Phase structure analysis of 2d CP(1) model with θ term by tensor network renormalization

H. Aizawa,

Kanazawa.U

Collaborator S.Takeda, Y.Yoshimura

2d CP(1) model

Toy model of 4d QCD

Common properties : Asymptotic freedom, confinement, θ terms, etc.

2d CP(1) model with θ terms

Action in continum

$$S = \int d^2x \left(\frac{1}{g^2} |D_\mu z(x)|^2 + \frac{i\theta}{2\pi} \varepsilon_{\mu\nu} \partial_\mu A_\nu \right)$$

Complex scalar

$$z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in \mathbb{C}^2$$

constraint

$$|z(x)|^2 = 1$$

U(1) gauge field

On the square lattice

$$S = -2\beta \sum_{x,\mu} [z^\dagger(x) z(x+\hat{\mu}) U_\mu(x) + z^\dagger(x+\hat{\mu}) z(x) U_\mu^{-1}(x)] - i\frac{\theta}{2\pi} \sum_x q(x)$$

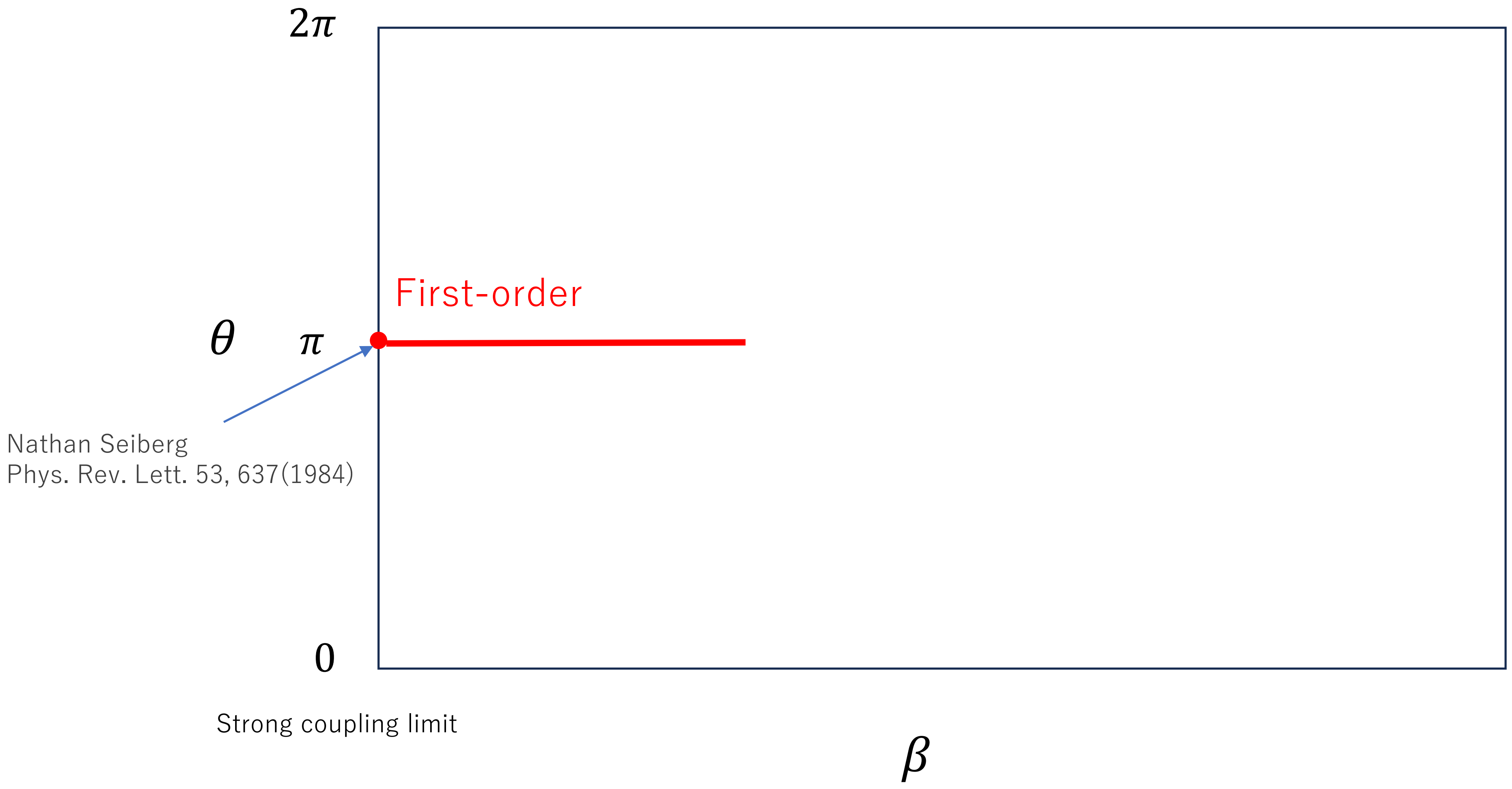
$$U_\mu(x) = e^{iA_\mu(x)}$$

$$q(x) = \frac{1}{i} \ln U_p(x)$$
$$= \{A_1(x) + A_2(x+\hat{1}) - A_1(x+\hat{2}) - A_2(x)\} \quad \text{mod} 2\pi$$

