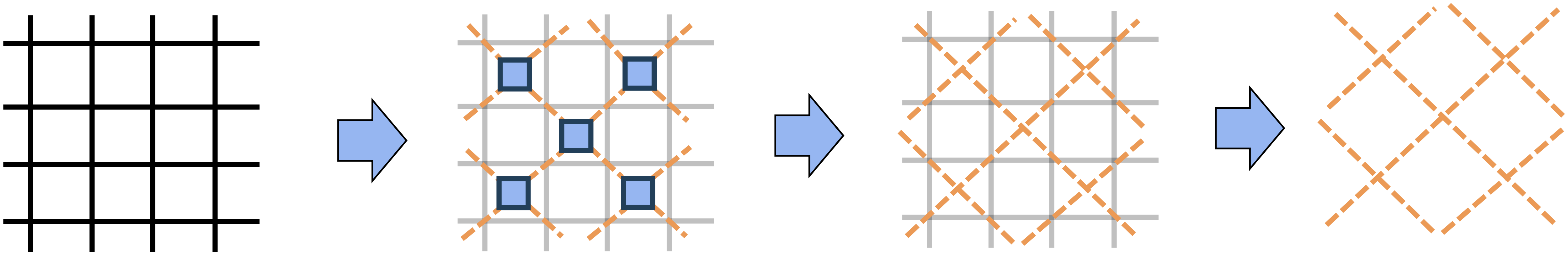
Tensor Renormalization Group

TRG

[Levin, Nave 2007]

Real space renormalization for one initial tensor

- From the translation invariance, we just focus on single tensor.
- Singular value decomposition (SVD)
  - Finite cut off for singular values : bond dimension
  - Approximation for TRG.
- Grassmann-TRG [Gu, Verstraete, Wen 1004.2563]
  - Fermion has less d.o.f. by Grassmann path integral.



Lattice action for Schwinger model

$$S = \sum_{n,\mu} \Bigg[-\frac{1}{g^2}\cos(A_p(n)) - \frac{i\theta}{2\pi}\tilde{A}_p(n)$$
$$+\frac{1}{2}\Big[\eta_\mu(n)\{\bar{\chi}(n)U_\mu(n)\chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu})U^\dagger_\mu(n)\chi(n)\} + m\bar{\chi}(n)\chi(n)\Big]\Bigg]$$

Staggered fermion

- 2d one staggered fermion
  ↔ 2-flavor Dirac fermion
- Lattice translation symmetry is broken
  - To use the staggered fermion, 2 sites for each direction is needed.
  - 2×2 lattice ↔ one unit site
  - To treat staggered fermion, we use two different tensors for initial tensors.
  - After 2-step iteration, the tensor will be unique.

Gauge field

$$\tilde{A}_p(n) = -i \log U_p(n)$$

- $\log U_p$ type θ term (2π periodicity of θ is realized.)
- We use Gauss-Legendre quadrature to discretize gauge field.

[Kuramashi, Yoshimura 1911.06480] (TRG for 2dim Maxwell with the θ term)

Previous studies for lattice Schwinger model with θ

Lagrange formalism

- Monte Carlo
  - (with sign problem) [Fukaya Onogi 0305004]
  - (Worldline formalism) [Gattringer et al. 1502.05476, 1708.00649]
  - (Bosonized theory) [Ohata 2303.05481]
- TRG
  - 1-flavor Schwinger is studied in [Shimizu, Kuramashi 1403.0642]. (Wilson fermions)
  - 2-flavor is studied in [Butt et al. 1911.01285], but massless. (Worldline formalism)

Hamilton formalism

- Quantum computation

  [Chakraborty et al. 2001.00485][Honda et al. 2105.03276][Honda et al. 2110.14105]
- DMRG [Angelides et al. 2303.11016][Dempsey et al. 2305.04437][Itou et al. 2307.16655]...
  - Some collaborations are studying the Schwinger model.
  - Germany, USA, Japan, ...

cf.) A. Matsumoto's talk (Quantum computing session, this morning)

Results

Our set up

$N_f = 2$ Schwinger model on lattice

- Parameters of this theory
  - β (inverse of the gauge coupling, $\beta = 1/(a^2 g^2)$)
  - m_0 (mass of the lattice fermion)
- Parameters of discretization
  - D (bond dimension)
  - K (number of points in Gauss-Legendre quadrature)
  - L^2 (volume) ← depends on the iteration number in TRG algorithm.
- Continuum limit
  - $\beta \to \infty$ with fixed β/L^2, βm_0^2
  - $D \to \infty$, $K \to \infty$
- What we calculate
  - Free energy density (dimensionless, normalized by the value of $\theta = 0$.)

$$f = -\left\{\frac{\log Z(\theta)}{g^2 V} - \frac{\log Z(\theta=0)}{g^2 V}\right\} = -\left\{\beta \frac{\log Z(\theta)}{L^2} - \beta \frac{\log Z(\theta=0)}{L^2}\right\}$$

Our set up

$N_f = 2$ Schwinger model on lattice ← parameters in this talk

- Parameters of this theory
  - β (inverse of the gauge coupling, $\beta = 1/(a^2 g^2)$) ← $\boldsymbol{\beta = 4}$
  - m_0 (mass of the lattice fermion)
- Parameters of discretization
  - D (bond dimension) ← $\boldsymbol{D = 120}$
  - K (number of points in Gauss-Legendre quadrature) ← $\boldsymbol{K = 25}$
  - L^2 (volume) ← depends on the iteration number in TRG algorithm. ← $\boldsymbol{L^2 = 2^{32}}$
- Continuum limit
  - $\beta \to \infty$ with fixed β/L^2, βm_0^2
  - $D \to \infty$, $K \to \infty$

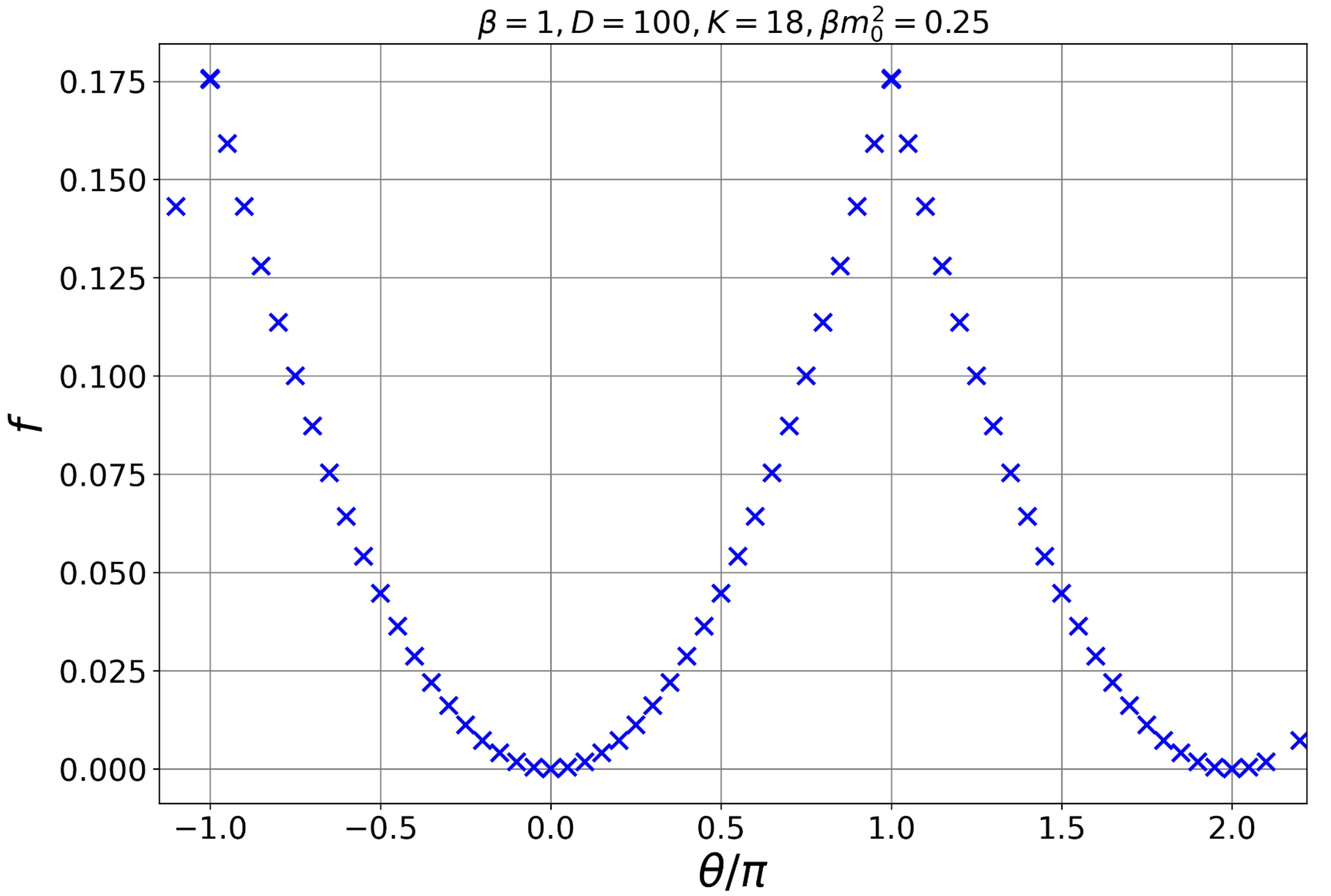
We take a very large volume, where the value of the free energy stabilize for the volume.

- What we calculate
  - Free energy density (dimensionless, normalized by the value of $\theta = 0$.)

$$f = -\left\{\frac{\log Z(\theta)}{g^2 V} - \frac{\log Z(\theta=0)}{g^2 V}\right\} = -\left\{\beta\frac{\log Z(\theta)}{L^2} - \beta\frac{\log Z(\theta=0)}{L^2}\right\}$$

2π periodicity

- Plot for free energy density vs θ
- 2π periodicity is obvious.