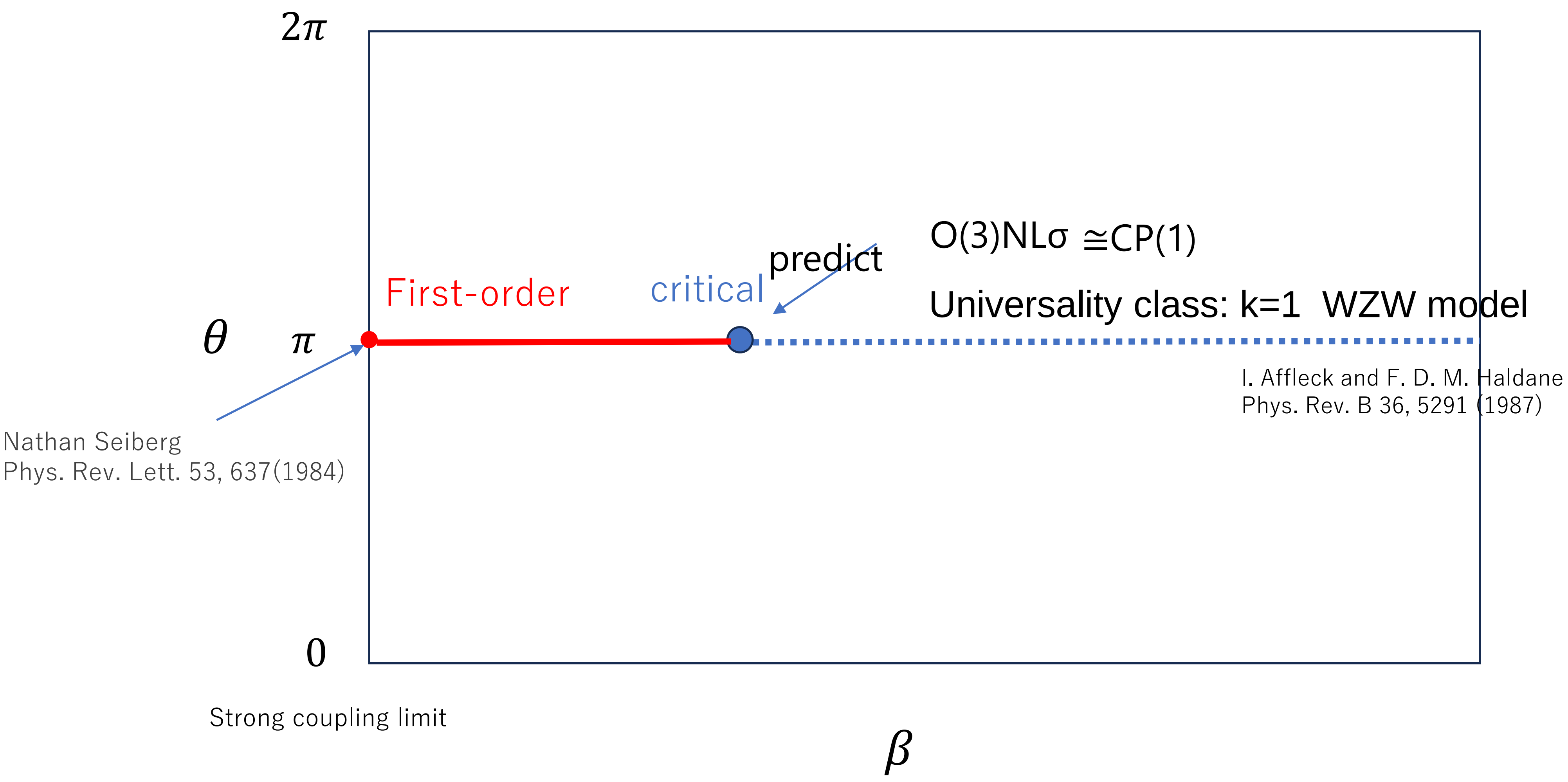
Nathan Seiberg
Phys. Rev. Lett. 53, 637

This model has sign problem

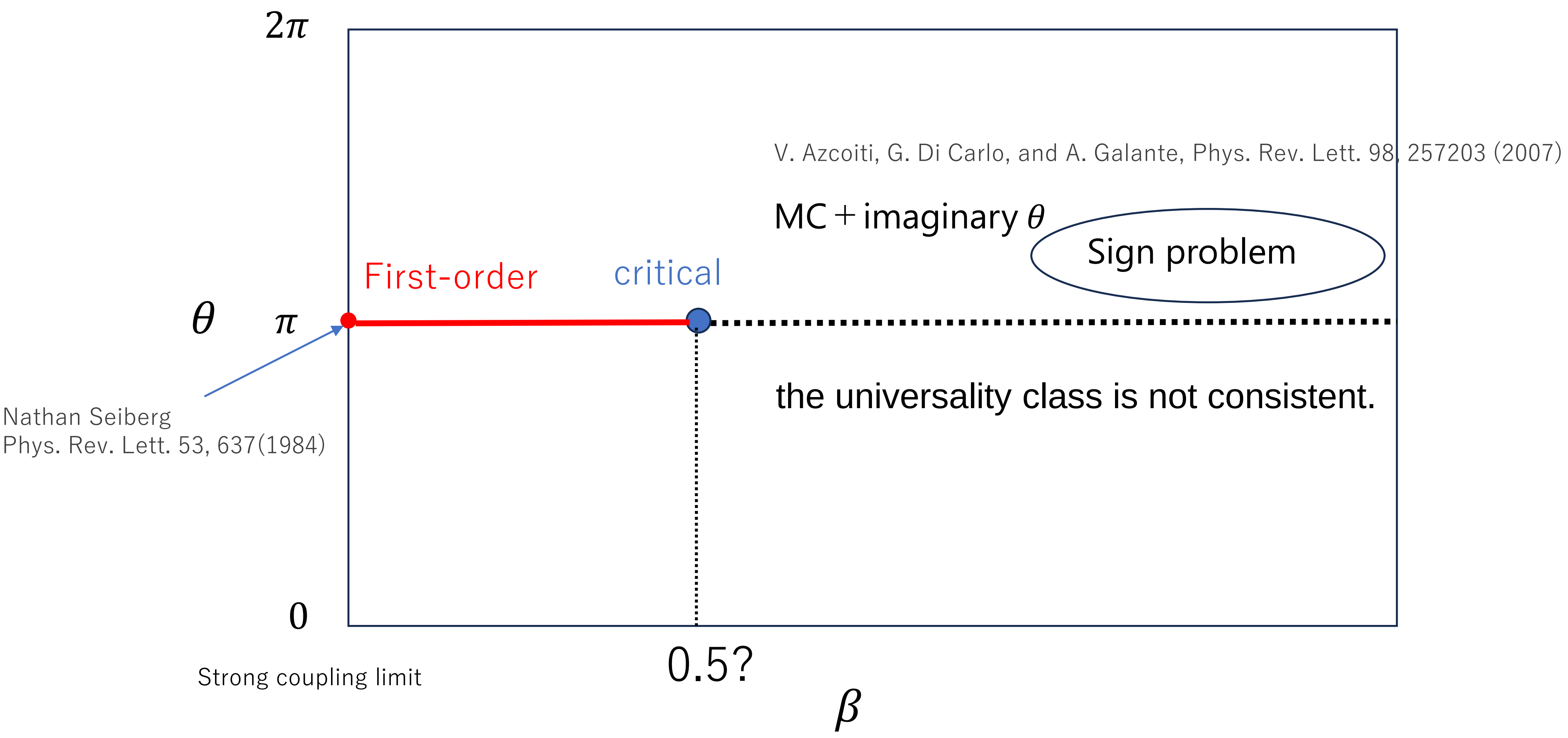
Previous study



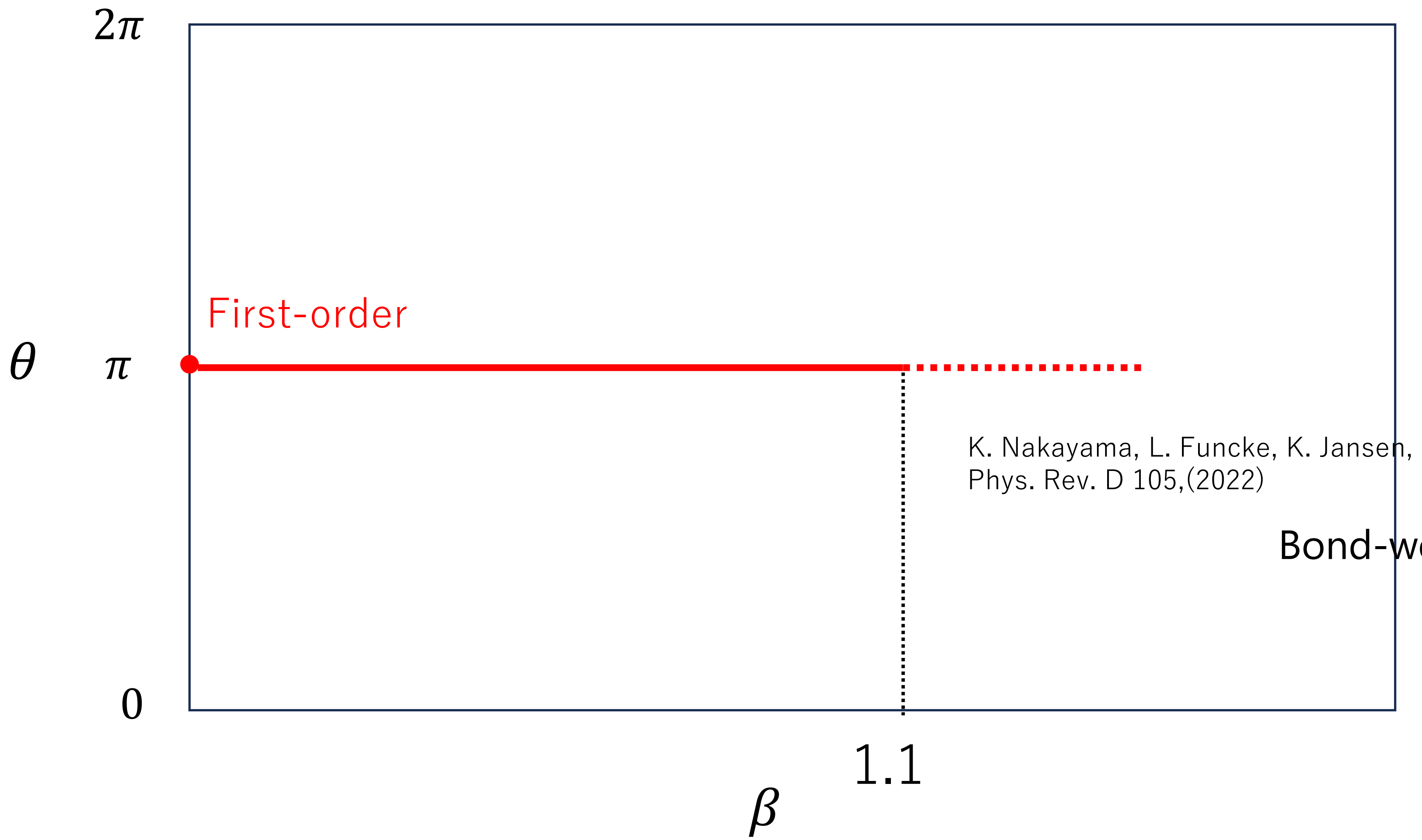
Previous study



Previous study



Previous study by using TRG



K. Nakayama, L. Funcke, K. Jansen, Y. Kao, and S. Kuhn
Phys. Rev. D 105,(2022)

Bond-weighted TRG

Making two improvements

- Initial tensor
- Phase structure analysis method

Initial tensor

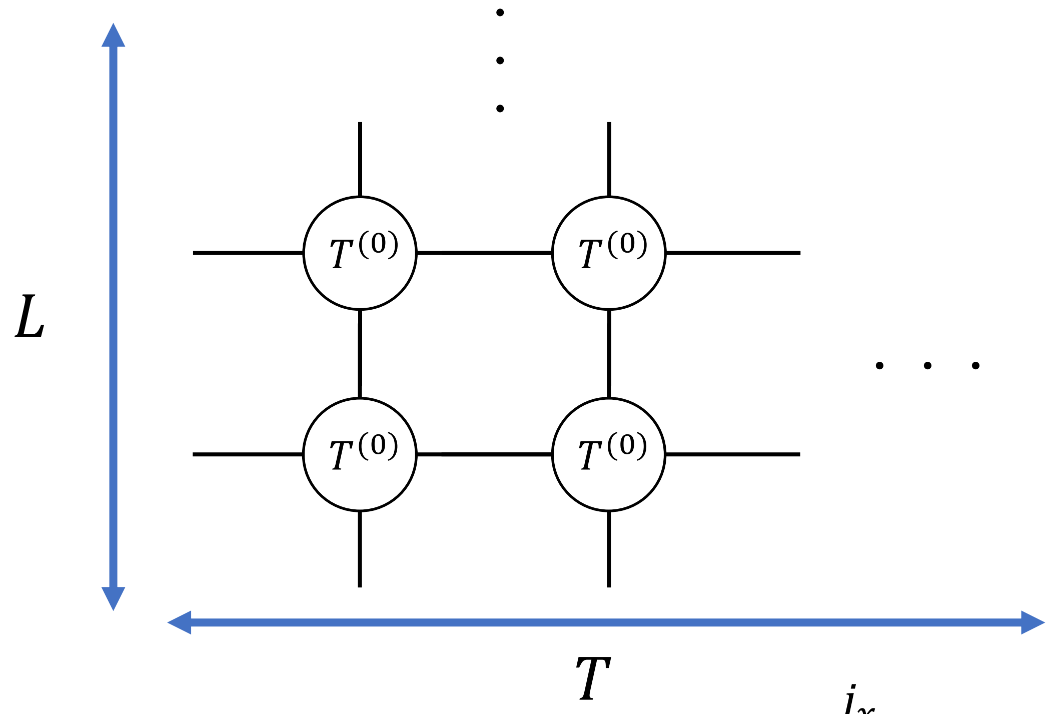
Partition function

$$Z = \int \prod_x dz(x) \int \prod_{x,\mu} dA_\mu(n) e^{-S[z,A_\mu]}$$

Tensor network rep.

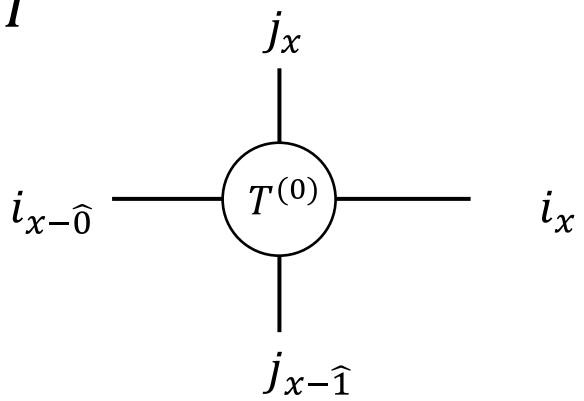
$$Z = \sum_{\{i_x, j_x\}} \prod_x^N T^{(0)}{}_{i_x, j_x, i_{x-\hat{0}}, j_{x-\hat{1}}}$$

$(N = L \times T)$



Initial tensor

$$T^{(0)}{}_{i_x, j_x, i_{x-\hat{0}}, j_{x-\hat{1}}} =$$



We need tensor that have finite index for numerical simulation

Initial tensor

Previous study

Using character expansion

$$e^{i\frac{\theta}{2\pi}q_p} = \sum_{k\in\mathbb{Z}} e^{ik(A_1+A_2-A_3-A_4)}C_k(\theta)$$

truncate

$$C_k(\theta) \propto \frac{1}{k}$$

Converge slowly

New tensor

Using quadrature

$$\int dz f(z) \approx \sum_{i=1}^{N_z} W_{z_i}^{(z)} f(z_i),$$

Scalar field Genz,Keister(1996)

$$\int dU f(A) \approx \sum_{a=1}^{N_A} W_{A_a}^{(A)} f(A_a)$$

Gauge field Ryo Sakai et al.(2018)

→ i, a become tensor index

Comparison of initial tensor

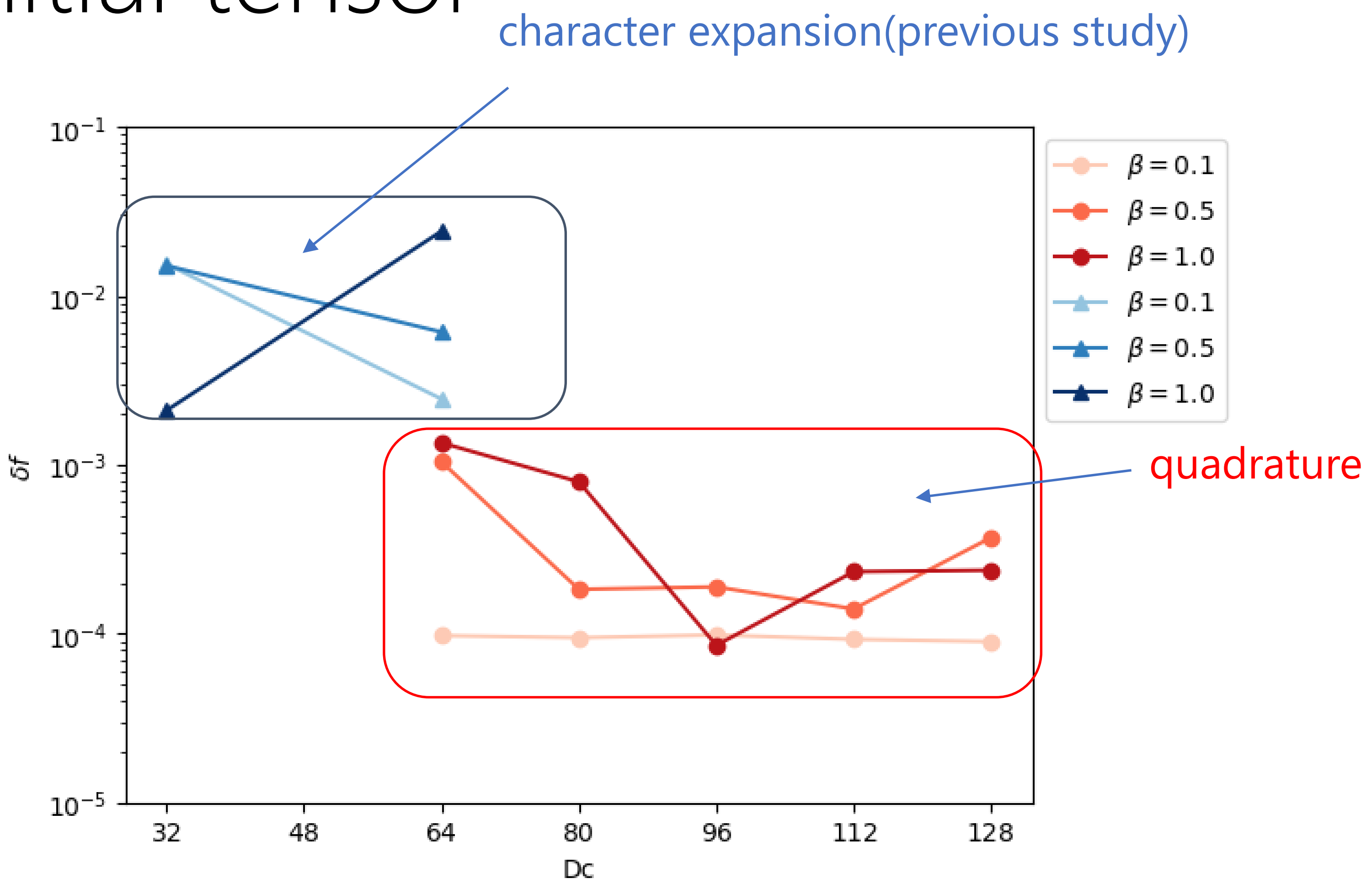
we investigate the error comparing the exact value on 2×2 lattice.

$$\delta f = \left| \frac{f_{tensor} - f_{exact}}{f_{exact}} \right|$$

Parameters

$N_z = 224,\ N_A = 120$

$\theta = \pi$



New initial tensor is better

Phase structure analysis method

Previous study

susceptibility $\chi = -\frac{1}{V}\frac{\partial^2 log Z}{\partial\theta^2}\Big|_{\theta=\pi}$

fitting Z near θ = π is needed

It is difficult to determine the fitting range

K. Nakayama, L. Funcke, K. Jansen, Y. Kao, and S. Kuhn
Phys. Rev. D 105,(2022)

In our study

We use central charge defined in 2d conformal field theory

Z.C. Gu and X.G. Wen
Phys. Rev. B 80, 155131 –(2009)

2d Conformal field theory

Virasoro algebra $$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

Algebra of 2d conformal transformation

$$L_0|h\rangle = h|h\rangle \qquad L_n|h\rangle = 0,\ n > 0 \qquad n, m \in \mathbb{Z}$$

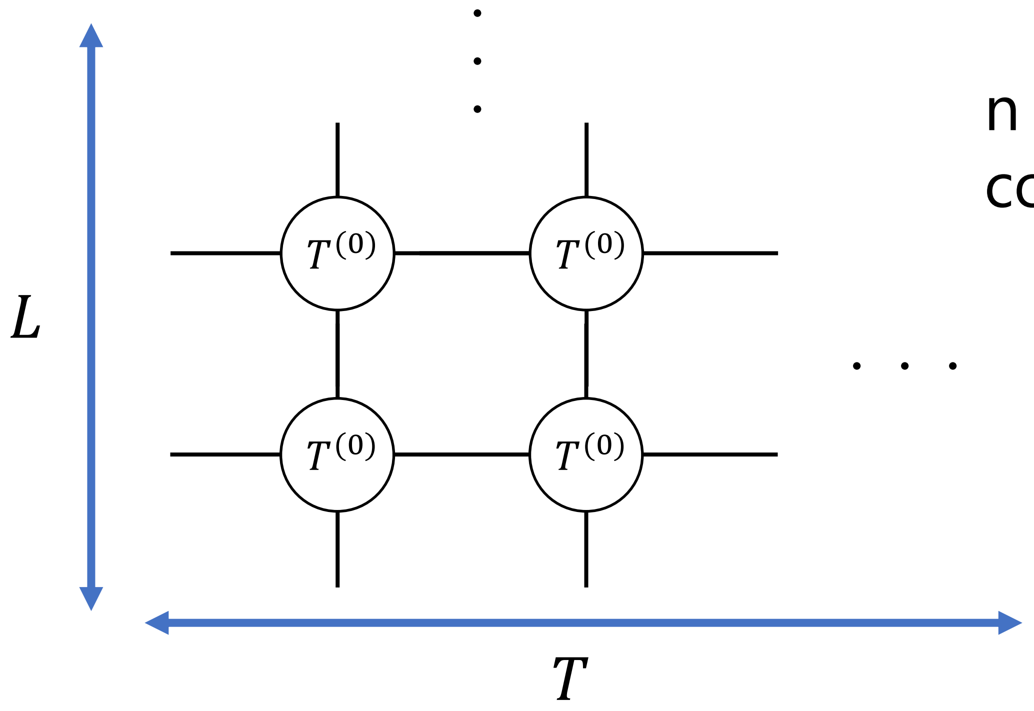
c:central charge

h:conformal weight

c and h Identify Universality

From the prediction of Haldane's conjecture, there should be a critical line with c=1 at θ=π.