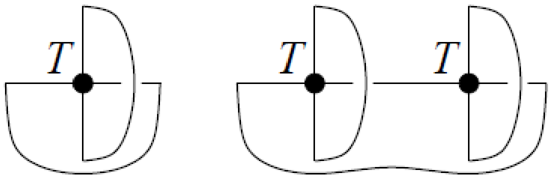


Degeneracy

Ground state degeneracy in TRG

- We can calculate ground state (or vacuum) degeneracy in TRG. [Gu, Wen 0903.1069]

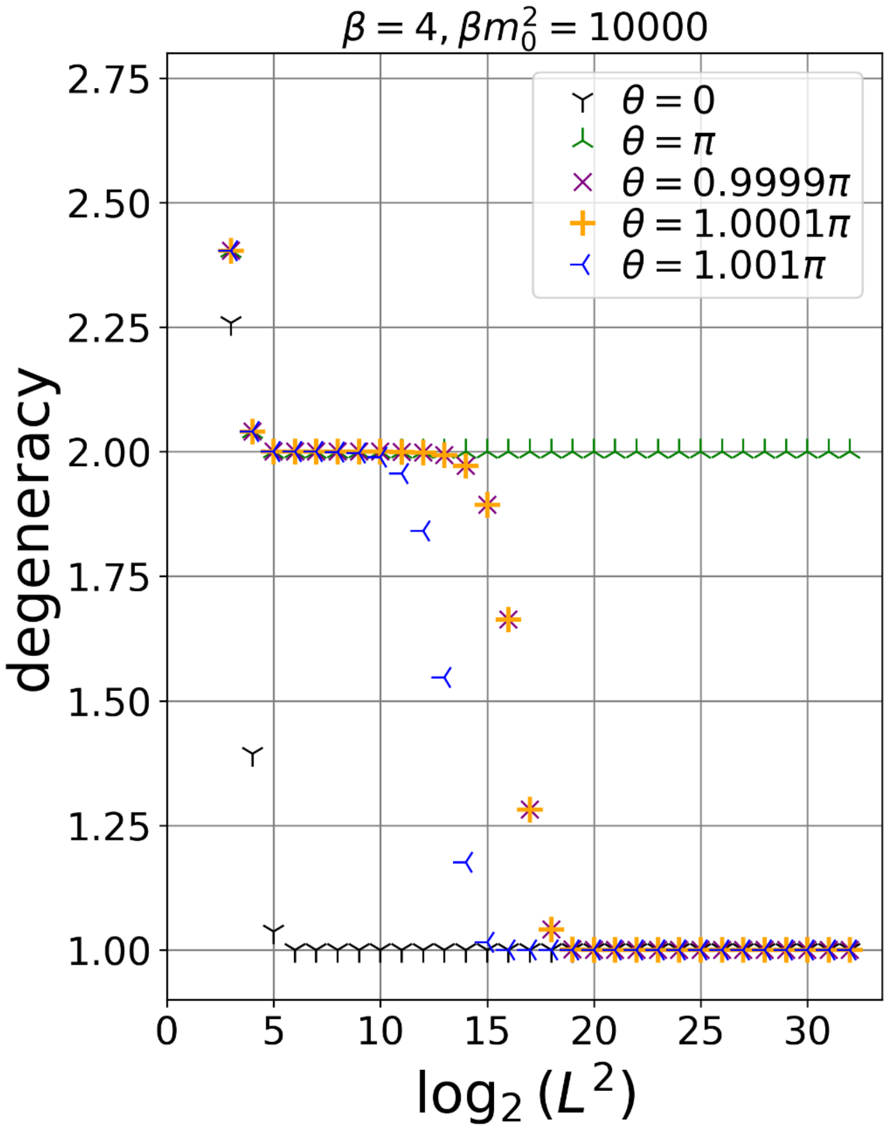
$$X_1 = \frac{(\sum_{ru} T_{ruru})^2}{\sum_{ruld} T_{rulu} T_{ldrd}} = \frac{(a)^2}{(b)}$$



- We checked 2-vacua degeneracy at $\theta = \pi$ for large mass parameters.
- $\theta = \pi \pm 0.0001\pi$ shows a single vacuum!

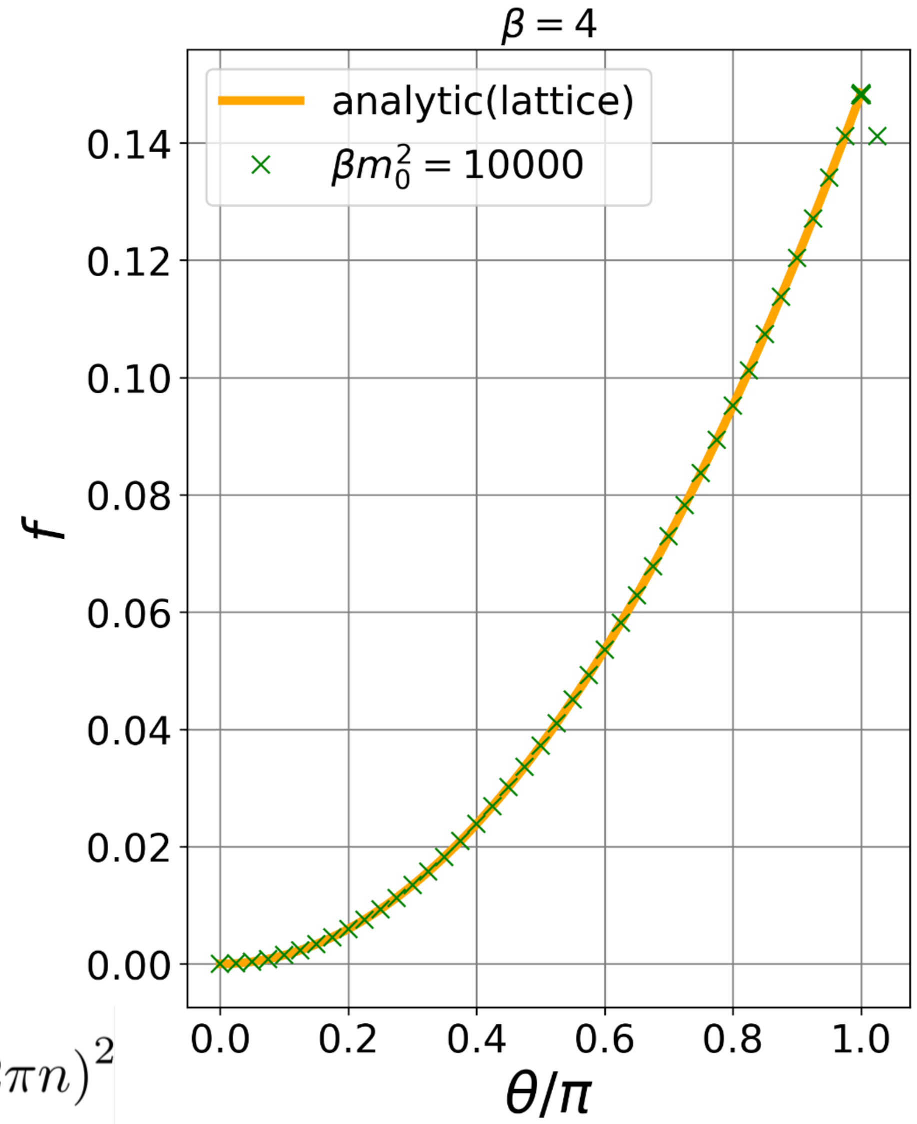
$\rightarrow$ 2π periodicity is obvious!

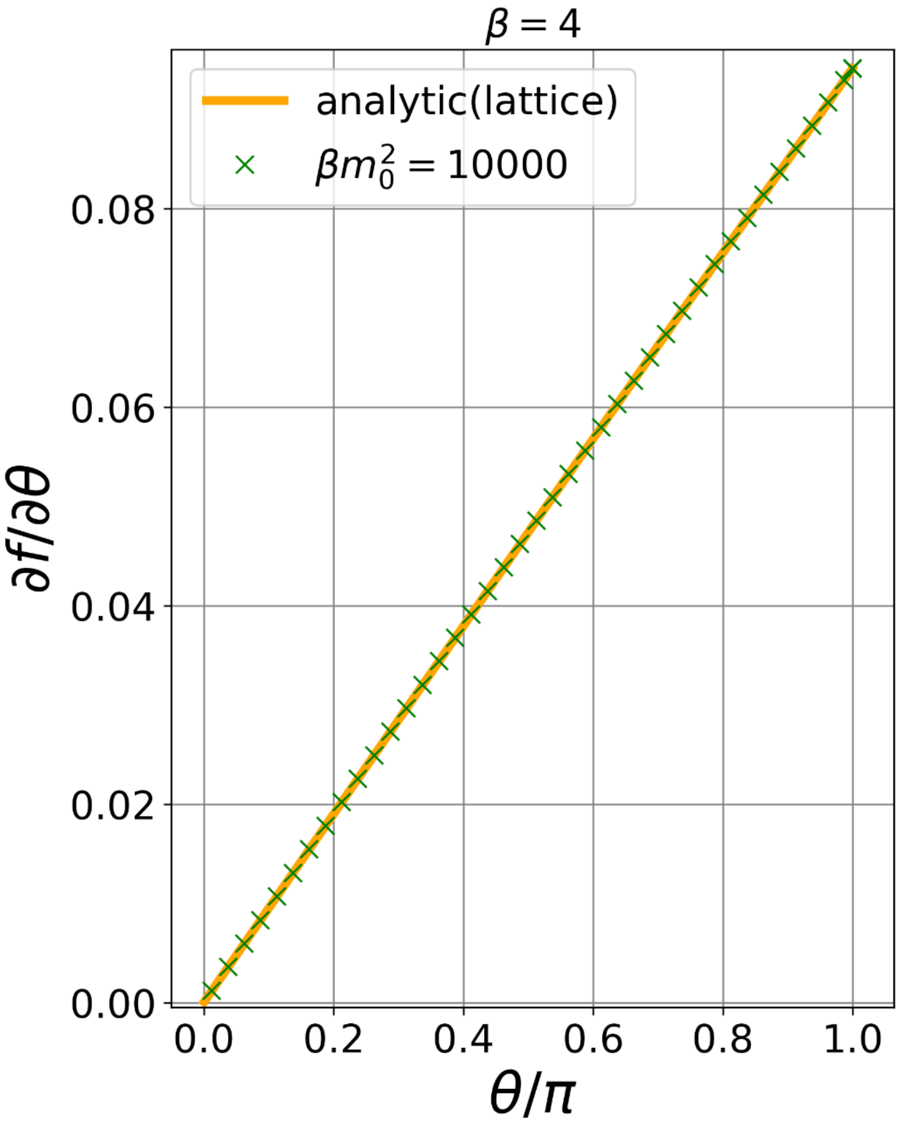
- In the following parts, we just focus on $\theta \in [0, \pi]$



Large mass limit

- Consistent with the analytic values of lattice action.
- The analytic value of the finite lattice spacing is also calculable. (slightly different from the continuum value)





$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \frac{1}{8\pi^2}(\theta - 2\pi n)^2$$