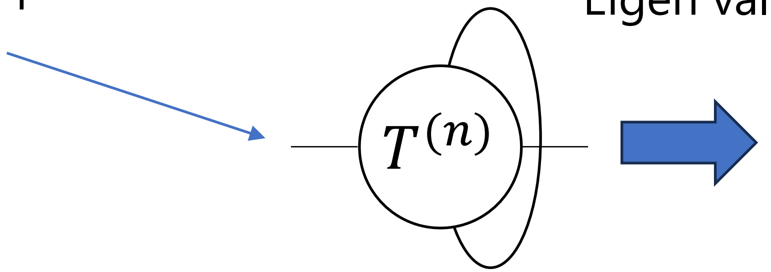
How to compute the central charge by TRG



Partition function

n times
coarse-graining

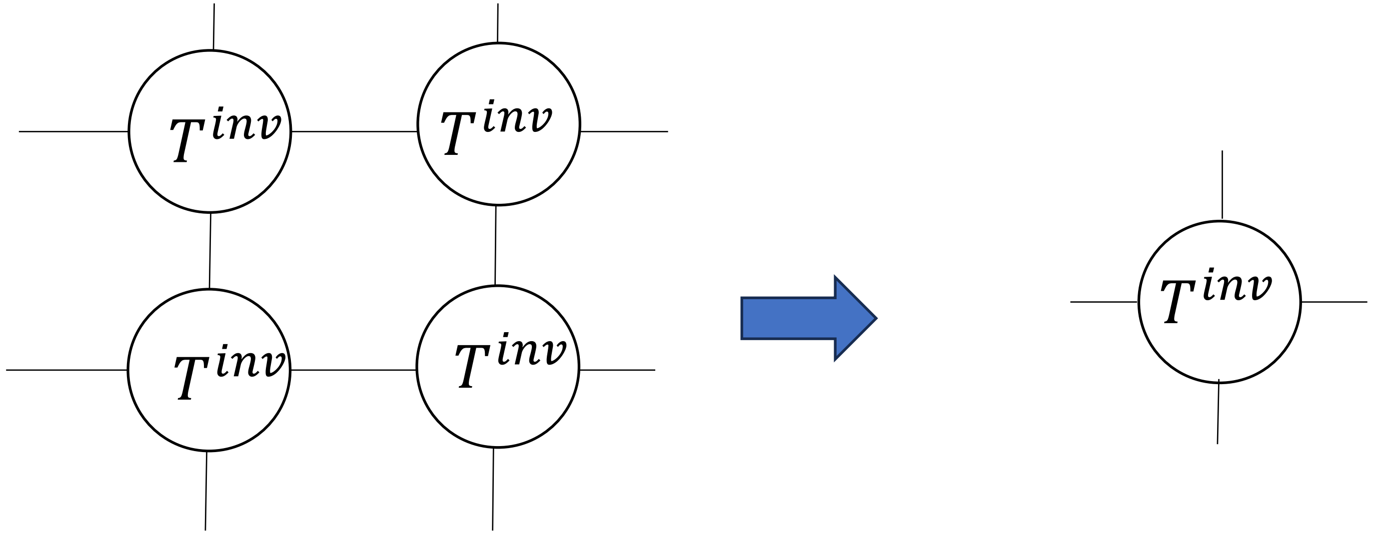
transfer matrix of a system with
periodic boundary conditions
in the spatial direction.



Eigen value

$$\lambda_i = e^{-E_i T}$$

When an invariant tensor is obtained under the TRG, the transfer matrix corresponds to the CFT one.

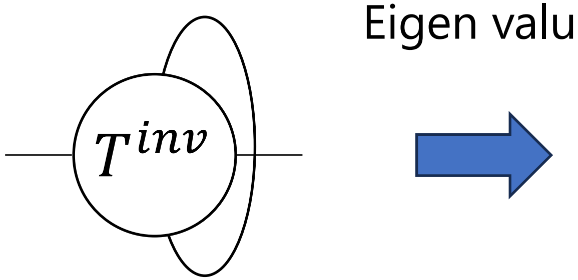


Central charge

$$c = \frac{6}{\pi}\log(\lambda_0)$$

Scaling dimension

$$x_i = h_i + \bar{h}_i = \frac{1}{2\pi}\log(\frac{\lambda_0}{\lambda_i})$$



Eigen value

$$\lambda_i = e^{2\pi(h_i+\bar{h}_i)+\frac{\pi c}{6}}$$

From transfer matrix of cft

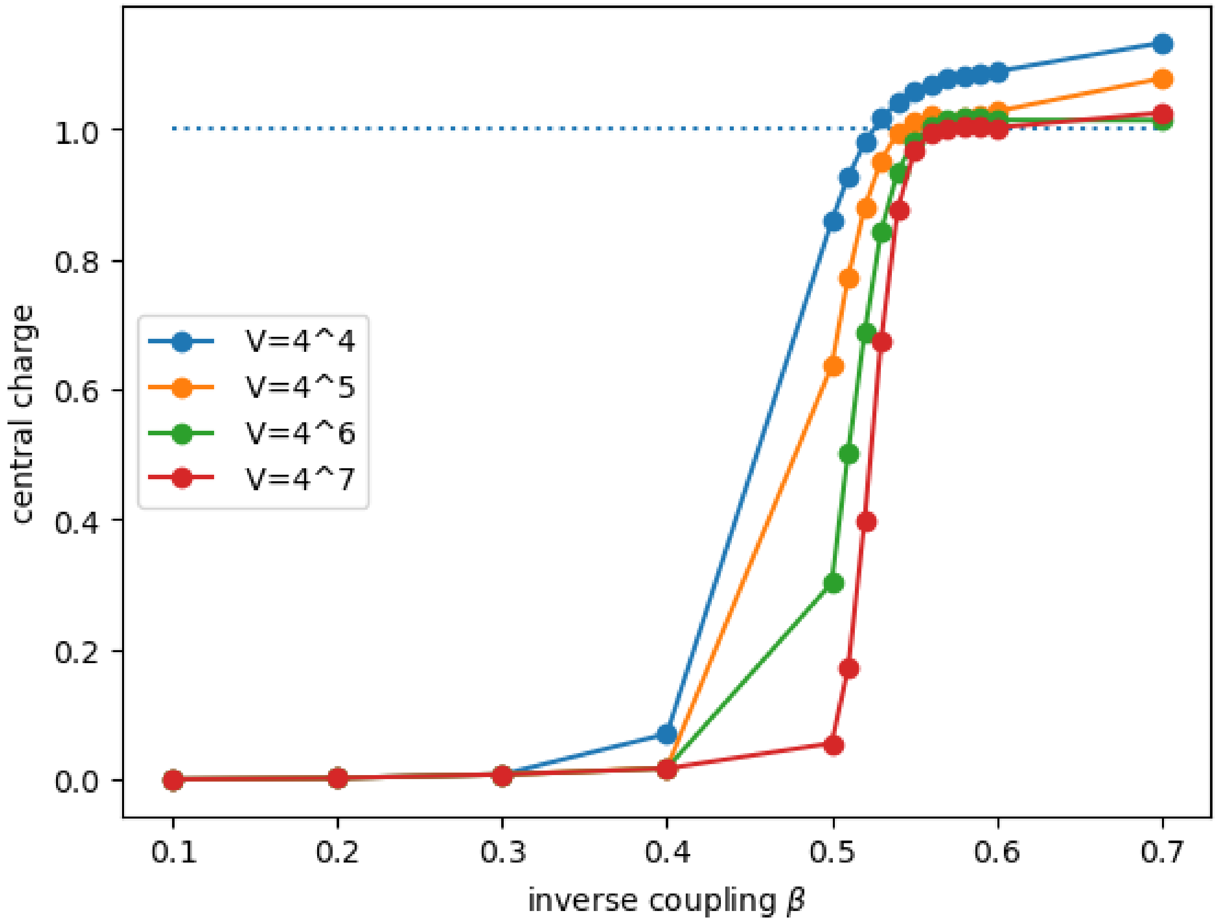
Result

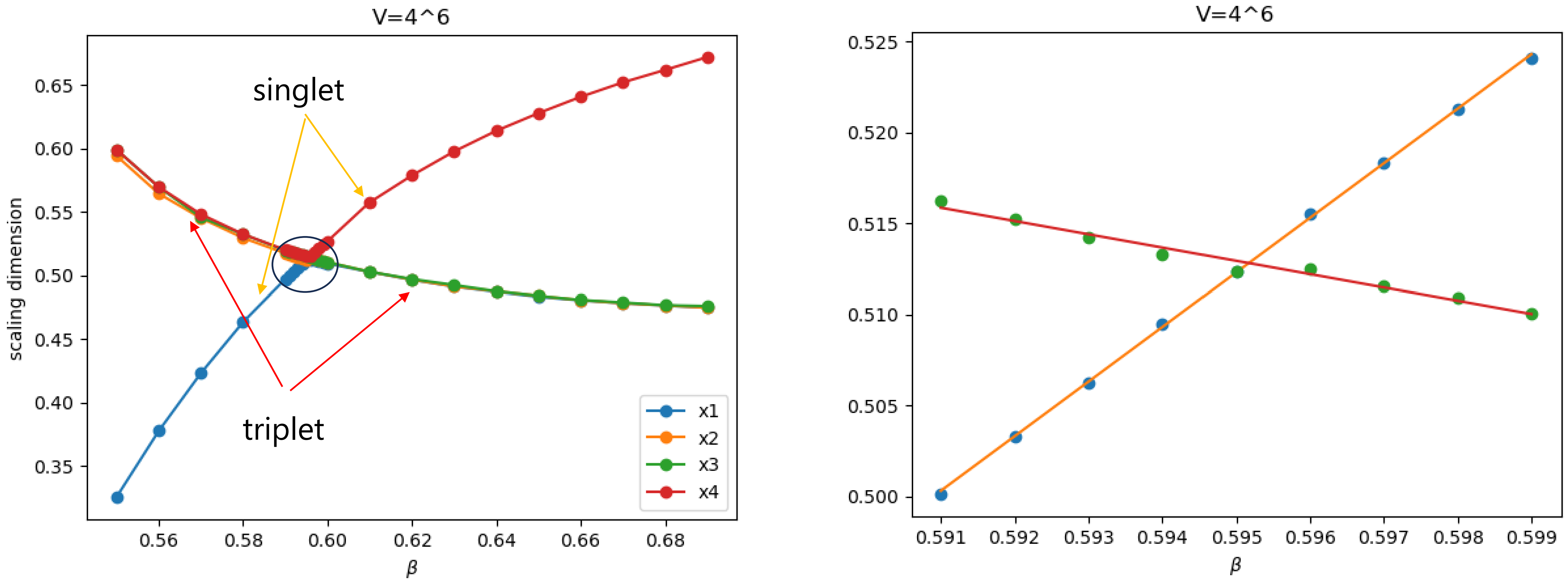
$c = 1$ for $\beta \geq 0.55$

Parameters

$N_z = 226, \;\; N_A = 120,$
$Dc = 128, \theta = \pi$

coarse-graining by bond-weight TRG,
$\mathrm{k} = -1/2$





the BKT transition point is roughly at β = 0.595

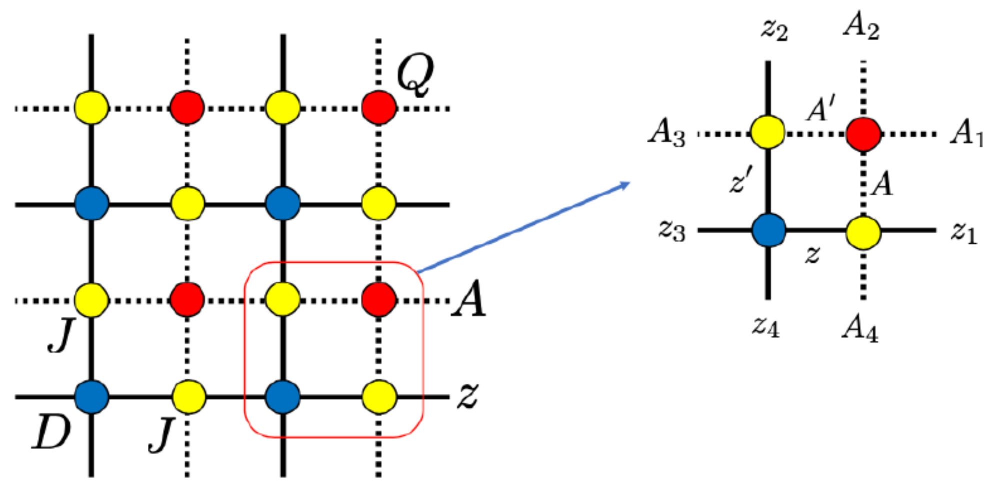
Summary

- We improve tensor network representation of the model by using quadrature. We confirm that the new tensor has a smaller error.

- We analyzed the phase structure by computing the central charge
  Our analysis shows that central charge is 1 for $\beta \geq 0.55$ and
  the critical point is found as predicted from Haldane's conjecture.

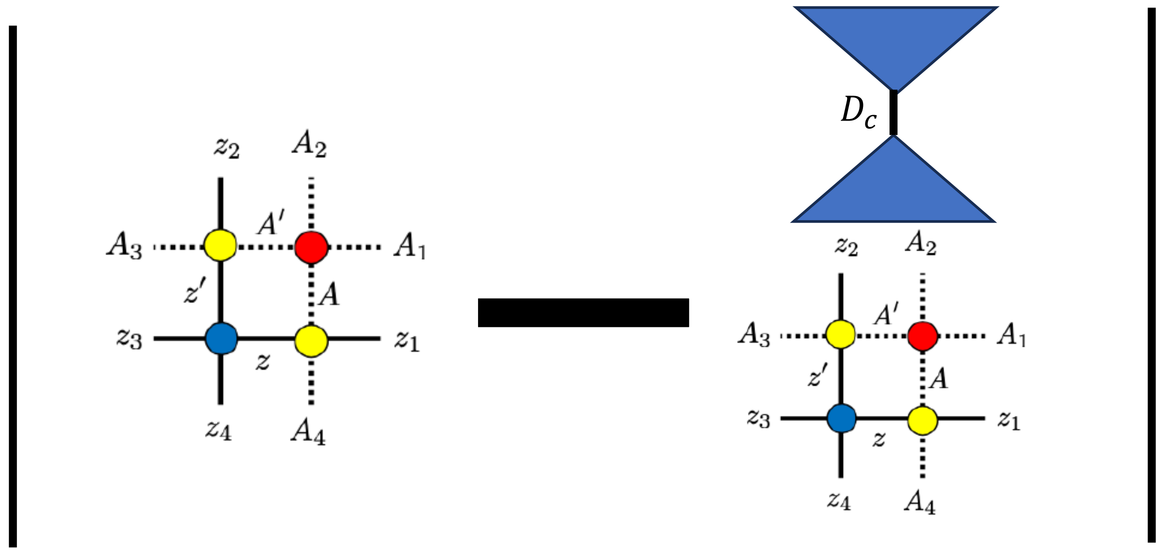
- This critical point corresponds to BKT transition, and the location of the transition point was roughly estimated as $\beta = 0.595$ by crossing of scaling dimensions.

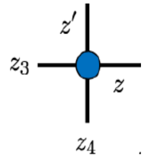
Buck Up



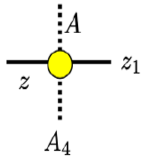
$$\int dz f(z) \approx \sum_{i=1}^{N_z} W_{z_i}^{(z)} f(z_i), \quad \int dU f(A) \approx \sum_{a=1}^{N_A} W_{A_a}^{(A)} f(A_a)$$

minimize

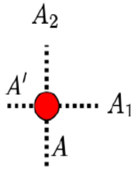




$$= \delta_{z_3,z_4}\delta_{z_3,z}\delta_{z_4,z'}$$

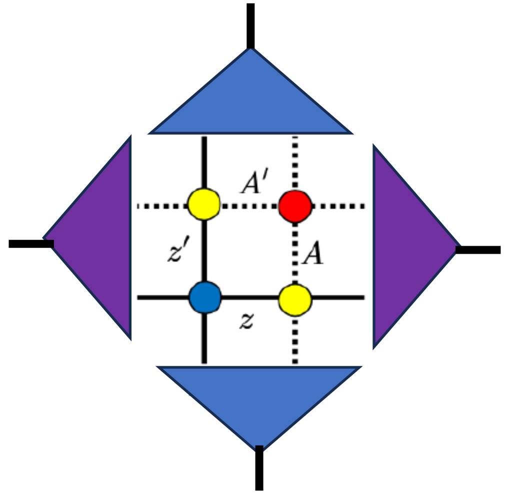


$$= \exp\left[2\beta\left(z^\dagger z_1 e^{iA} + z_1{\dagger} z e^{-iA}\right)\right]\delta_{A,A_4}$$



$$= \exp\left[i\frac{\theta}{2\pi}(A + A_1 - A_2 - A') \bmod 2\pi\right]$$

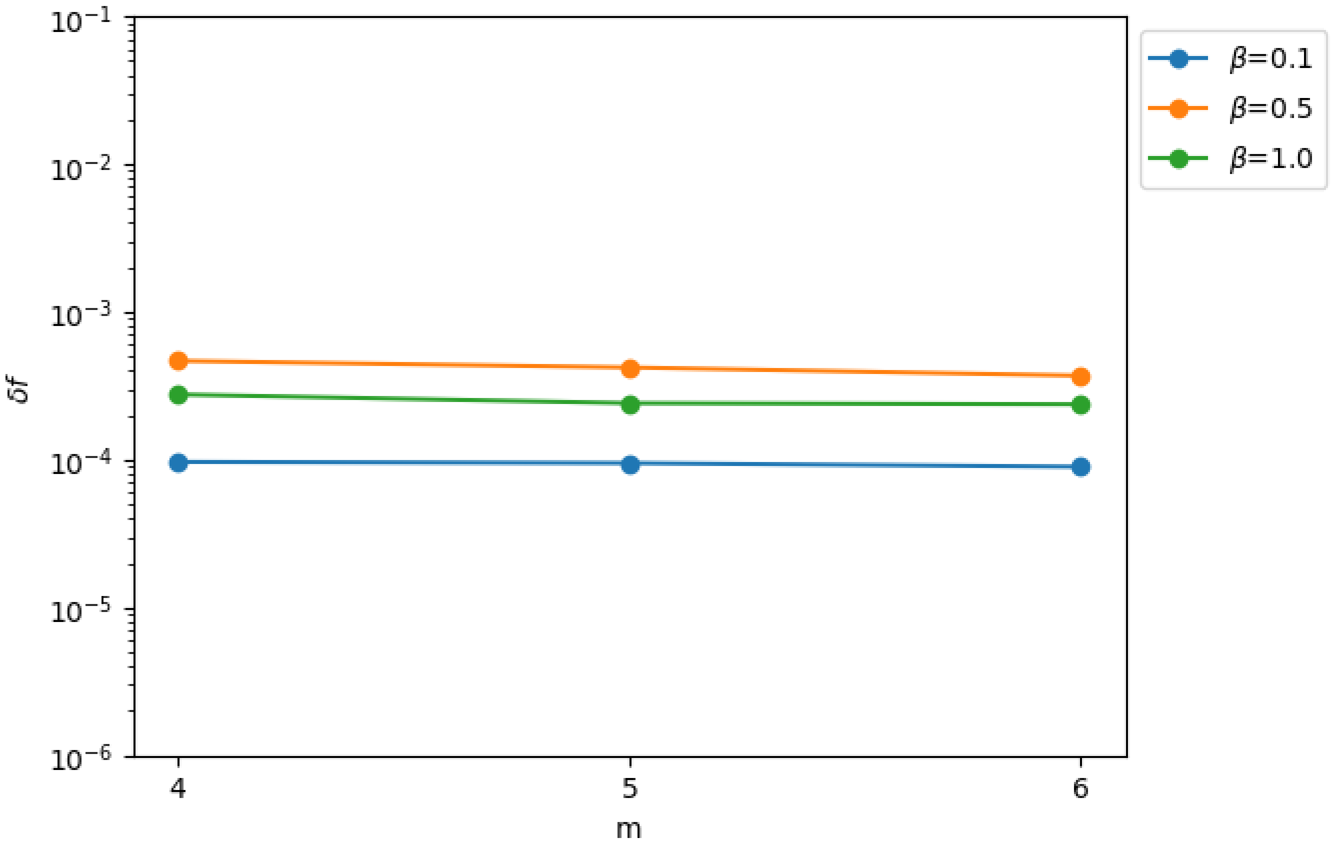
$$T^{(0)}{}_{i_x, j_x, i_{x-\hat{0}}, j_{x-\hat{1}}} =$$