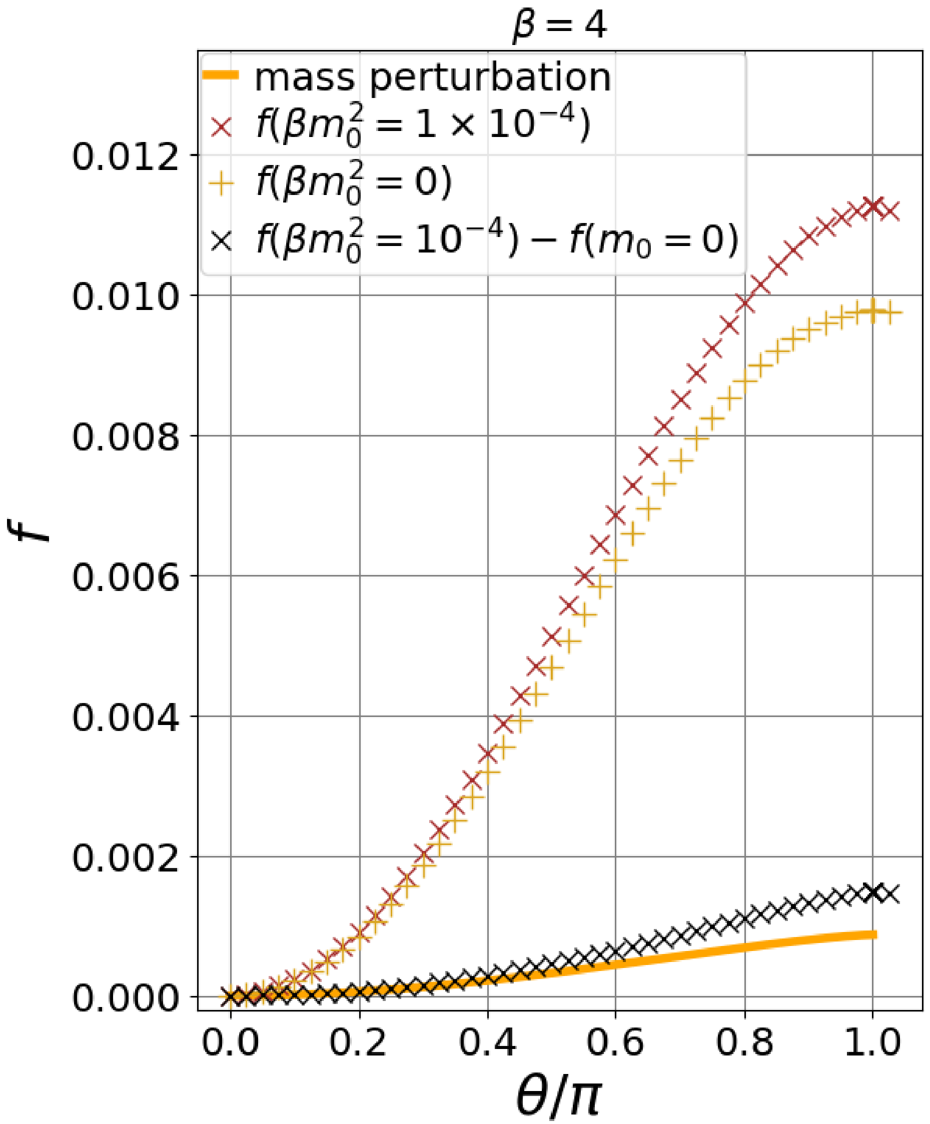
Small mass limit (1)

- There is a lattice artifact @ $\beta m_0^2 = 0$.
  - The free energy should not depend on θ in the massless case. (from the $U(1)_A$ anomaly)
  - This artifact will disappear in $\beta \to \infty$ limit.
  - The numerical results in a small mass cannot fit to the mass perturbation line.

- We subtract the lattice artifact from small mass results.

$$f(m_0) - f(m_0 = 0)$$

  - The subtracted result gets closer to the mass perturbation line. (**Black plots**)
  - The subtracted result is not consistent enough to the mass perturbation line.



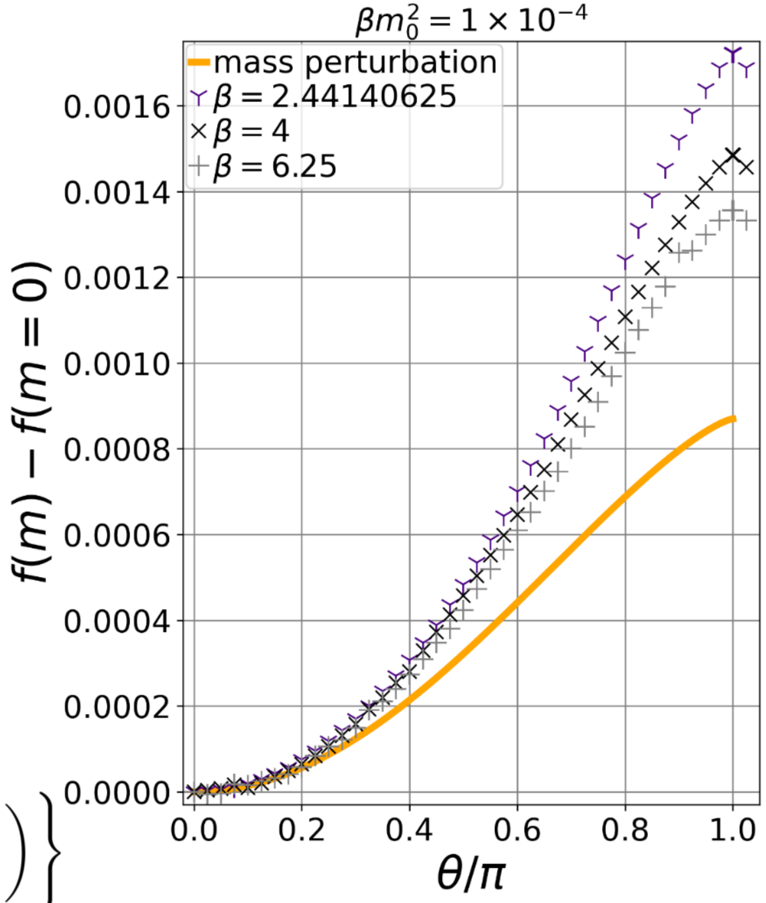
Small mass limit (2)

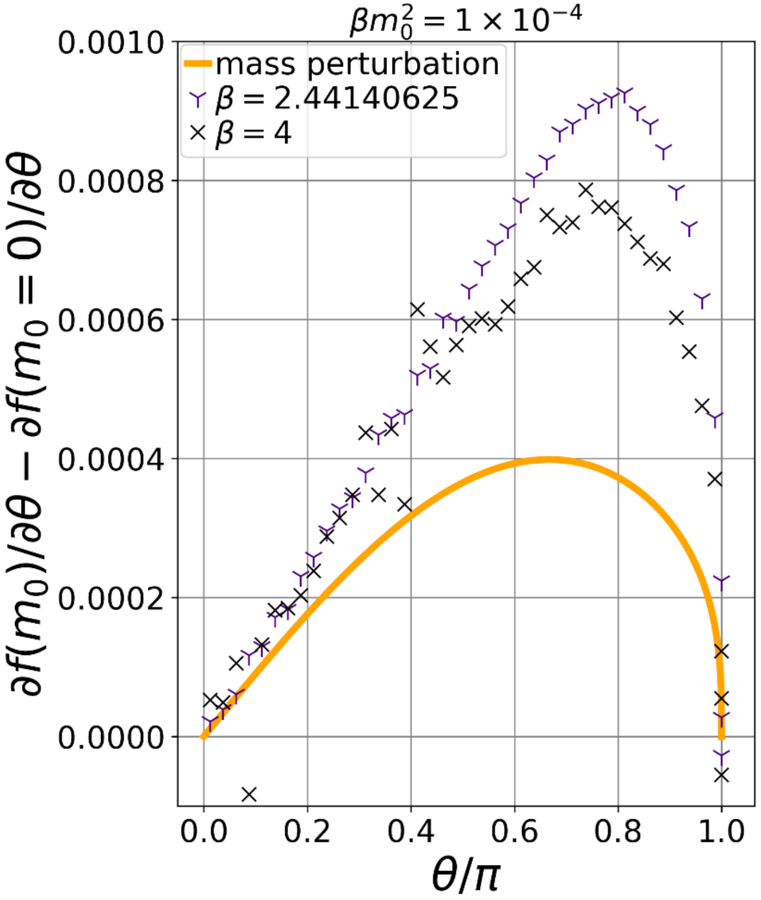
Check of the finite β effect

- In larger β, the numerical results are getting closer to the mass perturbation.
- The finite β effect seems to be severe in this parameter regime.
- In our calculation in $\beta = 4$, **we found discrepancy between our results and the mass perturbation** for any small mass parameters.

→ **Larger β calculations are required** to check the consistency with the mass perturbation. (For larger β, larger bond dimensions D are required.)

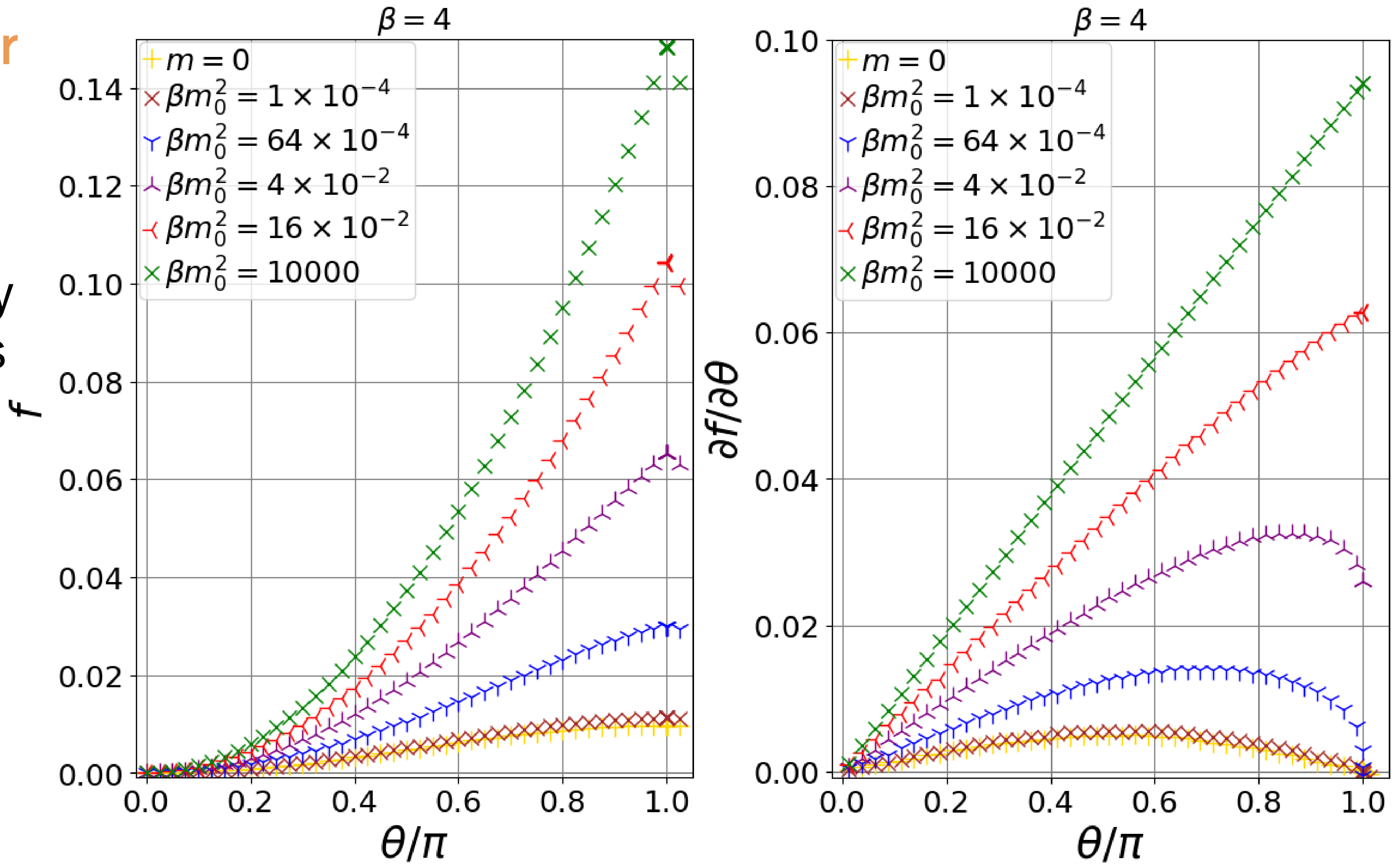
$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \left\{ (e^\gamma)^{\frac{4}{3}} \pi^{-\frac{5}{3}} 2^{\frac{1}{3}} \left(\frac{m^2}{g^2}\right)^{\frac{2}{3}} \cos^{\frac{4}{3}} \left(\frac{\theta - 2\pi n}{2}\right) \right\}$$





Intermediate mass

- Unknown parameter region by any analytical methods.
- The free energies change smoothly by the change of mass parameters
- In small mass regime, the lattice artifact exists.



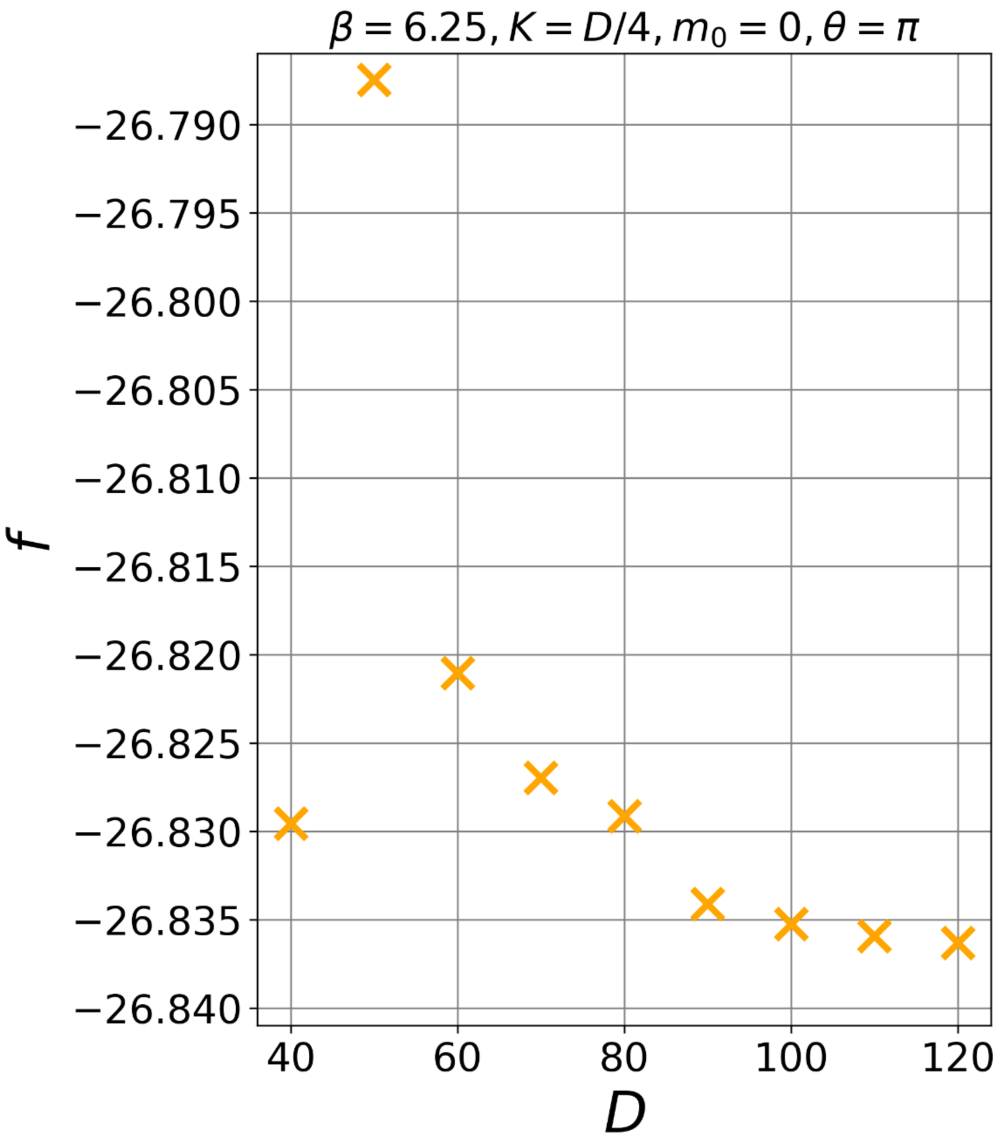
Conclusion

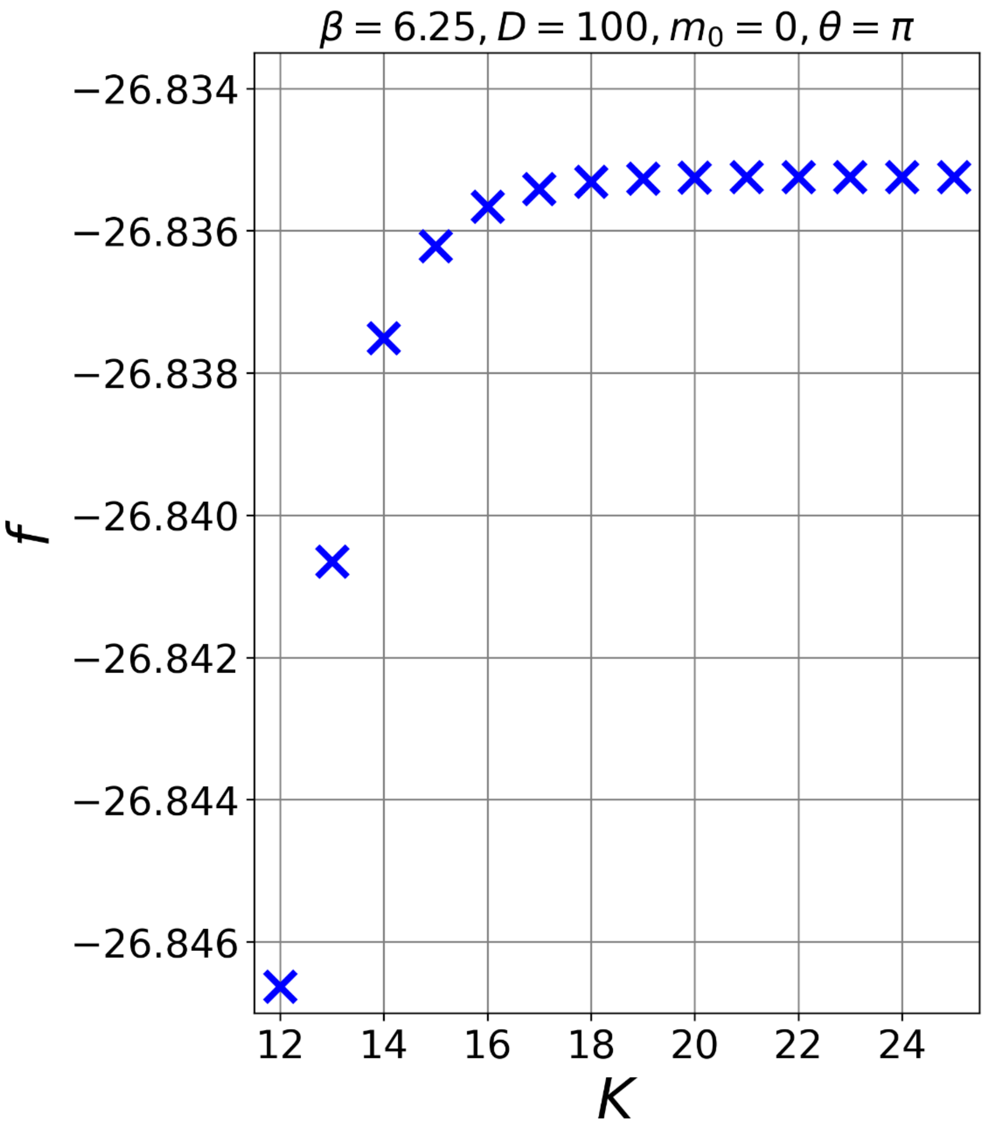
$N_f = 2$ Schwinger model in TRG

- **Schwinger model** : 2dim QED
  - 4dim **QCD**-like theory (chiral sym, vacuum structure, …)
  - **θ dependence of free energy** is also similar.
  - Good to calculate by TRG (Smaller d.o.f. than 4dim theory)
- We calculated θ dependence of the free energy by Grassmann-TRG.
  - **2π periodicity** of θ is obvious.
  - Large mass region is consistent.
  - Small mass region is not consistent enough. (finite β effect)
  - Finite mass effects for the intermediate mass regime.
- Future directions
  - Larger β calculations for small mass parameters (To check the consistency with the mass perturbation.) → Larger D calculation is required!