Parameters

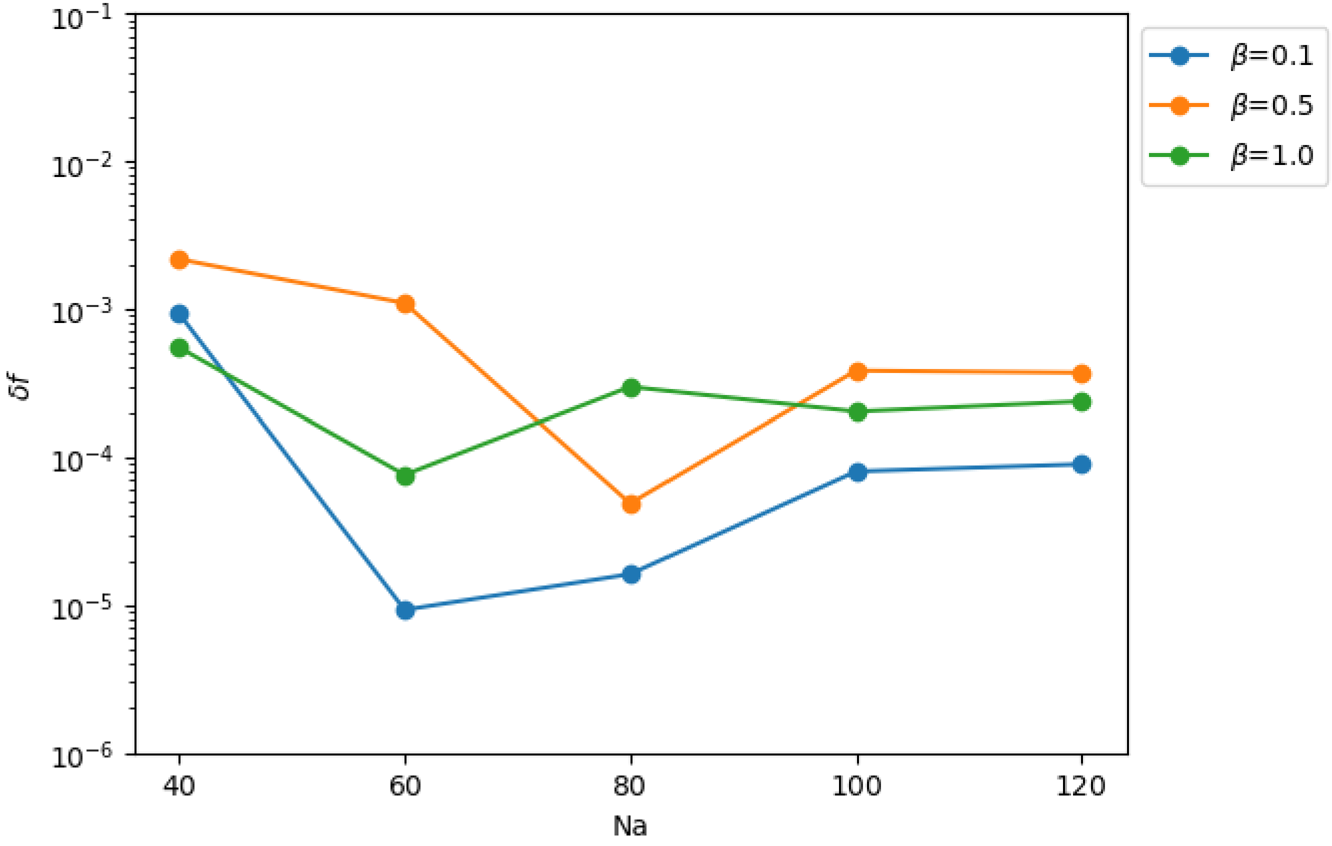
$N_A = 120,$
$Dc = 128,\ \theta = \pi$

$m = 4 \rightarrow N_z = 120,$
$m = 5 \rightarrow N_z = 168$
$m = 6 \rightarrow N_z = 224$



Parameters

$N_z = 224,$
$Dc = 128,\ \theta = \pi$



conformal transformation

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x)$$

infinitesimal transformation $x' = x + \epsilon$

$$ds^2 \rightarrow ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu$$

ϵ satisfy

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$$

Cauchy–Riemann equations (in 2d)

$$z, \overline{z} = x^1 \pm i x^2$$

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \ , \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1$$

$$z \rightarrow f(z) \ , \qquad \overline{z} \rightarrow \overline{f}(\overline{z})$$

conservation law

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0$$

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0.$$

$$T_{z\bar{z}} = T_{\bar{z}z} = 0$$

$$\partial_{\bar{z}} T_{zz} = 0 \quad \text{and} \quad \partial_z T_{\bar{z}\bar{z}} = 0$$

holomorphic

Anti-holomorphic

Virasoro algebra

$$Q = \frac{1}{2\pi i} \oint \left(dz\, T(z)\epsilon(z) + d\overline{z}\, \overline{T}(\overline{z})\overline{\epsilon}(\overline{z}) \right)$$

$$T(z) = \sum_{n \in \mathbf{Z}} z^{-n-2} L_n \ , \qquad \overline{T}(\overline{z}) = \sum_{n \in \mathbf{Z}} \overline{z}^{-n-2} \overline{L}_n \qquad \varepsilon(z) = \sum_{m \in \mathbb{Z}} z^{m+1} \varepsilon_n, \qquad \bar{\varepsilon}(\bar{z}) = \sum_{m \in \mathbb{Z}} \bar{z}^{m+1} \bar{\varepsilon}_n$$

$$Q = \sum_{n \in \mathbb{Z}} (\varepsilon_n L_n + \bar{\varepsilon}_n \bar{L}_n)$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

$$[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{\overline{c}}{12}(n^3 - n)\delta_{n+m,0}$$

$$[L_n, \overline{L}_m] = 0$$

$$[L_n, \phi(w)] = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w) = h(n+1)w^n \phi(w) + w^{n+1}\partial\phi(w)$$

$$|h\rangle = \phi(0)|0\rangle$$

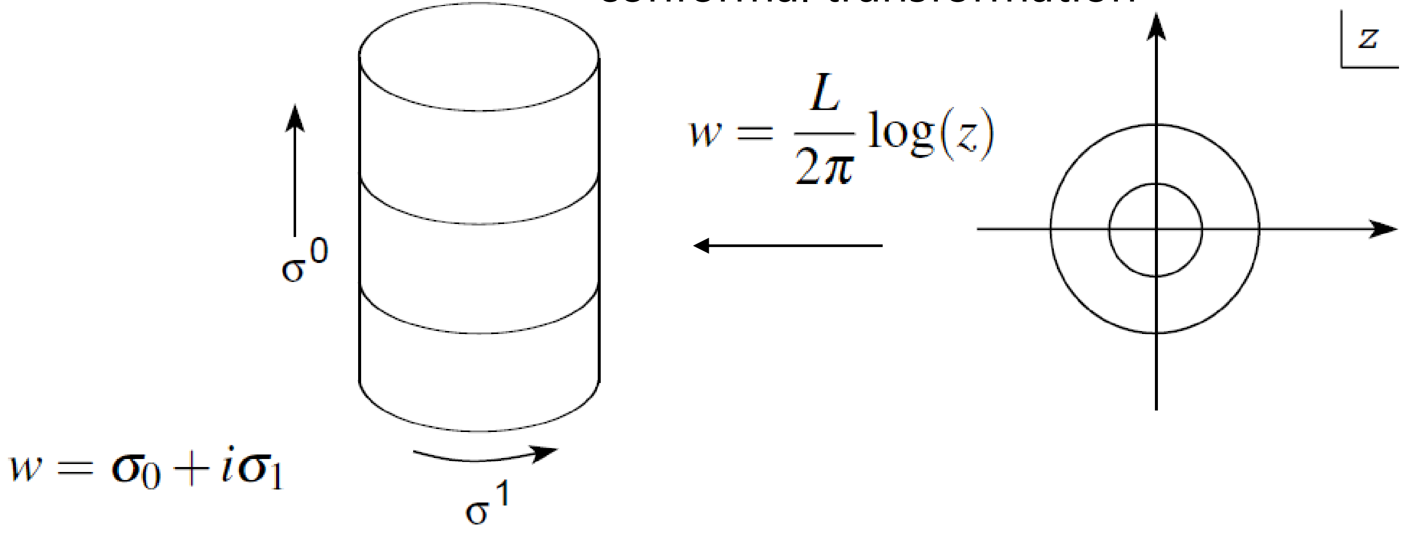
$$L_0|h\rangle = h|h\rangle \qquad L_n|h\rangle = 0,\ n > 0$$

level	dimension	field
0	h	ϕ
1	$h+1$	$\widehat{L}_{-1}\phi$
2	$h+2$	$\widehat{L}_{-2}\phi,\ \widehat{L}^2_{-1}\phi$
3	$h+3$	$\widehat{L}_{-3}\phi,\ \widehat{L}_{-1}\widehat{L}_{-2}\phi,\ \widehat{L}^3_{-1}\phi$
	$\cdots$	
N	$h+N$	$P(N)$ fields ,