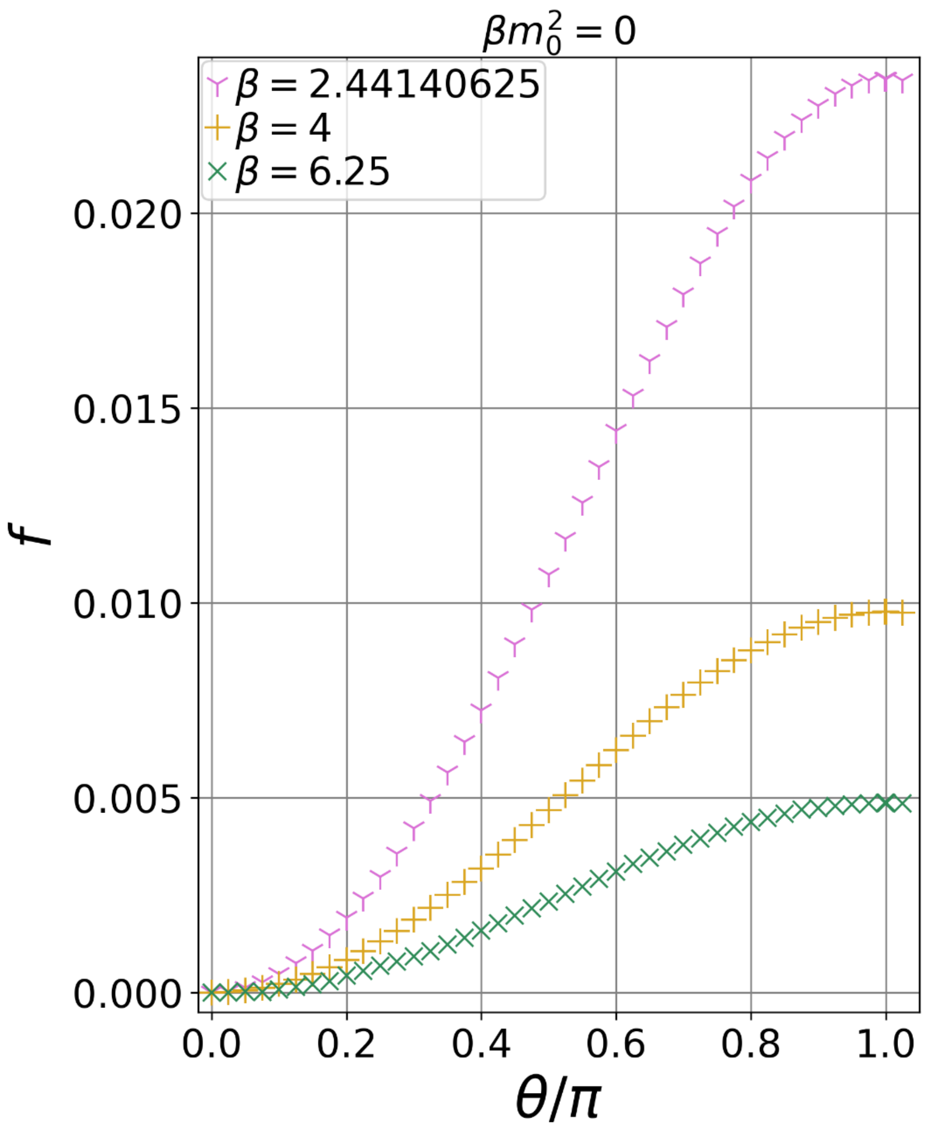
Back up

D, K dependence





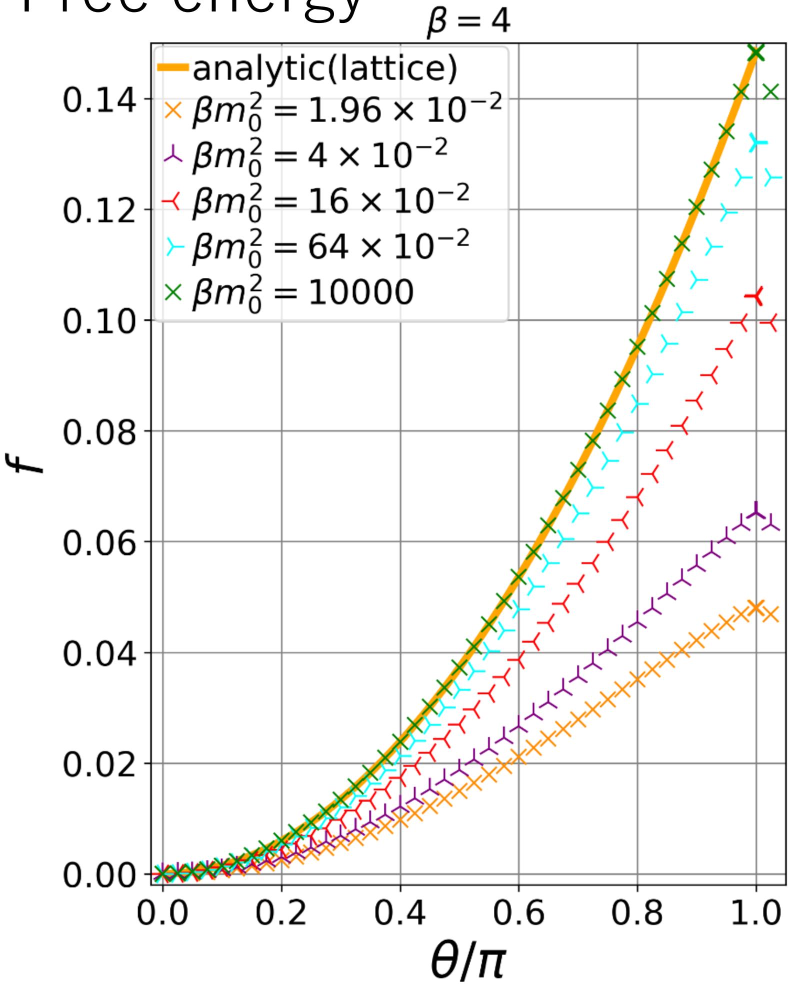
β dependence of the lattice artifact



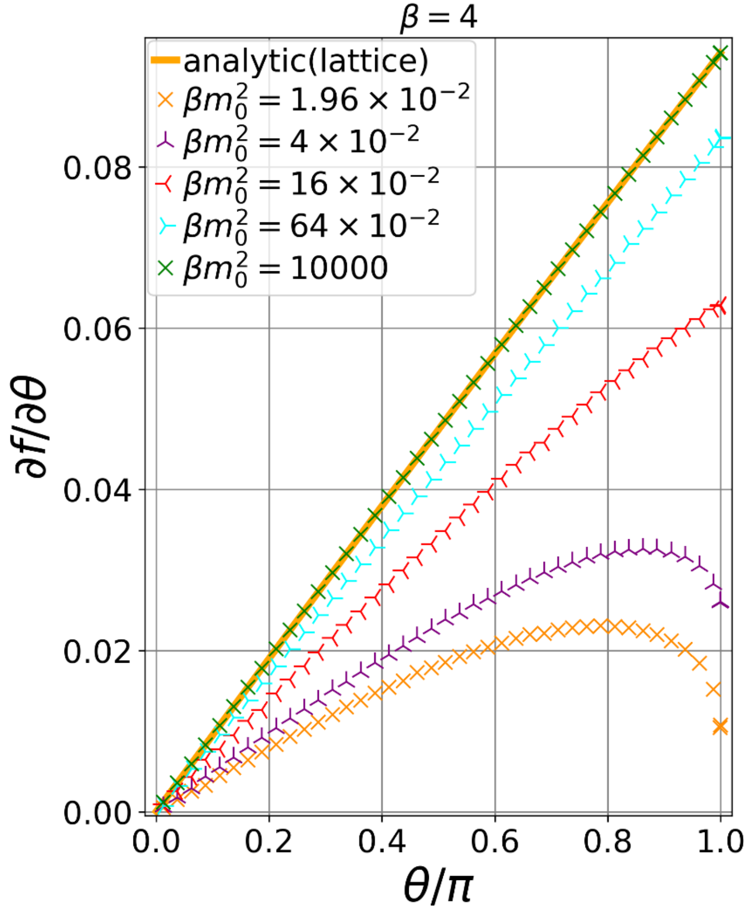
Intermediate mass (2)

Large mass regime

Free energy



Topological charge



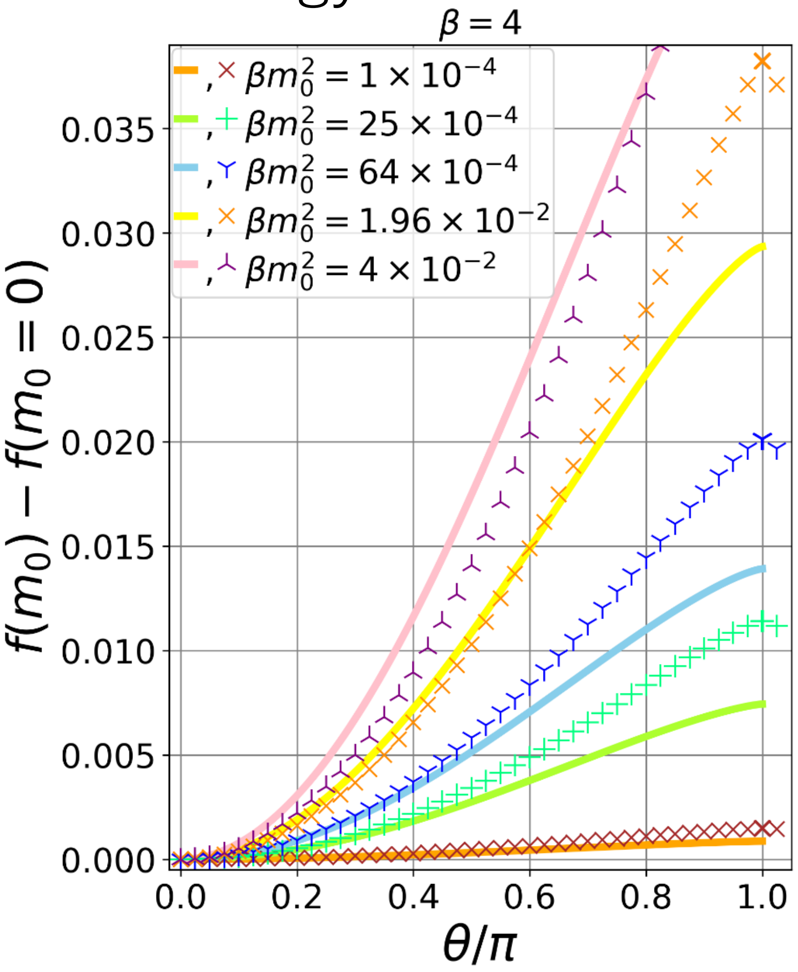
- Without the subtraction
- The plot of the topological charge $(\partial f/\partial\theta)$ is almost on straight lines in the large mass.
- The finite mass effect seems to appear as the different slope in the right figure.

$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \frac{1}{8\pi^2} (\theta - 2\pi n)^2$$

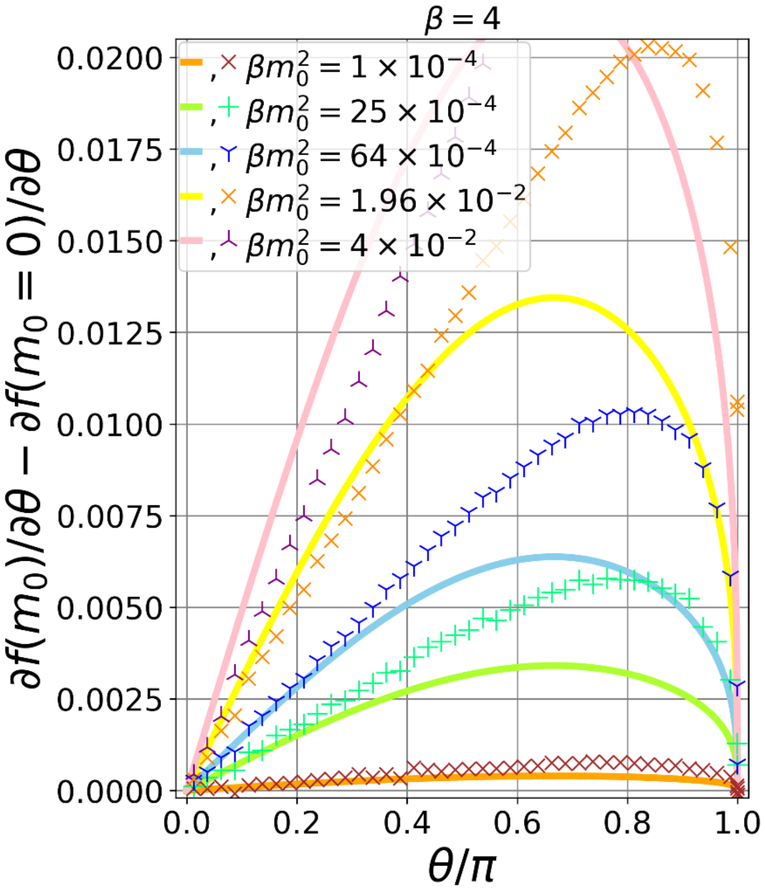
Intermediate mass (3)

Small mass regime

Free energy



Topological charge

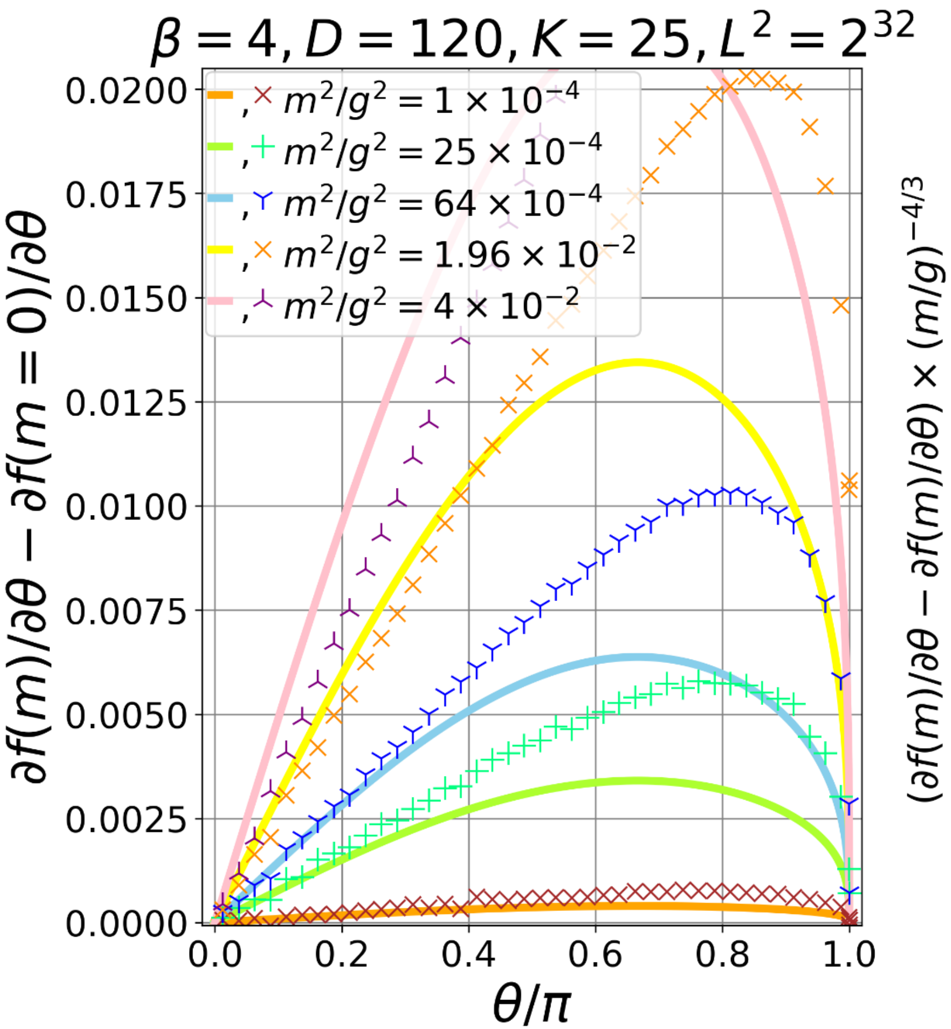


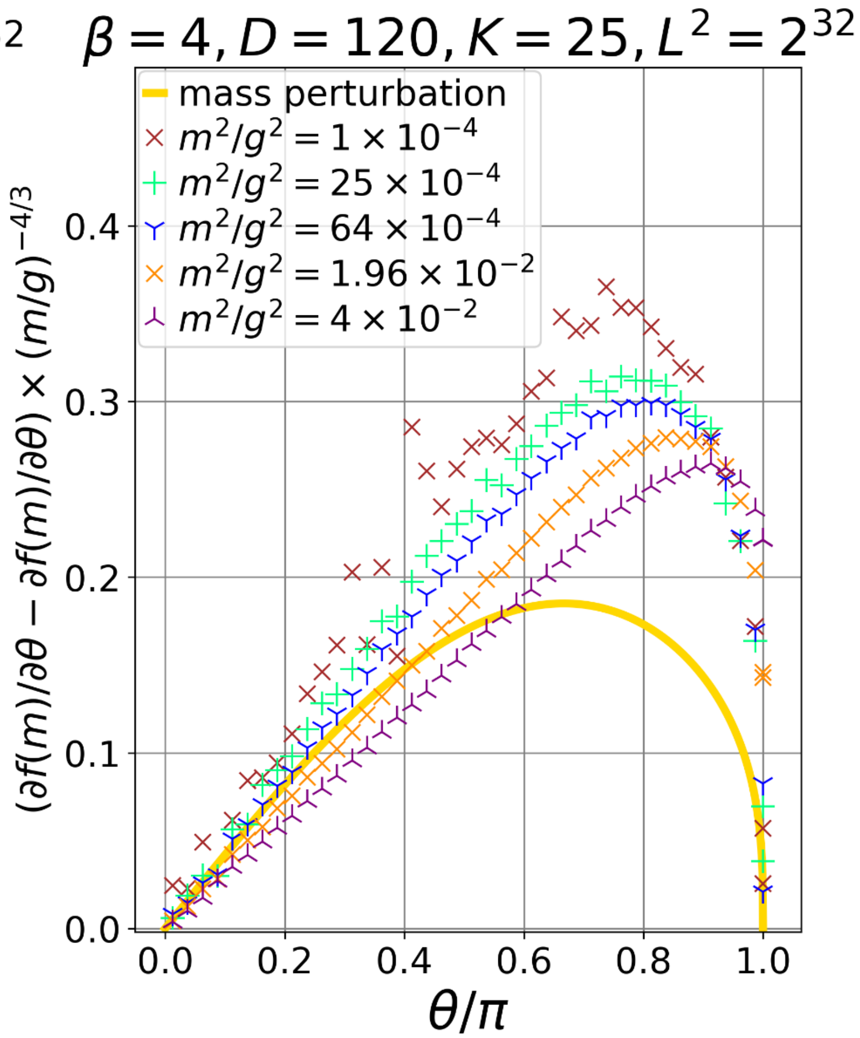
- **With the subtraction**
- We compare to the solid lines (mass perturbation lines).
- The consistency with the mass perturbation is not good in any mass parameters.
- Is this a finite mass effect or a finite β effect?

$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \left\{ (e^{\gamma})^{\frac{4}{3}} \pi^{-\frac{5}{3}} 2^{\frac{1}{3}} \left(\frac{m^2}{g^2}\right)^{\frac{2}{3}} \cos^{\frac{4}{3}}\left(\frac{\theta - 2\pi n}{2}\right) \right\}$$

Intermediate mass (4)