CFT on cylinder

conformal transformation

CFT on the complex plane



Hamiltonian on a cylinder

$$H_{cyl} = \int_0^L \frac{d\sigma_1}{2\pi}\left(T_{cyl}(w) + \bar{T}_{cyl}(\bar{w})\right)$$

Conformal transformation of the energy-momentum tensor

$$T(z) \to (\partial f)^2 \, T\big(f(z)\big) + \frac{c}{12} \, S(f, z)$$

$$S(f,z) = \frac{\partial_z f \, \partial_z^3 f - \frac{3}{2}(\partial_z^2 f)^2}{(\partial_z f)^2}$$

$$T_{cly}(w) = \left(\frac{2\pi}{L}\right)^2 \left[T(z)z^2 - \frac{c}{24}\right] = \left(\frac{2\pi}{L}\right)^2 \sum_n e^{-\frac{2\pi n}{L}w} \left(L_n - \frac{c}{24}\delta_{n,0}\right)$$

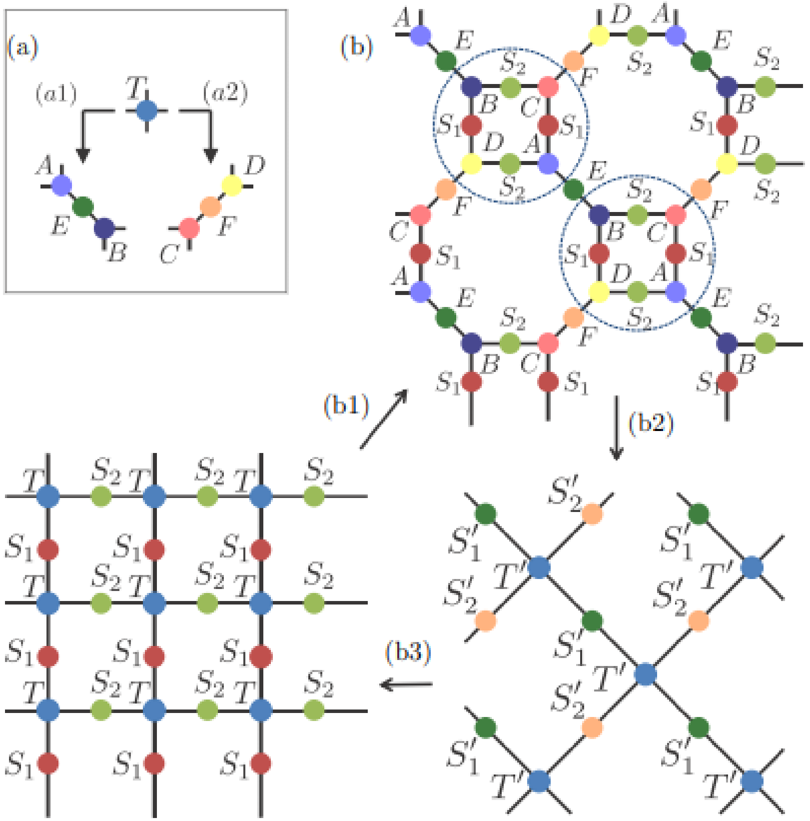
Hamiltonian on a cylinder

$$H_{cyl} = \frac{2\pi}{L}(L_0 + \bar{L}_0) - \frac{\pi c}{6L}$$

transfer matrix

Eigen value

$$e^{-H_{cyl}T} \Rightarrow e^{\frac{2\pi T}{L}(h_n + \bar{h}_n) + \frac{\pi c T}{6L}}$$



$$A_{(x_0,y_0),i} = U_{1(x_0,y_0),i}\sigma_{1ii}^{(1-k)/2},$$

$$E_{i,j} = \delta_{ij}\sigma_{1ii}^{k},$$

$$B_{i,(x_1,y_1)} = \sigma_{1ii}^{(1-k)/2}V_{1i,(x_1,y_1)},$$

$$C_{(x_0,y_1),i} = U_{2(x_0,y_1),i}\sigma_{2ii}^{(1-k)/2},$$

$$F_{i,j} = \delta_{ij}\sigma_{2ii}^{k},$$

$$D_{i,(x_1,y_0)} = \sigma_{2ii}^{(1-k)/2}V_{2i,(x_1,y_0)}.$$

D. Adachi, T. Okubo, and S. Todo,
Bond-weighted Tensor Renormalization Group,
Phys. Rev. B 105, L060402(2022)

$$T_{x_0,x_1,y_0,y_1} \approx \sum_{i}^{\chi} U_{1(x_0,y_0),i}\sigma_{1ii}V_{1i,(x_1,y_1)},$$

$$T_{x_0,x_1,y_0,y_1} \approx \sum_{i}^{\chi} U_{2(x_0,y_1),i}\sigma_{2ii}V_{2i,(x_1,y_0)},$$

$$T'_{x_0,x_1,y_0,y_1} = \sum_{i_0,i_1,i_2,i_3}\left[B_{x_0,(i_0,i_2)}C_{(i_0,i_3),y_0}D_{y_1,(i_1,i_2)} \times A_{(i_1,i_3),x_1}S_{2i_0,i_0}S_{2i_1,i_1}S_{1i_2,i_2}S_{1i_3,i_3}\right],$$