Small mass regime



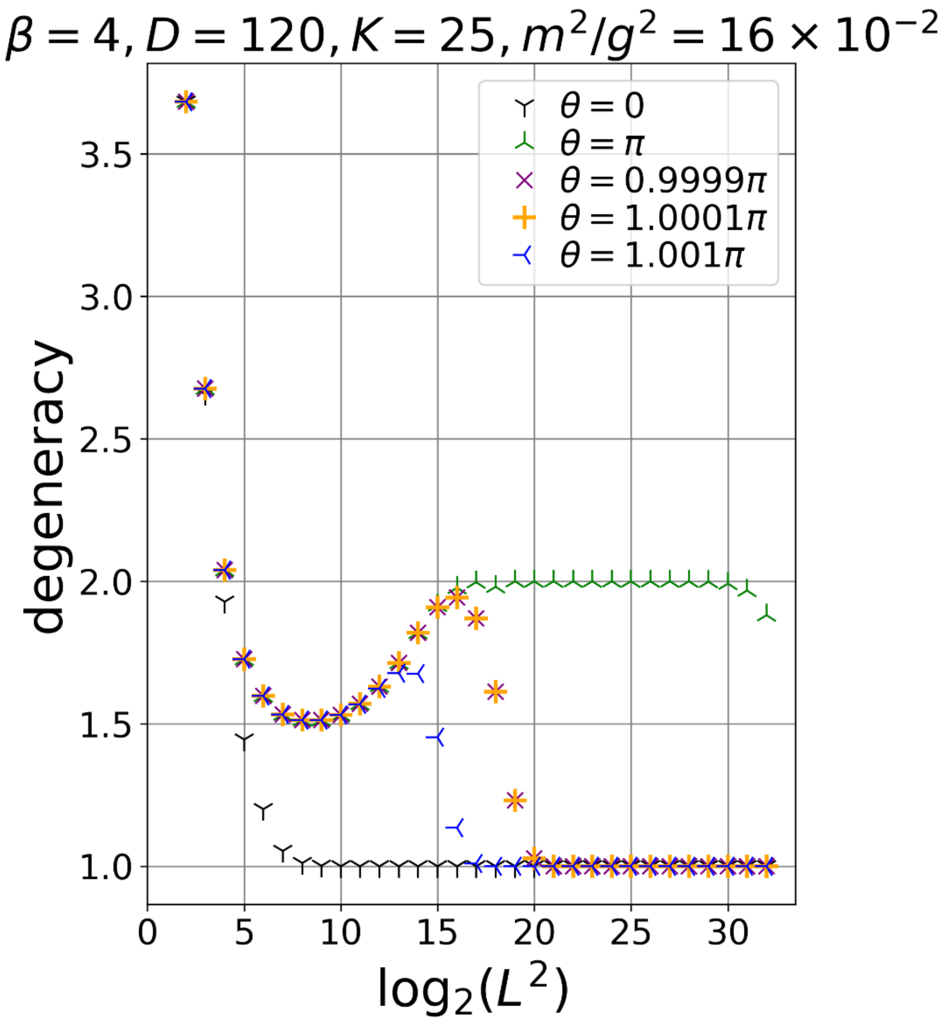


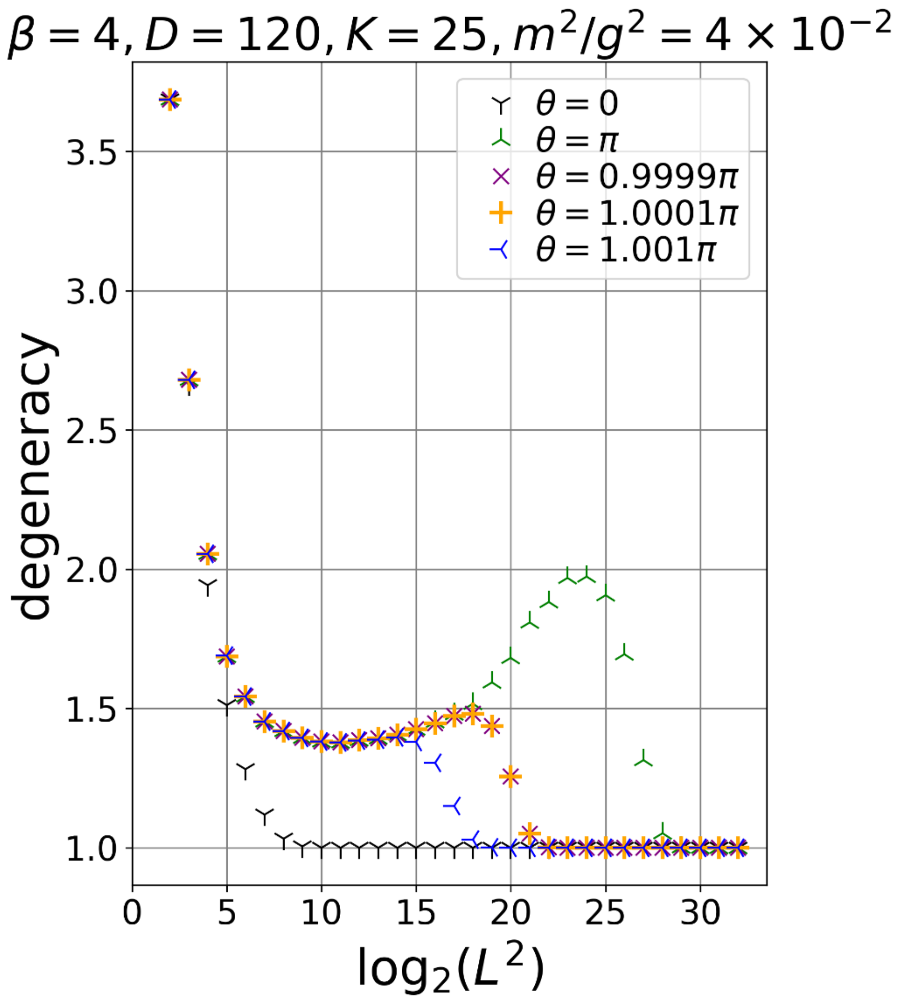
- The plot of $\partial f/\partial\theta$ and its plot scaled by mass.
- The places of the maximum values are different.

  (It is constant in the mass perturbation.)
- The mass scaling of the numerical results are different from the mass perturbation.

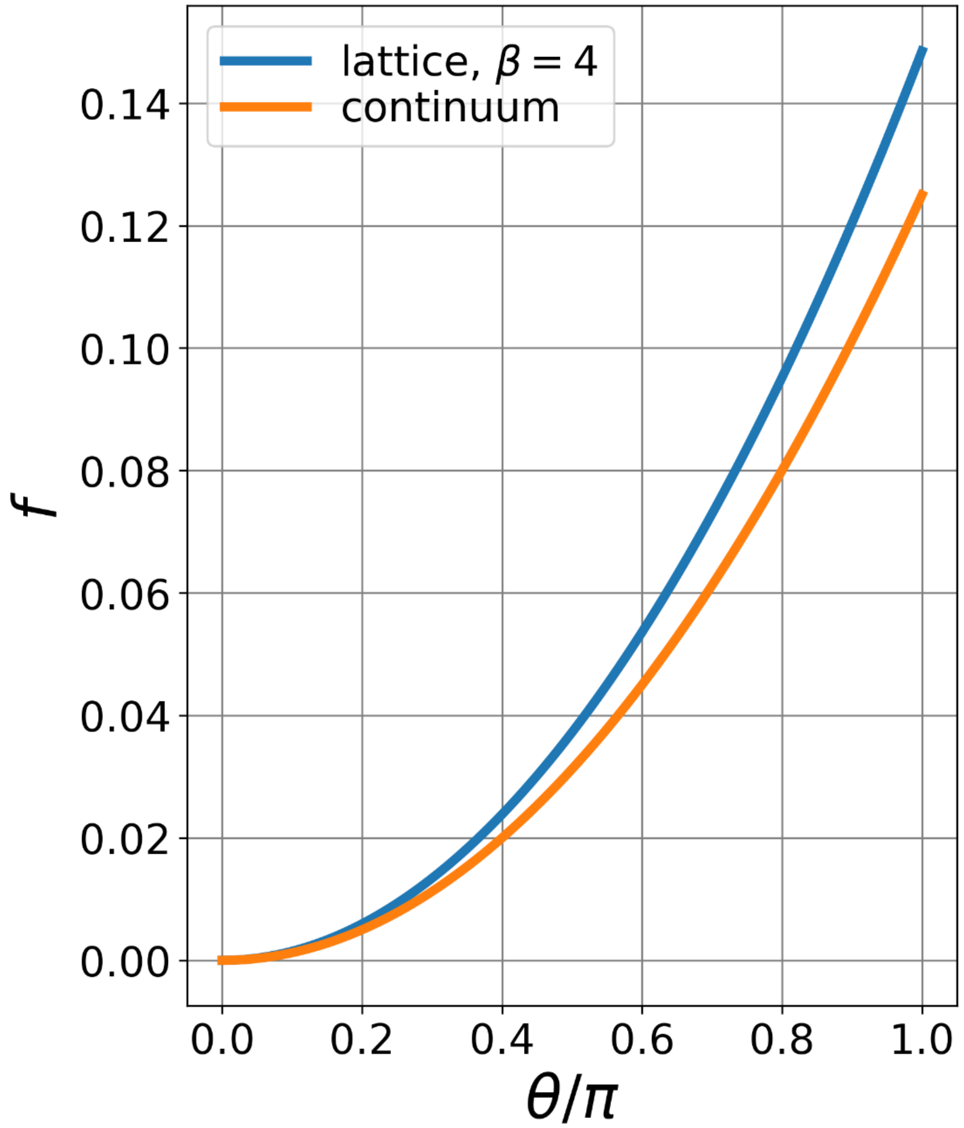
Degeneracy (2)

- We could not saw two vacua in $m^2/g^2 \lesssim 10^{-3}$. → Finite D effect?

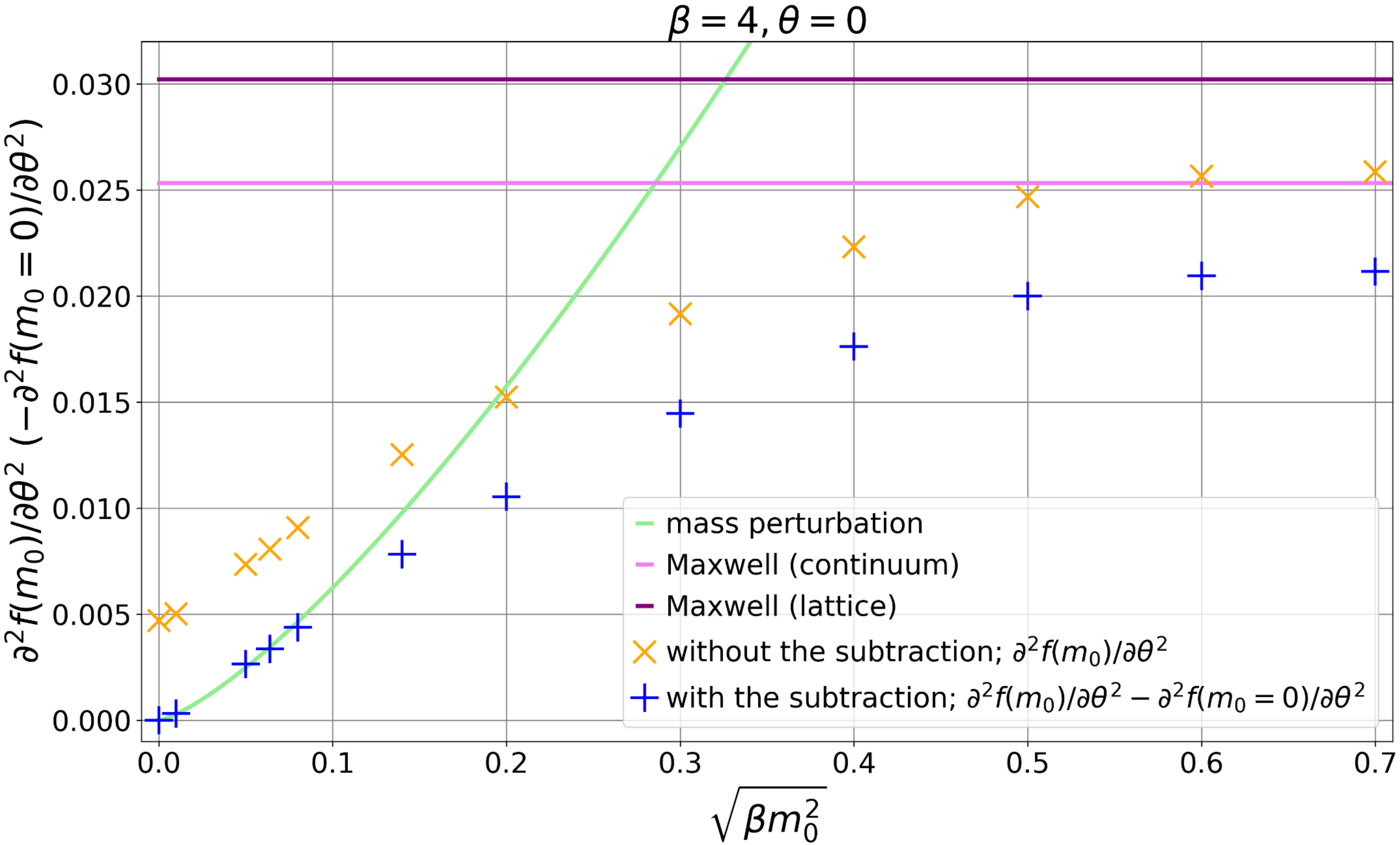




Large mass limit



Topological susceptibility @$\theta = 0$

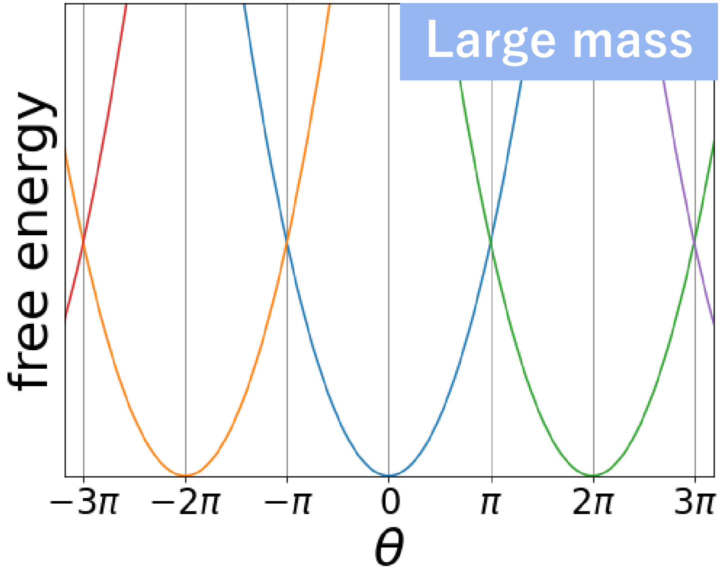


θ dependence of the free energy (1)

4d **QCD** ($N_f \geq 2$ case)

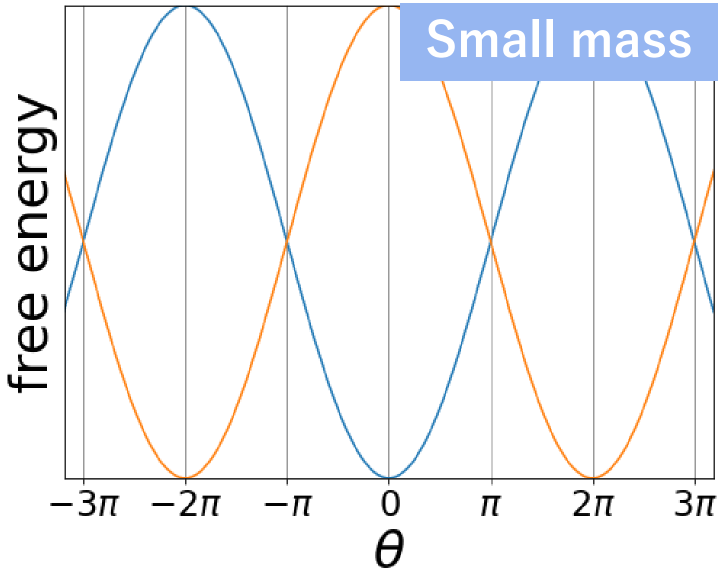
Large mass

- $SU(N)$ Yang-Mills theory, gapped
- Free energy can be calculated in large N. [Witten 1980]
- Free energy : $$F = -\log Z[\theta] \propto \min_k (\theta + 2\pi k)^2$$



Small mass

- IR EFT is a pion theory ($SU(N_f)$ non-linear sigma model)
- For a fermion mass, the pion mass can be introduced perturbatively.
  $$U = e^{i\pi(x)} \in U(N_f)$$
  - Mass term: $m\bar{\psi}\psi \quad \leftrightarrow \quad \mathrm{tr}[mU + U^\dagger m^\dagger]$
  - Free energy : $$F = -\log Z[\theta] \propto -\min_k |m| \cos\left(\frac{\theta + 2\pi k}{N_f}\right)$$



Pion theory (1)

IR effective field theory (EFT) for 4d QCD (massless)

- SSB : $\frac{U(N_f)_L \times U(N_f)_R}{U(1)_A} \to U(N_f)_V$
- NG boson = pion $\in \frac{U(N_f)_L \times U(N_f)_R}{U(1)_A \times U(N_f)_V} \sim SU(N_f)$
- Pion theory : non-linear sigma model (with $SU(N_f)_{N_c}$ WZW term)

$$S_\pi = \int d^4x \, \frac{f_\pi^2}{4} \mathrm{tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] - \int \frac{N_c}{240\pi^2} \mathrm{tr}\left[\left(U^\dagger dU\right)^5\right]$$

$$U = \mathrm{e}^{i\frac{\pi(x)}{f_\pi}} \in SU(N_f)$$

f_π : the pion decay constant (∼QCD scale → Not CFT)

- In 2d, no SSB (Coleman-Mermin-Wagner theorem). What happened?

Pion theory (2)

Bosonization (for the Schwinger model)

- In 2d, we can use the bosonization technique. [Coleman 1976],[Witten 1984]
  - fermion ↔ boson
- Bosonized Schwinger model = pion theory
  - $SU(N_f)$ NLSM with WZW term + η' meson + $U(1)$ gauge theory

$$S=\int d^2x\left\{\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+\frac{iN_f}{4\pi}(\phi+\theta)\,\epsilon^{\mu\nu}F_{\mu\nu}+\frac{N_f}{8\pi}\partial_\mu\phi\partial^\mu\phi+\frac{1}{8\pi}\mathrm{tr}\left[\partial_\mu U\partial^\mu U^\dagger\right]\right\}-\frac{i}{12\pi}\int \mathrm{tr}[(UdU^\dagger)^3]$$

$U=\mathrm{e}^{i\pi(x)}\in SU(N_f)$ ϕ: η' meson ($U(1)_A$ part, heavier than the pions)

  - 4d QCD also includes η' meson. (It is decoupled in the IR limit.)
  - In IR limit (integrating out η' and photon), $SU(N_f)_1$ WZW model (CFT, c=N_f-1)
- Very similar theory to 4dim QCD!

Pion theory (2)

Bosonization (for the **Schwinger model**)

- In 2d, we can use the **bosonization** technique. [Coleman 1976],[Witten 1984]
  - fermion ↔ boson
- Bosonized **Schwinger model** = pion theory
  - $SU(N_f)$ NLSM with WZW term + η' meson + $U(1)$ gauge theory

$$S = \int d^2x \left\{ \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{iN_f}{4\pi} (\phi + \theta) \epsilon^{\mu\nu} F_{\mu\nu} + \frac{N_f}{8\pi} \partial_\mu \phi \partial^\mu \phi + \frac{1}{8\pi} \mathrm{tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] \right\} - \frac{i}{12\pi} \int \mathrm{tr}[(U dU^\dagger)^3]$$

$U = \mathrm{e}^{i\pi(x)} \in SU(N_f)$ $\quad$ ϕ: η' meson ($U(1)_A$ part, heavier than the pions)

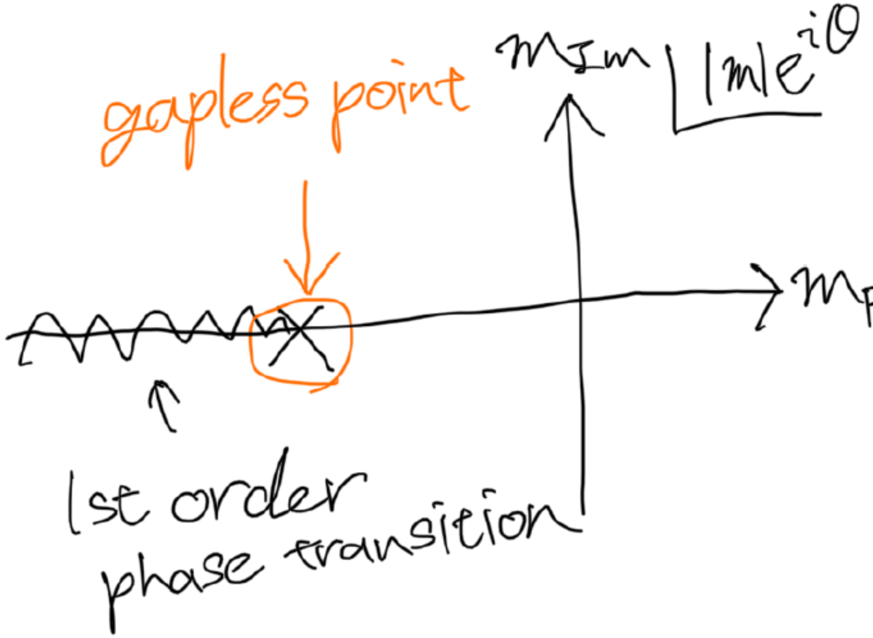
  - 4d **QCD** also includes η' meson. (It is decoupled in the IR limit.)
  - In IR limit (integrating out η' and photon), $SU(N_f)_1$ WZW model (CFT, c=$N_f - 1$)
- Very similar theory to 4dim QCD!

Phase transition @$\theta = \pi$

[Gaiotto, Komargodski, Seiberg 1708.06806]

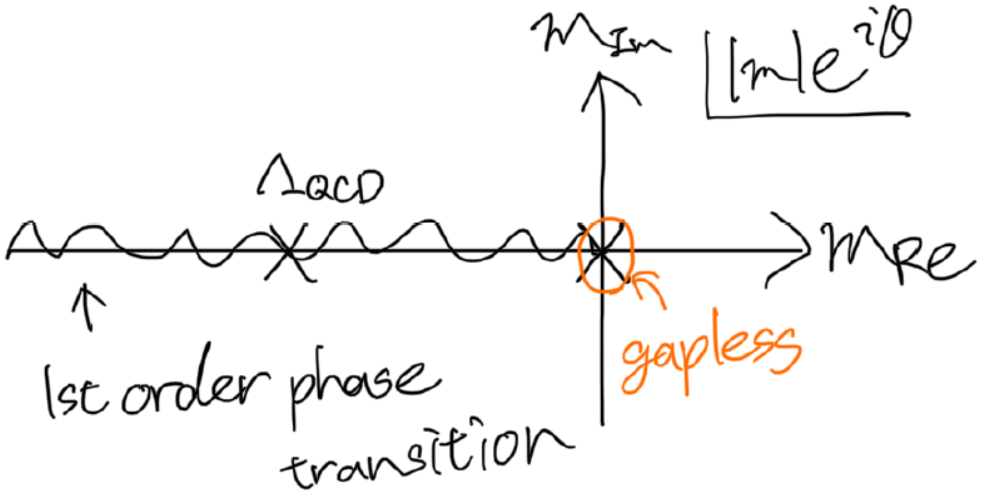
$N_f = 1$ case

- No massless mode
- If $\theta = \pi$, η meson can be massless.
- In large mass, two vacua degenerate at $\theta = \pi$.



$N_f > 1$ case

- Massless pions
- In the small mass region, mass perturbation works.
- In all mass, 2 vacua degenerate at $\theta = \pi$.
- First order phase transition at $\theta = \pi$ for all fermion mass.



Why calculate Schwinger model by TRG?

To calculate θ dependence of free energy

- DMRG
  - 1+1 dim Hamilton formalism.
  - With boundary calculation ← **2π periodicity of θ can be broken** by boundary effects!
  - Periodic boundary takes a higher cost.
- TRG
  - It is easy to take periodic boundary condition (**good for 2π periodicity**)
  - Free energy calculation is easy.

  → **TRG is better!**
  - Can be generalized for higher dimensions. (In principle, we can apply it to 4d QCD.)

  Some negative points
  - Lower bond dimension than DMRG.
    - Discretize for 2-directions. → Bond dimension for each direction is lower than DMRG.
  - Correlation function is more difficult.