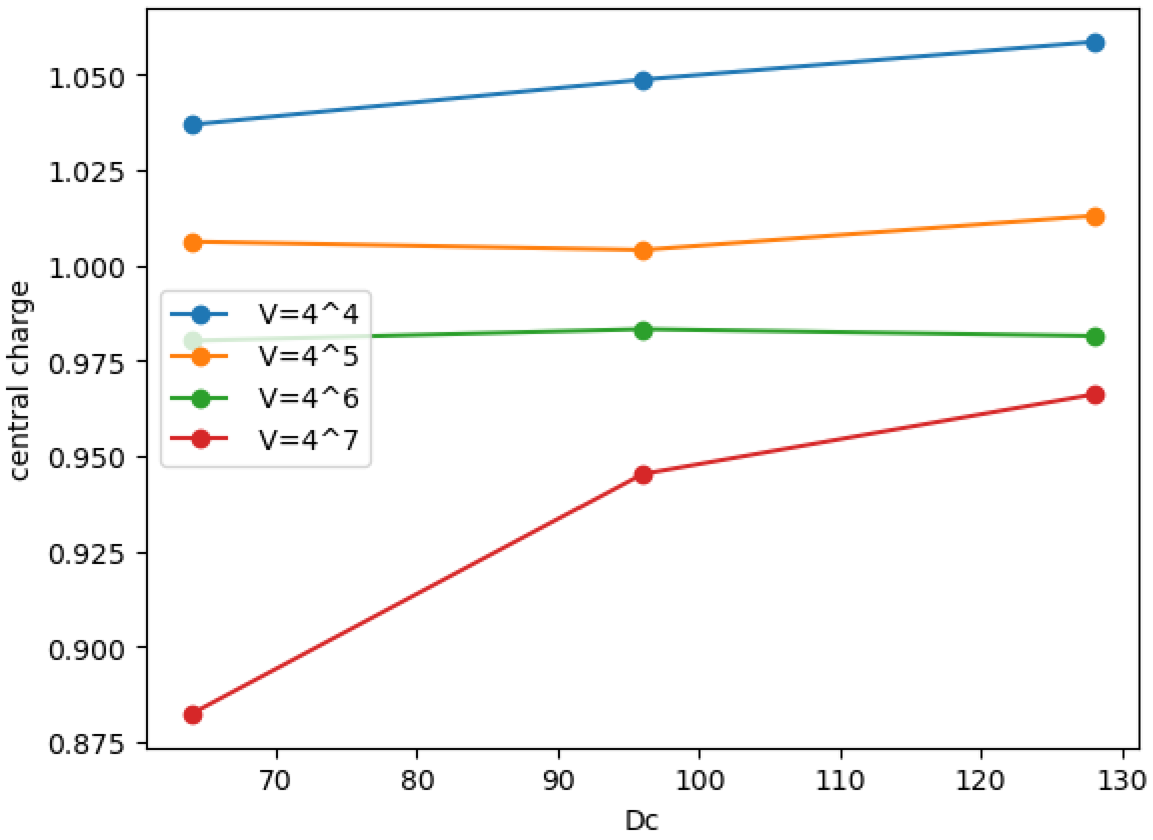
$$S'_1 = E,$$

$$S'_2 = F.$$

Parameters

$N_z = 226, \;\; N_A = 120, \theta = \pi$

coarse-graining by bond-weight TRG,
$\mathrm{k} = -1/2$

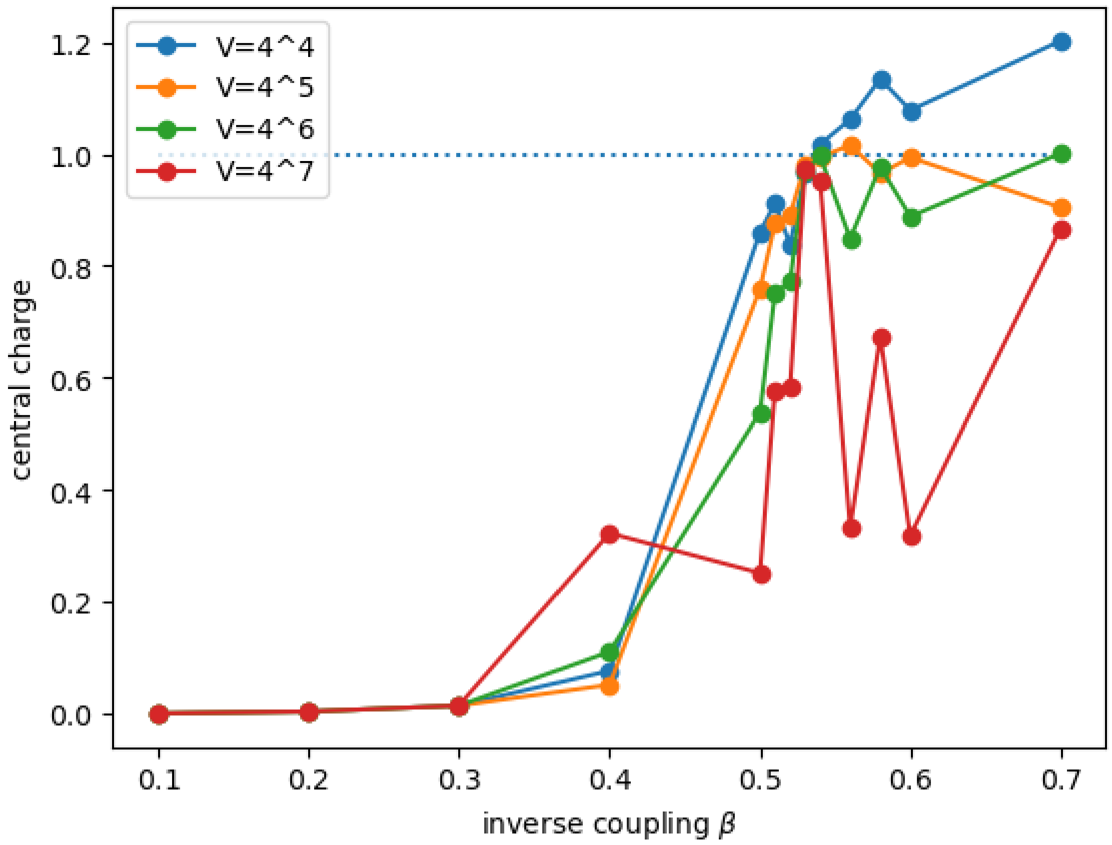


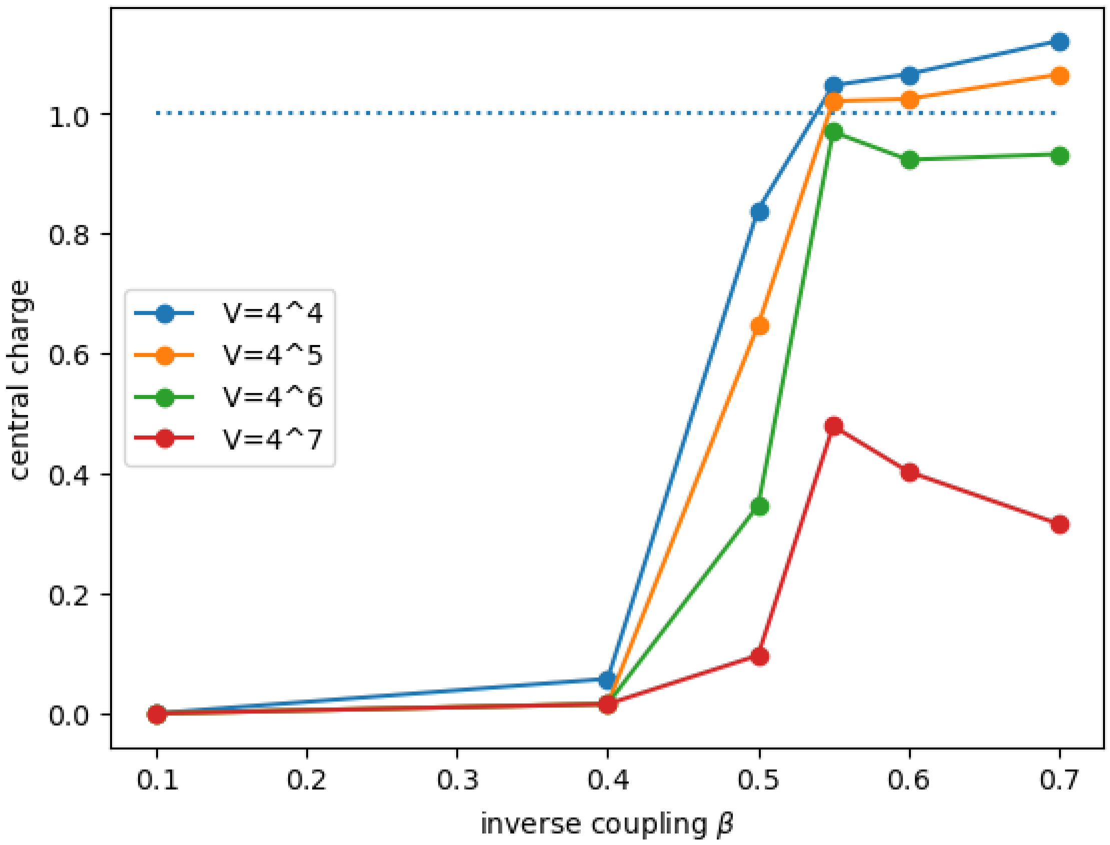
Parameters (previous tensor)

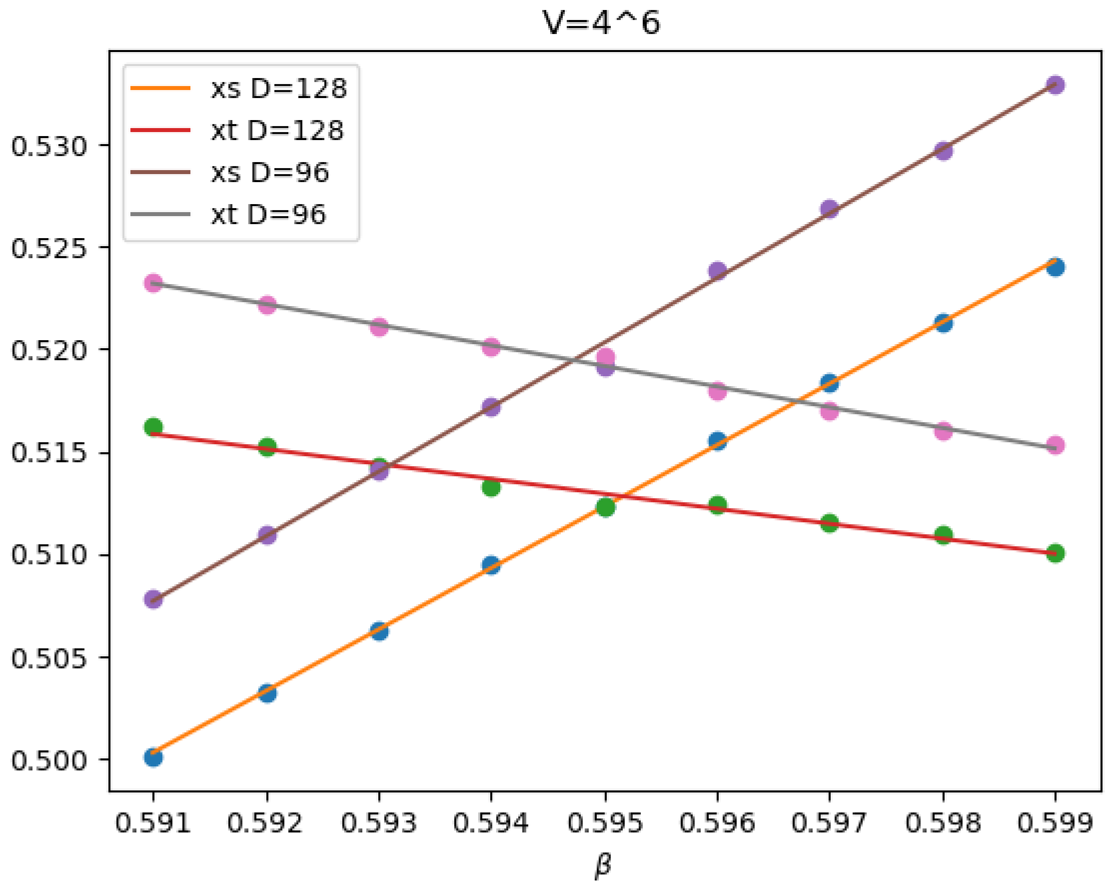
$kmax = 2(16), pmax = 4$
$Dc = 64,\ \theta = \pi$

Parameters (new tensor)

$N_z = 224,, N_A = 120,$
$Dc = 64,\ \theta = \pi$



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xs D=128
xt D=128
xs D=96
xt D=96
0.530
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